EXISTENCE RESULTS FOR NONLOCAL AND NONSMOOTH HEMIVARIATIONAL INEQUALITIES

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Received 4 May 2005; Accepted 10 May 2005

We consider an elliptic hemivariational inequality with nonlocal nonlinearities. Assuming only certain growth conditions on the data, we are able to prove existence results for the problem under consideration. In particular, no continuity assumptions are imposed on the nonlocal term. The proofs rely on a combined use of recent results due to the authors on hemivariational inequalities and operator equations in partially ordered sets.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < \infty$, denote the usual Sobolev spaces with their dual spaces V^* and V_0^*, respectively. In this paper, we deal with the following quasilinear hemivariational inequality:

$$u \in V_0 : \langle -\Delta_p u, v - u\rangle + \int_\Omega j^o(u; v - u)\,dx \geq \langle \mathcal{F}u, v - u\rangle, \quad \forall\, v \in V_0, \tag{1.1}$$

where $j^o(s;r)$ denotes the generalized directional derivative of the locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$ at s in the direction r given by

$$j^o(s;r) = \limsup_{y \to s,\ t \downarrow 0} \frac{j(y + tr) - j(y)}{t}, \tag{1.2}$$

(cf., e.g., [3, Chapter 2]), $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < \infty$, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V_0 and V_0^*. The mapping $\mathcal{F} : V_0 \to V_0^*$ on the right-hand side of (1.1) comprises the nonlocal term and is generated by a function $F : \Omega \times L^p(\Omega) \to \mathbb{R}$ through

$$\mathcal{F}u := F(\cdot, u). \tag{1.3}$$

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 79532, Pages 1–13
DOI 10.1155/JIA/2006/79532

While elliptic hemivariational inequalities in the form (1.1) with $\mathcal{F}u$ replaced by a given element $f \in V_0^*$ have been treated recently, for example, in [2] under the assumption that appropriately defined super- and subsolutions are available, the novelty of the problem under consideration is that the term on the right-hand side of (1.1) is nonlocal and not necessarily continuous in u. Moreover, we do not assume the existence of super- and subsolutions.

Our main goal is to prove existence results for problem (1.1) only under the assumption that certain growth conditions on the data are satisfied.

Problem (1.1) includes various special cases, such as the following. for example.

(i) For $j : \mathbb{R} \to \mathbb{R}$ smooth, (1.1) is the weak formulation of the nonlocal Dirichlet problem

$$u \in V_0 : -\Delta_p u + j'(u) = \mathcal{F}u \quad \text{in } V_0^*. \tag{1.4}$$

(ii) If $j : \mathbb{R} \to \mathbb{R}$ is not necessarily smooth, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with its Nemytskij operator G, then the following (local) hemivariational inequality of the form

$$u \in V_0 : \langle -\Delta_p u, v-u\rangle + \int_\Omega j^o(u; v-u)\, dx \geq \langle Gu, v-u\rangle, \quad \forall\, v \in V_0, \tag{1.5}$$

is a special case of (1.1) by defining $F(x,u) := g(x,u(x))$.

(iii) If $j : \mathbb{R} \to \mathbb{R}$ is convex, then (1.1) is equivalent to the following inclusion:

$$u \in V_0 : -\Delta_p u + \partial j(u) \ni \mathcal{F}u \quad \text{in } V_0^*, \tag{1.6}$$

where $\partial j(s)$ denotes the usual subdifferential of j at s in the sense of convex analysis.

(iv) As for an example of a (discontinuous) nonlocal $\mathcal{F}$ that will be treated later, we consider F defined by

$$F(x,u) = [|x|] + \gamma \int_\Omega [u(x)]\,dx, \tag{1.7}$$

where γ is some positive constant, and $[\cdot] : \mathbb{R} \to \mathbb{Z}$ is the integer function which assigns to each $s \in \mathbb{R}$ the greatest integer $[s] \in \mathbb{Z}$ satisfying $[s] \leq s$.

The plan of the paper is as follows. In Section 2 we formulate the hypotheses and the main result. In Section 3 we deal with an auxiliary hemivariational inequality which arises from (1.1) by replacing $\mathcal{F}u$ on the right-hand side by a given $f \in V_0^*$. The preliminary results about this auxiliary problem are of independent interest. Finally, in Section 4 we prove our main result and give an example.

