GENERALIZED PARTIALLY RELAXED PSEUDOMONOTONE VARIATIONAL INEQUALITIES AND GENERAL AUXILIARY PROBLEM PRINCIPLE

RAM U. VERMA

Received 30 April 2004; Accepted 29 August 2004

Let $T : K \to H$ be a nonlinear mapping from a nonempty closed invex subset K of an infinite-dimensional Hilbert space H into H. Let $f : K \to R$ be proper, invex, and lower semicontinuous on K and let $h : K \to R$ be continuously Fréchet-differentiable on K with h', the gradient of h, (η,α)-*strongly* monotone, and (η,β)-*Lipschitz* continuous on K. Suppose that there exist an $x^* \in K$, and numbers $a > 0$, $r \geq 0$, $\rho(a < \rho < \alpha)$ such that for all $t \in [0,1]$ and for all $x \in K^*$, the set S^* defined by $S^* = \{(h,\eta) : h'(x^* + t(x - x^*))(x - x^*) \geq \langle h'(x^* + t\eta(x,x^*)),\eta(x,x^*)\rangle\}$ is nonempty, where $K^* = \{x \in K : \|x - x^*\| \leq r\}$ and $\eta : K \times K \to H$ is (λ)-*Lipschitz* continuous with the following assumptions. (i) $\eta(x,y) + \eta(y,x) = 0$, $\eta(x,y) = \eta(x,z) + \eta(z,y)$, and $\|\eta(x,y)\| \leq r$. (ii) For each fixed $y \in K$, map $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology. If, in addition, h' is continuous from H equipped with weak topology to H equipped with strong topology, then the sequence $\{x^k\}$ generated by the general auxiliary problem principle converges to a solution x^* of the variational inequality problem (VIP): $\langle T(x^*),\eta(x,x^*)\rangle + f(x) - f(x^*) \geq 0$ for all $x \in K$.

1. Introduction

A tremendous amount of work, applying the auxiliary problem principle in finite- as well as in infinite-dimensional Hilbert space settings, on the approximation-solvability of various classes of variational inequalities and complementarity problems, especially finite-dimensional cases, has been carried out in recent years. During the course of these investigations, there has been a significant progress in developing more generalized classes of mappings in the context of new iterative algorithms. In this paper, we intend based on a general auxiliary problem principle to present the approximation-solvability of a class of variational inequality problems (VIP) involving partially relaxed pseudomonotone mappings along with some modified results on Fréchet-differentiable functions that play a pivotal role in the development of a general framework for the auxiliary problem principle. Results thus obtained generalize/complement investigations of Argyros and

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 90295, Pages 1–12
DOI 10.1155/JIA/2006/90295

Verma [1], El Farouq [7], Verma [20], and others. For more details on general variational inequality problems and the auxiliary problem principle, we refer to [1–23].

Let H be an infinite-dimensional real Hilbert space with the inner product $\langle x,y\rangle$ and norm $\|x\|$ for all $x,y\in H$. We consider the variational inequality problem (VIP) as follows: determine an element $x^*\in K$ such that

$$\langle T(x^*),\eta(x,x^*)\rangle + f(x) - f(x^*) \geq 0 \quad \forall x\in K, \tag{1.1}$$

where K is a nonempty closed invex subset of H, and $\eta: K\times K\to H$ is any mapping with some additional conditions.

When $\eta(x,x^*) = x - x^*$, the VIP (1.1) reduces to the VIP: determine an element $x^*\in K$ such that

$$\langle T(x^*),x-x^*\rangle + f(x) - f(x^*) \geq 0 \quad \forall x\in K, \tag{1.2}$$

where K is a nonempty closed convex subset of H.

When $f=0$ in (1.2), it reduces to the following: find an element $x^*\in K$ such that

$$\langle T(x^*),x-x^*\rangle \geq 0 \quad \forall x\in K. \tag{1.3}$$

Now we recall the following auxiliary result for the approximation solvability of nonlinear variational inequality problems based on iterative procedures.

Lemma 1.1. *For elements $u,v,w\in H$,*

$$\|u\|^2 + \langle u,\eta(v,w)\rangle \geq -\frac{1}{4}\|\eta(v,w)\|^2. \tag{1.4}$$

Lemma 1.2. *For $u,v\in H$,*

$$\langle u,v\rangle = \frac{\|u+v\|^2 - \|u\|^2 - \|v\|^2}{2}. \tag{1.5}$$

Now recall and in some cases upgrade the existing notions in the literature. Let $\eta: H\times H\to H$ be any mapping.

Definition 1.3. A mapping $T: H\to H$ is called

(i) (η)-*monotone* if for each $x,y\in H$, there exists,

$$\langle T(x)-T(y),\eta(x,y)\rangle \geq 0; \tag{1.6}$$

(ii) (η,r)-*strongly* monotone if there exists a positive constant r such that

$$\langle T(x)-T(y),\eta(x,y)\rangle \geq r\|x-y\|^2 \quad \forall x,y\in H; \tag{1.7}$$

(iii) (r)-*expansive* if

$$\|T(x)-T(y)\| \geq r\|\eta(x,y)\|; \tag{1.8}$$

(iv) expansive if $r=1$ in (iii),

(v) (η,γ)-*cocoercive* if there exists a constant $\gamma > 0$ such that

$$\langle T(x)-T(y),\eta(x,y)\rangle \geq r\|T(x)-T(y)\|^2 \quad \forall x,y\in H; \tag{1.9}$$

(vi) (η)-*pseudomonotone* if

$$\langle T(y),\eta(x,y)\rangle \geq 0 \Longrightarrow \langle T(x),\eta(x,y)\rangle \geq 0; \tag{1.10}$$

(vii) (η,b)-*strongly* pseudomonotone if

$$\langle T(y),\eta(x,y)\rangle \geq 0 \Longrightarrow \langle T(x),\eta(x,y)\rangle \geq b\|x-y\|^2 \quad \forall x,y\in H; \tag{1.11}$$

(viii) (η,c)-*pseudococoercive* if there exists a constant $c > 0$ such that

$$\langle T(y),\eta(x,y)\rangle \geq 0 \Longrightarrow \langle T(x),\eta(x,y)\rangle \geq c\|T(x)-T(y)\|^2 \quad \forall x,y\in H; \tag{1.12}$$

(ix) (η)-*quasimonotone* if

$$\langle T(y),\eta(x,y)\rangle > 0 \Longrightarrow \langle T(x),\eta(x,y)\rangle \geq 0 \quad \forall x,y\in H; \tag{1.13}$$

(x) (η,L)-*relaxed* (also called weakly monotone) if there is a positive constant L such that

$$\langle T(x)-T(y),\eta(x,y)\rangle \geq (-L)\|x-y\|^2 \quad \forall x,y\in H; \tag{1.14}$$

(xi) (η)-*hemicontinuous* if for all $x,y,w\in H$, the function

$$t\in[0,1] \longrightarrow \langle T(y+t\eta(x,y)),w\rangle \tag{1.15}$$

is continuous;

(xii) (η,β)-*Lipschitz* continuous if there exists a constant $\beta \geq 0$ such that

$$\|T(x)-T(y)\| \leq \beta\|\eta(x,y)\|; \tag{1.16}$$

(xiii) (η,γ)-*partially* relaxed monotone if there exists a positive constant γ such that

$$\langle T(x)-T(y),\eta(z,y)\rangle \geq (-\gamma)\|z-x\|^2 \quad \forall x,y,z\in H; \tag{1.17}$$

(xiv) (η,γ)-*partially* relaxed pseudomonotone if there exists a positive constant γ such that

$$\langle T(y),\eta(z,y)\rangle \geq 0 \Longrightarrow \langle T(x),\eta(z,y)\rangle \geq (-\gamma)\|z-x\|^2 \quad \forall x,y,z\in H. \tag{1.18}$$

LEMMA 1.4. *Let $T: H \to H$ be (η,α)-cocoercive and let $\eta: H\times H \to H$ be a mapping such that*

(i) $\|\eta(x,y)\| \leq \lambda\|x-y\|$;

(ii) $\eta(x,y)+\eta(y,x) = 0$;

(iii) $\eta(x,y) = \eta(x,z)+\eta(z,y)$.

Then T is $(\eta,-(\lambda^2/4\alpha))$-partially relaxed monotone.