2. Hypotheses and main result

We denote the norms in $L^p(\Omega), V_0$, and V_0^* by $\|\cdot\|_p$, $\|\cdot\|_{V_0}$, and $\|\cdot\|_{V_0^*}$, respectively. Let $\partial j : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ denote Clarke's generalized gradient of j defined by

$$\partial j(s) := \{\zeta \in \mathbb{R}\} \mid j^o(s;r) \geq \zeta r, \quad \forall r \in \mathbb{R}, \tag{2.1}$$

(cf., e.g., [3, Chapter 2]). Denote by λ_1 the first Dirichlet eigenvalue of $-\Delta_p$ which is positive (see [5]) and given by the variational characterization

$$\lambda_1 = \inf_{0\neq u\in V_0} \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega |u|^p\,dx}. \tag{2.2}$$

Further, let $L^p(\Omega)$ be equipped with the natural partial ordering of functions defined by $u \le w$ if and only if $w-u$ belongs to the positive cone $L^p_+(\Omega)$ of all nonnegative elements of $L^p(\Omega)$. This induces a corresponding partial ordering also in the subspace V of $L^p(\Omega)$.

We assume the following hypothesis for j and F.

(H1) The function $j:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz and its Clarke's generalized gradient ∂j satisfies the following conditions:

(i) there exists a constant $c_1 \ge 0$ such that

$$\xi_1 \le \xi_2 + c_1(s_2-s_1)^{p-1} \tag{2.3}$$

for all $\xi_i\in\partial j(s_i)$, $i=1,2$, and for all s_1, s_2 with $s_1<s_2$;

(ii) there are a $\varepsilon\in(0,\lambda_1)$ and a constant $c_2\ge 0$ such that

$$\xi\in\partial j(s): |\xi| \le c_2 + (\lambda_1-\varepsilon)|s|^{p-1}, \quad \forall\, s\in\mathbb{R}. \tag{2.4}$$

(H2) The function $F:\Omega\times L^p(\Omega)\to\mathbb{R}$ is assumed to satisfy the following.

(i) $x\mapsto F(x,u)$ is measurable in $x\in\Omega$ for all $u\in L^p(\Omega)$, and for almost every (a.e.) $x\in\Omega$ the function $u\mapsto F(x,u)$ is increasing, that is, $F(x,u)\le F(x,v)$ whenever $u\le v$.

(ii) There exist constants $c_3>0$, $\mu\ge 0$ and $\alpha\in[0,p-1]$ such that

$$\|\mathcal{F}u\|_q \le c_3 + \mu\|u\|_p^\alpha, \quad \forall\, u\in L^p(\Omega), \tag{2.5}$$

where $q\in(1,\infty)$ is the conjugate real to p satisfying $1/p+1/q=1$, and $\mu\ge 0$ may be arbitrarily if $\alpha\in[0,p-1)$, and $\mu\in[0,\varepsilon)$ if $\alpha=p-1$, where ε is the constant in (H1)(ii).

The main result of the present paper is given by the following theorem.

Theorem 2.1. *Let hypotheses (H1) and (H2) be satisfied. Then problem (1.1) possesses solutions, and the solution set of all solutions of (1.1) is bounded in V_0 and has minimal and maximal elements.*

The proof of Theorem 2.1 requires several preliminary results which are of interest in its own and which will be provided in Section 3. In Section 4 we recall an abstract existence result for an operator equation in ordered Banach spaces, which together with the results of Section 3 form the main tools in the proof of Theorem 2.1. We will assume throughout the rest of the paper that the hypotheses of Theorem 2.1 are satisfied

3. Preliminaries

Let $f \in V_0^*$ be given. In this section, we consider the following auxiliary hemivariational inequality:

$$u \in V_0 : \langle -\Delta_p u, v-u\rangle + \int_\Omega j^o(u; v-u)\,dx \ge \langle f, v-u\rangle, \quad \forall\, v \in V_0. \tag{3.1}$$

In the next sections, we are going to prove the existence of solutions of (3.1), the existence of extremal solutions of (3.1), and the monotone dependence of these extremal solutions.

3.1. An existence result for (3.1). The existence of solutions of (3.1) follows by standard arguments and is given here only for the sake of completeness and for providing the necessary tools that will be used later. The main ingredient is the following surjectivity result for multivalued pseudomonotone and coercive operators, see, for example, [6, Theorem 2.6] or [7, Chapter 32].

Proposition 3.1. *Let X be a real reflexive Banach space with dual space X^*, and let the multivalued operator $\mathcal{A} : X \to 2^{X^*}$ be pseudomonotone and coercive. Then $\mathcal{A}$ is surjective, that is, range $\mathcal{A} = X^*$.*

For convenience, let us recall the notion of multivalued pseudomonotone operators (cf., e.g., [6, Chapter 2]).

Definition 1. Let X be a real reflexive Banach space. The operator $\mathcal{A} : X \to 2^{X^*}$ is called *pseudomonotone* if the following conditions hold.

(i) The set $\mathcal{A}(u)$ is nonempty, bounded, closed, and convex for all $u \in X$.

(ii) $\mathcal{A}$ is upper semicontinuous from each finite-dimensional subspace of X to the weak topology on X^*.