Proof. Since $T : H \to H$ is (η,α)-*cocoercive*, we have

$$\begin{aligned}\langle T(x)-T(y),\eta(z,y)\rangle &= \langle T(x)-T(y),\eta(z,x)\rangle + \langle T(x)-T(y),\eta(x,y)\rangle \\ &\geq \alpha\|T(x)-T(y)\|^2 + \langle T(x)-T(y),\eta(z,x)\rangle \\ &= \alpha\left(\|T(x)-T(y)\|^2 + \frac{1}{\alpha}\langle T(x)-T(y),\eta(z,x)\rangle\right) \\ &\geq -\left\{\frac{1}{4\alpha}\|\eta(z,x)\|^2\right\} \geq -\left\{\frac{\lambda^2}{4\alpha}\|z-x\|^2\right\}.\end{aligned} \tag{1.19}$$

□

Definition 1.5. A mapping $T : H \to H$ is said to be μ-*cocoercive* [2] if for each $x, y \in H$, there exists

$$\langle T(x)-T(y),x-y\rangle \geq \mu\|T(x)-T(y)\|^2, \tag{1.20}$$

where μ is a positive constant.

Example 1.6. Let $T : K \to H$ be nonexpansive. Then $I - T$ is 1/2-*cocoercive*, where I is the identity mapping on H. For if $x, y \in K$, we have

$$\begin{aligned}\|(I-T)(x)-(I-T)(y)\|^2 &= \|x-y-(T(x)-T(y))\|^2 \\ &= \|x-y\|^2 - 2\langle x-y,T(x)-T(y)\rangle + \|T(x)-T(y)\|^2 \\ &\leq 2\{\|x-y\|^2 - \langle x-y,T(x)-T(y)\rangle\} \\ &= 2\langle x-y,(I-T)(x)-(I-T)(y)\rangle,\end{aligned} \tag{1.21}$$

that is,

$$\langle (I-T)(x)-(I-T)(y),x-y\rangle \geq \frac{1}{2}\|(I-T)(x)-(I-T)(y)\|^2. \tag{1.22}$$

A subset K of H is said to be invex if there exists a function $\eta : K \times K \to H$ such that whenever $x, y \in K$ and $t \in [0,1]$, it follows that

$$x + t\eta(y,x) \in K. \tag{1.23}$$

A function $f : K \to R$ is called invex if whenever $x, y \in K$ and $t \in [0,1]$, it follows that

$$f(x+t\eta(y,x)) \leq (1-t)f(x) + tf(y). \tag{1.24}$$

2. Some auxiliary results

This section deals with some auxiliary results [2] and their modified versions crucial to the approximation-solvability of VIP (1.1). Let $h : H \to R$ be a continuously Fréchet-differentiable mapping on a Hilbert space H. It follows that $h'(x) \in L(H,R)$—the space of all bounded linear operators from H into R. From now on, we will denote the real number $h'(x)(y)$ by $\langle h'(x), y\rangle$ for all $x, y \in H$.

Lemma 2.1. *Let H be a real Hilbert space and let K be a nonempty closed invex subset of H. Let h', the gradient of $h : K \to R$, be (η,α)-strongly monotone on K and let the following assumptions hold.*

(i) *There exist an $x^* \in K$ and a number $r \geq 0$ such that for all $x \in K^*$ and $t \in [0,1]$, the mapping $\eta : K \times K \to H$ satisfies*

$$||\eta(x,y)|| \leq r. \tag{2.1}$$

(ii) *The set S^* defined by*

$$S^* = \{(h,\eta) : h'(x^* + t(x - x^*))(x - x^*) \geq \langle h'(x^* + t\eta(x,x^*)), \eta(x,x^*)\rangle\} \tag{2.2}$$

is nonempty, where $h : K \to R$ is a continuously Fréchet-differentiable mapping, and the set K^* is defined by

$$K^* = \{x \in K : ||x - x^*|| \leq r\}. \tag{2.3}$$

Then for all $x \in K^*$ and $(h,\eta) \in S^*$,

$$h(x) - h(x^*) - \langle h'(x^*), \eta(x,x^*)\rangle \geq \frac{\alpha}{2}||x - x^*||^2. \tag{2.4}$$

Lemma 2.2. *Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let h', the gradient of $h : K \to R$, be (α)-strongly monotone on K and let $h : K \to R$ be a continuously Fréchet-differentiable mapping. Then for all $x, x^* \in K$,*

$$h(x) - h(x^*) - \langle h'(x^*), x - x^*\rangle \geq \frac{\alpha}{2}||x - x^*||^2. \tag{2.5}$$

Lemma 2.3. *Let H be a real Hilbert space and let K be a nonempty closed invex subset of H. Let h', the gradient of $h : K \to R$, be (η,δ)-Lipschitz continuous on K and let the following assumptions hold.*

(i) *There exist an $x^* \in K$ and a number $q \geq 0$ such that for all $x \in K_1$ and $t \in [0,1]$, the mapping $\eta : K \times K \to H$ satisfies*

$$||\eta(x,y)|| \leq q. \tag{2.6}$$

(ii) *The set S_1 defined by*

$$S_1 = \{(h,\eta) : h'(x^* + t(x - x^*))(x - x^*) \leq \langle h'(x^* + t\eta(x,x^*)), \eta(x,x^*)\rangle\} \tag{2.7}$$

is nonempty, where $h : K \to R$ is a continuously Fréchet-differentiable mapping, and the set K_1 is defined by

$$K_1 = \{x \in K : ||x - x^*|| \leq q\}. \tag{2.8}$$

Then for all $x \in K_1$ and $(h,\eta) \in S_1$,

$$h(x) - h(x^*) - \langle h'(x^*), \eta(x,x^*)\rangle \leq \frac{\delta}{2}||x - x^*||^2. \tag{2.9}$$

3. General auxiliary problem principle

In this section, we present the approximation-solvability of the VIP (1.1) using the convergence analysis for the general auxiliary problem principle.

Algorithm 3.1. *For arbitrarily chosen initial point $x^0 \in K$, determine an iterate x^{k+1} such that*

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), \eta(x, x^{k+1}) \rangle + \rho\big(f(x) - f(x^{k+1})\big) \geq 0, \tag{3.1}$$

for all $x \in K$, where $h : K \to R$ is continuously Fréchet-differentiable, $f : K \to R$ is proper, invex, and lower semicontinuous, $\rho > 0$, and $\eta : K \times K \to H$ is any mapping.

Algorithm 3.2. *For arbitrarily chosen initial point $x^0 \in K$, determine an iterate x^{k+1} such that*

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), x - x^{k+1} \rangle + \rho\big(f(x) - f(x^{k+1})\big) \geq 0, \tag{3.2}$$

for all $x \in K$, where $h : K \to R$ is continuously Fréchet-differentiable, $\rho > 0$, and K is a nonempty closed convex subset of H.

Algorithm 3.3. *For arbitrarily chosen initial point $x^0 \in K$, determine an iterate x^{k+1} such that*

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), x - x^{k+1} \rangle \geq 0, \tag{3.3}$$

for all $x \in K$, where $h : K \to R$ is continuously Fréchet-differentiable, $\rho > 0$, and K is a nonempty closed convex subset of H.

We now present, based on Algorithm 3.1, the approximation solvability of the VIP (1.1) in a Hilbert space setting.

Theorem 3.4. *Let H be a real infinite-dimensional Hilbert space and let K be a nonempty closed invex subset of H. Let $T : K \to H$ be (η, γ)-partially relaxed pseudomonotone. Let $f : K \to R$ be proper, invex, and lower semicontinuous on K, let $h : K \to R$ be continuously Fréchet-differentiable on K with h', the gradient of h, (η, α)-strongly monotone, and (η, β)-Lipschitz continuous, and let h' be continuous from H equipped with weak topology to H equipped with strong topology. Suppose that the following assumptions hold.*

(i) *There exist a $y^* \in K$ and numbers $a > 0$, $r \geq 0$, $\rho(a < \rho < \alpha/2\gamma)$ such that for all $t \in [0,1]$ and for all $x \in K^*$, the set S^* defined by*

$$S^* = \{(h, \eta) : h'(y^* + t(x - y^*))(x - y^*) \geq \langle h'(y^* + t\eta(x, y^*)), \eta(x, y^*) \rangle\} \tag{3.4}$$

is nonempty, where

$$K^* = \{x \in K : ||x - y^*|| \leq r\} \subset K. \tag{3.5}$$

(ii) *The mapping $\eta : K \times K \to H$ is (λ)-Lipschitz continuous.*

(iii) *$\eta(u, v) + \eta(v, u) = 0$ and $\eta(u, v) = \eta(u \cdot w) + \eta(w, v)$.*

(iv) *For each fixed $y \in K$, the map $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.*

(v) $\|\eta(u,v)\| \leq r$.