(iii) If $(u_n) \subset X$ with $u_n \rightharpoonup u$, and if $u_n^* \in \mathcal{A}(u_n)$ is such that $\limsup \langle u_n^*, u_n - u\rangle \le 0$, then to each element $v \in X$, there exists $u^*(v) \in \mathcal{A}(u)$ with

$$\liminf \langle u_n^*, u_n - v\rangle \ge \langle u^*(v), u - v\rangle. \tag{3.2}$$

The existence result for (3.1) reads as the following lemma.

Lemma 3.2. *The hemivariational inequality (3.1) possesses solutions for each $f \in V_0^*$.*

Proof. We introduce the function $J : L^p(\Omega) \to \mathbb{R}$ defined by

$$J(v) = \int_\Omega j(v(x))\,dx, \quad \forall\, v \in L^p(\Omega). \tag{3.3}$$

Using the growth condition (H1)(ii) and Lebourg's mean value theorem, we note that the function J is well-defined and Lipschitz continuous on bounded sets in $L^p(\Omega)$, thus locally Lipschitz. Moreover, the Aubin-Clarke theorem (see [3, page 83]) ensures that, for each $u \in L^p(\Omega)$ we have

$$\xi \in \partial J(u) \Longrightarrow \xi \in L^q(\Omega) \text{ with } \xi(x) \in \partial j(u(x)) \quad \text{for a.e. } x \in \Omega. \tag{3.4}$$

Consider now the multivalued operator $\mathcal{A} : V_0 \to 2^{V_0^*}$ defined by

$$\mathcal{A}(v) = -\Delta_p v + \partial(J|_{V_0})(v), \quad \forall v \in V_0, \tag{3.5}$$

where $J|_{V_0}$ denotes the restriction of J to V_0. It is well known that $-\Delta_p : V_0 \to V_0^*$ is continuous, bounded, strictly monotone, and thus, in particular, pseudomonotone. It has been shown in [2] that the multivalued operator $\partial(J|_{V_0})$ is bounded and pseudomonotone in the sense given above. Since $-\Delta_p$ and $\partial(J|_{V_0})$ are pseudomonotone, it follows that the multivalued operator $\mathcal{A}$ is pseudomonotone. Thus in view of Proposition 3.1 the operator $\mathcal{A}$ is surjective provided $\mathcal{A}$ is coercive. By making use of the equivalent norm in V_0 which is $\|u\|_{V_0}^p = \int_\Omega |\nabla u|^p\,dx$, and the variational characterization of the first eigenvalue of $-\Delta_p$, the coercivity can readily be seen as follows: For any $v \in V_0$ and any $w \in \partial(J|_{V_0})(v)$ we obtain by applying (H1) the estimate

$$\begin{aligned}\frac{1}{\|v\|_{V_0}}\langle -\Delta_p v + w, v\rangle &\geq \frac{1}{\|v\|_{V_0}}\left[\int_\Omega |\nabla v|^p dx - \int_\Omega \left(c_2 + (\lambda_1 - \varepsilon)|v|^{p-1}\right)|v|\,dx\right]\\ &\geq \frac{1}{\|v\|_{V_0}}\left[\|v\|_{V_0}^p - \frac{\lambda_1 - \varepsilon}{\lambda_1}\|v\|_{V_0}^p - c\|v\|_p\right],\end{aligned} \tag{3.6}$$

for some constant $c > 0$, which proves the coercivity of $\mathcal{A}$. Applying Proposition 3.1 we obtain that there exists $u \in V_0$ such that $f \in \mathcal{A}(u)$, that is, there is an $\xi \in \partial J(u)$ such that $\xi \in L^q(\Omega)$ with $\xi(x) \in \partial j(u(x))$ for a.e. $x \in \Omega$ and

$$-\Delta_p u + \xi = f \quad \text{in } V_0^*, \tag{3.7}$$

where

$$\langle \xi, \varphi\rangle = \int_\Omega \xi(x)\varphi(x)dx \quad \forall \varphi \in V_0, \tag{3.8}$$

and thus by definition of Clarke's generalized gradient ∂j from (3.8), we get

$$\langle \xi, \varphi\rangle = \int_\Omega \xi(x)\,\varphi(x)\,dx \leq \int_\Omega j^o(u(x);\varphi(x))\,dx \quad \forall \varphi \in V_0. \tag{3.9}$$

Due to (3.7) and (3.9) we conclude that $u \in V_0$ is a solution of the auxiliary hemivariational inequality (3.1). □

3.2. Existence of extremal solutions of (3.1). In this section, we show that problem (3.1) has extremal solutions which are defined as in the following definition.

Definition 2. A solution u^* of (3.1) is called the *greatest solution* if for any solution u of (3.1), $u \leq u^*$. Similarly, u_* is the *least solution* if for any solution u, one has $u_* \leq u$. The least and greatest solutions of the hemivariational inequality (3.1) are called the *extremal* ones.