Then an iterate x^{k+1} is a unique solution to (3.1).

If, in addition, $x^* \in K$ is a solution to VIP (1.1) and $\|T(x^k) - T(x^*)\| \to 0$, then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges weakly to x^*.

Proof. First to show that x^{k+1} is a unique solution to (3.1), assume that y^{k+1} is another distinct solution to (3.1). Since h' is (η,α)-*strongly* monotone, it follows applying (3.1) that

$$-\langle h'(x^{k+1}) - h'(y^{k+1}), \eta(x^{k+1}, y^{k+1})\rangle \geq 0, \tag{3.6}$$

or

$$||x^{k+1} - y^{k+1}||^2 \leq 0, \tag{3.7}$$

a contradiction.

Since $x^* \in K$ is a solution to the VIP (1.1), we define a function Δ^* by

$$\Delta^*(x) := h(x^*) - h(x) - \langle h'(x), \eta(x^*, x)\rangle. \tag{3.8}$$

Then applying Lemma 2.1, we have

$$\Delta^*(x) := h(x^*) - h(x) - \langle h'(x), \eta(x^*, x)\rangle \geq \frac{\alpha}{2}||x^* - x||^2. \tag{3.9}$$

It follows that

$$\Delta^*(x^{k+1}) := h(x^*) - h(x^{k+1}) - \langle h'(x^{k+1}), \eta(x^*, x^{k+1})\rangle. \tag{3.10}$$

Now we can write

$$\begin{aligned}
\Delta^*(x^k) - \Delta^*(x^{k+1}) &= h(x^{k+1}) - h(x^k) - \langle h'(x^k), \eta(x^{k+1}, x^k)\rangle \\
&\quad + \langle h'(x^{k+1}) - h'(x^k), \eta(x^*, x^{k+1})\rangle \\
&\geq \frac{\alpha}{2}||x^{k+1} - x^k||^2 + \langle h'(x^{k+1}) - h'(x^k), \eta(x^*, x^{k+1})\rangle \\
&\geq \frac{\alpha}{2}||x^{k+1} - x^k||^2 + \rho\langle T(x^k), \eta(x^{k+1}, x^*)\rangle \\
&\quad + \rho(f(x^{k+1}) - f(x^*)),
\end{aligned} \tag{3.11}$$

for $x = x^*$ in (3.1).

Therefore, we have

$$\Delta^*(x^k) - \Delta^*(x^{k+1}) \geq \frac{\alpha}{2}||x^{k+1} - x^k||^2 + \rho\langle T(x^k), \eta(x^{k+1}, x^*)\rangle + \rho(f(x^{k+1}) - f(x^*)). \tag{3.12}$$

If we replace x by x^{k+1} in (1.1), we obtain

$$\langle T(x^*), \eta(x^{k+1}, x^*)\rangle + f(x^{k+1}) - f(x^*) \geq 0. \tag{3.13}$$

Since T is (η,γ)-*partially* relaxed pseudomonotone, it implies in light of (3.13) that

$$\Delta^*(x^k)-\Delta^*(x^{k+1})\geq\frac{\alpha}{2}||x^{k+1}-x^k||^2-\rho\gamma||x^{k+1}-x^k||^2=\left(\frac{\alpha}{2}-\rho\gamma\right)||x^{k+1}-x^k||^2 \tag{3.14}$$

for $\rho<(\alpha/2\gamma)$.

It follows that the sequence $\{\Delta^*(x^k)\}$ is a strictly decreasing sequence except for $x^{k+1}=x^k$, and in that situation x^k is a solution to (1.1). Since the difference of two consecutive terms tends to zero as $k\to\infty$, it implies that

$$||x^{k+1}-x^k||\longrightarrow 0 \quad \text{as } k\longrightarrow\infty. \tag{3.15}$$

On the top of that, in light of Lemma 2.1, we have

$$||x^*-x^k||^2\leq\frac{2}{\alpha}\Delta^*(x^k), \tag{3.16}$$

and so the sequence $\{x^k\}$ is bounded. Let x' be a cluster point of the sequence $\{x^k\}$, that is, there exists a subsequence $\{x^{kj}\}$ of the sequence $\{x^k\}$ such that $\{x^{kj}\}$ converges weakly to x'. Since h' is (η,β)-*Lipschitz* continuous and $a<\rho$, it follows using (3.1) that for some $x\in K$, we have

$$\begin{aligned}\langle\rho T(x^k),\eta(x,x^{k+1})\rangle+\rho(f(x)-f(x^{k+1}))&\geq-\langle h'(x^{k+1})-h'(x^k),\eta(x,x^{k+1})\rangle\\&\geq-\beta||x^{k+1}-x^k||\,||\eta(x,x^{k+1})||,\end{aligned} \tag{3.17}$$

or

$$\langle T(x^k),\eta(x,x^{k+1})\rangle+f(x)-f(x^{k+1})\geq-\frac{\beta}{a}||x^{k+1}-x^k||\,||\eta(x,x^{k+1})||. \tag{3.18}$$

Since $T(x^{kj})$ converges strongly to $T(x^*)$ and $\|x^{kj+1}-x^{kj}\|\to 0$, and f is invex and lower semicontinuous (and hence f is weakly lower semicontinuous), it follows from (3.18) that

$$\langle T(x^*),\eta(x,x')\rangle+f(x)-f(x')\geq 0 \quad \forall x\in K, \tag{3.19}$$

while

$$\begin{gathered}\langle T(x'),\eta(x^{kj},x')\rangle+f(x)-f(x^{kj})\longrightarrow 0,\\ \langle T(x'),\eta(x^{kj},x')\rangle\longrightarrow 0.\end{gathered} \tag{3.20}$$

At this stage, if $T(x')=0$, then x' is a solution to the VIP (1.1); and if $T(x')\neq 0$, then we express it in the form

$$\eta(y^{kj},x^{kj})=-\frac{\langle T(x'),\eta(x^{kj},x')\rangle T(x')}{||T(x')||^2}. \tag{3.21}$$

It follows that

$$\langle T(x'),\eta(y^{kj},x')\rangle = 0, \tag{3.22}$$

and thus, we have

$$||\eta(y^{kj},x^{kj})|| \longrightarrow 0. \tag{3.23}$$

It follows that

$$y^{kj} \rightharpoonup x'. \tag{3.24}$$

Applying (3.22), we have

$$0 = \langle T(x'),\eta(y^{kj},x')\rangle = \langle T(x'),\eta(y^{kj},x^*)\rangle + \langle T(x'),\eta(x^*,x')\rangle. \tag{3.25}$$

Since $T(x') \neq 0$, it follows that $y^{kj} \rightharpoonup x^*$ and $x^* = x'$, a solution to the VIP (1.1). □

Corollary 3.5. *Let H be a real infinite-dimensional Hilbert space and let K be a nonempty closed invex subset of H. Let $T : K \to H$ be (η,γ)-pseudococoercive. Let $f : K \to R$ be proper, invex, and lower semicontinuous on K, let $h : K \to R$ be continuously Fréchet-differentiable on K with h', the gradient of h, (η,α)-strongly monotone and (η,β)-Lipschitz continuous, and let h' be continuous from H equipped with weak topology to H equipped with strong topology. Suppose that the following assumptions hold.*

(i) *There exist a $y^* \in K$ and numbers $a > 0$, $r \geq 0$, $\rho(a < \rho < \alpha/2\gamma)$ such that for all $t \in [0,1]$ and for all $x \in K^*$, the set S^* defined by*

$$S^* = \{(h,\eta) : h'(y^* + t(x-y^*))(x-y^*) \geq \langle h'(y^* + t\eta(x,y^*)),\eta(x,y^*)\rangle\} \tag{3.26}$$

is nonempty, where

$$K^* = \{x \in K : ||x - y^*|| \leq r\} \subset K. \tag{3.27}$$

(ii) *The mapping $\eta : K \times K \to H$ is (λ)-Lipschitz continuous.*

(iii) *$\eta(u,v) + \eta(v,u) = 0$ and $\eta(u,v) = \eta(u \cdot w) + \eta(w,v)$.*

(iv) *For each fixed $y \in K$, the map $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.*

(v) *$\|\eta(u,v)\| \leq r$.*

Then an iterate x^{k+1} is a unique solution to (3.1).

If $x^* \in K$ is a solution to VIP (1.1), then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges weakly to x^*.

For $f = 0$ and $\eta(u,v) = u - v$ in Corollary 3.5, it reduces to the following corollary.

Corollary 3.6. *Let H be a real infinite-dimensional Hilbert space and let K be a nonempty closed convex subset of H. Let $T : K \to H$ be (γ)-pseudococoercive. Let $h : K \to R$ be continuously Fréchet-differentiable on K with h', the gradient of h, (α)-strongly monotone, and (β)-Lipschitz continuous, and let h' be continuous from H equipped with weak topology to H equipped with strong topology. Then an iterate x^{k+1} is a unique solution to (3.3). If $x^* \in K$ is* *a solution to VIP (1.3), then the sequence $\{x^k\}$ generated by Algorithm 3.3 converges weakly to x^*.*

Note that Corollary 3.6 is proved in [7, Theorem 4.1] with an additional imposition of the uniform continuity on the mapping T, but we feel that the uniform continuity is not required for the convergence purposes.

Theorem 3.7. *Let H be a real infinite-dimensional Hilbert space and let K be a nonempty closed invex subset of H. Let $T : K \to H$ be (η,γ)-partially relaxed pseudomonotone. Let $f : K \to R$ be proper, invex, and lower semicontinuous on K, let $h : K \to R$ be continuously Fréchet-differentiable on K with h', the gradient of h, (η,α)-strongly monotone, and (η,β)-Lipschitz continuous, and let h' be continuous from H equipped with weak topology to H equipped with strong topology. Suppose that the following assumptions hold.*

(i) *There exist a $y^* \in K$ and numbers $a > 0$, $r \geq 0$, $q \geq 0$, $\rho(a < \rho < \alpha/2\gamma)$ such that for all $t \in [0,1]$ and for all $x \in K^*$, the set S^* defined by*

$$S^* = \{(h,\eta) : h'(y^* + t(x - y^*))(x - y^*) \geq \langle h'(y^* + t\eta(x,y^*)),\eta(x,y^*)\rangle\} \tag{3.28}$$

is nonempty, where

$$K^* = \{x \in K : ||x - y^*|| \leq r\} \subset K. \tag{3.29}$$

(ii) *The mapping $\eta : K \times K \to H$ is (λ)-Lipschitz continuous.*

(iii) *$\eta(u,v) + \eta(v,u) = 0$ and $\eta(u,v) = \eta(u \cdot w) + \eta(w,v)$.*

(iv) *For each fixed $y \in K$ the map $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.*

(v) *$\|\eta(u,v)\| \leq r$.*

Then an iterate x^{k+1} is a unique solution to (3.1). If $x^* \in K$ is a solution to VIP (1.1) and $\|T(x^k) - T(x^*)\| \to 0$, then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges weakly to x^*.

In addition, assume that

(vi) *there exist a $y^* \in K$ such that for all $x \in K_1$, the set S_1 defined by*

$$S_1 = \{(h,\eta) : h'(y^* + t(x - y^*))(x - y^*) \leq \langle h'(y^* + t\eta(x,y^*)),\eta(x,y^*)\rangle\} \tag{3.30}$$

is nonempty, where

$$K_1 = \{x \in K : ||x - y^*|| \leq q\} \subset K, \tag{3.31}$$

with $\|\eta(x,y^*)\| \leq q$.

Then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges to x^*.

Proof. Since based on the proof of Theorem 3.4, x' is a weak cluster point of the sequence $\{x^k\}$, we define a function Λ^* by

$$\Lambda^*(x^k) = h(x') - h(x^k) - \langle h'(x^k),\eta(x',x^k)\rangle. \tag{3.32}$$

Applying Lemmas 2.1 and 2.3, we have the following:

$$\Lambda^*(x^k) = h(x') - h(x^k) - \langle h'(x^k), \eta(x', x^k)\rangle \geq \frac{\alpha}{2}||x' - x^k||^2. \tag{3.33}$$

$$\Lambda^*(x^k) = h(x') - h(x^k) - \langle h'(x^k), \eta(x', x^k)\rangle \leq \frac{\beta}{2}||x' - x^k||^2. \tag{3.34}$$

It follows from (3.34) that

$$\lim_{n\to\infty} \Lambda^*(x^k) = 0. \tag{3.35}$$

Applying (3.35) to (3.33), it follows that the entire sequence $\{x^k\}$ generated by Algorithm 3.1 converges to x'. □

References

[1] I. K. Argyros and R. U. Verma, *On general auxiliary problem principle and nonlinear mixed variational inequalities*, Nonlinear Functional Analysis and Applications **6** (2001), no. 2, 247–256.

[2] ———, *Generalized partial relaxed monotonicity and solvability of nonlinear variational inequalities*, Panamerican Mathematical Journal **12** (2002), no. 3, 85–104.

[3] G. Cohen, *Auxiliary problem principle and decomposition of optimization problems*, Journal of Optimization Theory and Applications **32** (1980), no. 3, 277–305.

[4] ———, *Auxiliary problem principle extended to variational inequalities*, Journal of Optimization Theory and Applications **59** (1988), no. 2, 325–333.

[5] J. Eckstein, *Nonlinear proximal point algorithms using Bregman functions, with applications to convex programming*, Mathematics of Operations Research **18** (1993), no. 1, 202–226.

[6] J. Eckstein and D. P. Bertsekas, *On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators*, Mathematical Programming Series A **55** (1992), no. 3, 293–318.

[7] N. El Farouq, *Pseudomonotone variational inequalities: convergence of the auxiliary problem method*, Journal of Optimization Theory and Applications **111** (2001), no. 2, 305–326.

[8] ———, *Pseudomonotone variational inequalities: convergence of proximal methods*, Journal of Optimization Theory and Applications **109** (2001), no. 2, 311–326.

[9] S. Karamardian, *Complementarity problems over cones with monotone and pseudomonotone maps*, Journal of Optimization Theory and Applications **18** (1976), no. 4, 445–454.

[10] S. Karamardian and S. Schaible, *Seven kinds of monotone maps*, Journal of Optimization Theory and Applications **66** (1990), no. 1, 37–46.

[11] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, Revue Française d'Informatique Recherche Opérationnelle **4** (1970), Ser. R-3, 154–158 (French).

[12] Z. Naniewicz and P. D. Panagiotopoulos, *Mathematical theory of hemivariational inequalities and applications*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 188, Marcel Dekker, New York, 1995.

[13] R. T. Rockafellar, *Augmented Lagrangians and applications of the proximal point algorithm in convex programming*, Mathematics of Operations Research **1** (1976), no. 2, 97–116.

[14] ———, *Monotone operators and the proximal point algorithm*, SIAM Journal Control Optimization **14** (1976), no. 5, 877–898.

[15] R. U. Verma, *Nonlinear variational and constrained hemivariational inequalities involving relaxed operators*, Zeitschrift für Angewandte Mathematik und Mechanik **77** (1997), no. 5, 387–391.

[16] ———, *Approximation-solvability of nonlinear variational inequalities involving partially relaxed monotone (PRM) mappings*, Advances in Nonlinear Variational Inequalities **2** (1999), no. 2, 137–148.

[17] ———, *A new class of iterative algorithms for approximation-solvability of nonlinear variational inequalities*, Computers & Mathematics with Applications **41** (2001), no. 3-4, 505–512.

[18] ———, *General auxiliary problem principle involving multivalued mappings*, Nonlinear Functional Analysis and Applications **8** (2003), no. 1, 105–110.

[19] ———, *Generalized strongly nonlinear variational inequalities*, Revue Roumaine de Mathématiques Pures et Appliquées. Romanian Journal of Pure and Applied Mathematics **48** (2003), no. 4, 431–434.

[20] ———, *Nonlinear implicit variational inequalities involving partially relaxed pseudomonotone mappings*, Computers & Mathematics with Applications **46** (2003), no. 10-11, 1703–1709.

[21] ———, *Partial relaxed monotonicity and general auxiliary problem principle with applications*, Applied Mathematics Letters **16** (2003), no. 5, 791–796.

[22] ———, *Partially relaxed cocoercive variational inequalities and auxiliary problem principle*, Journal of Applied Mathematics and Stochastic Analysis **2004** (2004), no. 2, 143–148.

[23] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. II/B. Nonlinear Monotone Operators*, Springer, New York, 1990.

Ram U. Verma: Department of Mathematics, University of Toledo, Toledo, OH 43606, USA
E-mail address: verma99@msn.com