Here we prove the following extremality result.

Lemma 3.3. *The hemivariational inequality (3.1) possesses extremal solutions.*

Proof. Let us introduce the set $\mathcal{S}$ of all solutions of (3.1). The proof will be given in steps (a), (b) and (c).

(a) Claim: $\mathcal{S}$ is compact in V_0.

First, let us show that $\mathcal{S}$ is bounded in V_0. By taking $v = 0$ in (3.1), we get

$$\langle -\Delta_p u, u\rangle \le \langle f, u\rangle + \int_\Omega j^o(u;-u)\,dx, \tag{3.10}$$

which yields by applying (H1)(ii)

$$\|u\|_{V_0}^p \le \|f\|_{V_0^*}\|u\|_{V_0} + c\|u\|_p + (\lambda_1 - \varepsilon)\|u\|_p^p, \tag{3.11}$$

for some constant $c \ge 0$. By means of Young's inequality, we get for any $\eta > 0$,

$$\|u\|_{V_0}^p \le \|f\|_{V_0^*}\|u\|_{V_0} + c(\eta) + \eta\|u\|_p^p + (\lambda_1 - \varepsilon)\|u\|_p^p, \tag{3.12}$$

which yields for $\eta < \varepsilon$ and setting $\tilde{\varepsilon} = \varepsilon - \eta$ the estimate

$$\|u\|_{V_0}^p \le \|f\|_{V_0^*}\|u\|_{V_0} + c(\eta) + \frac{\lambda_1 - \tilde{\varepsilon}}{\lambda_1}\|u\|_{V_0}^p, \tag{3.13}$$

and hence the boundedness of $\mathcal{S}$ in V_0.

Let $(u_n) \subset \mathcal{S}$. Then there is a subsequence (u_k) of (u_n) with

$$u_k \rightharpoonup u \quad \text{in } V_0, \quad u_k \longrightarrow u \quad \text{in } L^p(\Omega), \qquad u_k(x) \longrightarrow u(x) \text{ a.e. in } \Omega. \tag{3.14}$$

Since the u_k solve (3.1), we get with $v = u$ in (3.1)

$$\langle -\Delta_p u_k - f, u - u_k\rangle + \int_\Omega j^o(u_k; u - u_k)\,dx \ge 0, \tag{3.15}$$

and thus

$$\langle -\Delta_p u_k, u_k - u\rangle \le \langle f, u_k - u\rangle + \int_\Omega j^o(u_k; u - u_k)\,dx. \tag{3.16}$$

Due to (3.14) and due to the fact that $(s,r) \mapsto j^o(s;r)$ is upper semicontinuous, we get by applying Fatou's lemma

$$\limsup_k \int_\Omega j^o(u_k; u - u_k)\,dx \le \int_\Omega \limsup_k j^o(u_k; u - u_k)\,dx = 0. \tag{3.17}$$

In view of (3.17), we thus obtain from (3.14) and (3.16)

$$\limsup_k \langle -\Delta_p u_k, u_k - u\rangle \le 0. \tag{3.18}$$

Since the operator $-\Delta_p$ enjoys the (S_+)-property, the weak convergence of (u_k) in V_0 along with (3.18) imply the strong convergence $u_k \to u$ in V_0, see, for example, [1, Theorem D.2.1]. Moreover, the limit u belongs to $\mathcal{S}$ as can be seen by passing to the lim sup on the left-hand side of the following inequality:

$$\langle -\Delta_p u_k - f, v - u_k\rangle + \int_\Omega j^o(u_k; v - u_k)\,dx \geq 0, \tag{3.19}$$

where we have used Fatou's lemma and the strong convergence of (u_k) in V_0. This completes the proof of Claim (a).

(b) Claim: $\mathcal{S}$ is a directed set

The solution set $\mathcal{S}$ is called *upward directed* if for each pair $u_1, u_2 \in \mathcal{S}$ there exists a $u \in \mathcal{S}$ such that $u_k \leq u$, $k = 1,2$. Similarly, $\mathcal{S}$ is called *downward directed* if for each pair $u_1, u_2 \in \mathcal{S}$ there exists a $u \in \mathcal{S}$ such that $u \leq u_k$, $k = 1,2$, and $\mathcal{S}$ is called *directed* if it is both upward and downward directed. Let us show that $\mathcal{S}$ is upward directed. To this end we consider the following auxiliary hemivariational inequality

$$u \in V_0 : \langle -\Delta_p u - f + \lambda B(u), v - u\rangle + \int_\Omega j^o(u; v - u)\,dx \geq 0, \quad \forall\, v \in V_0, \tag{3.20}$$

where $\lambda \geq 0$ is a free parameter to be chosen later, and the operator B is the Nemytskij operator given by the following cut-off function $b : \Omega \times \mathbb{R} \to \mathbb{R}$:

$$b(x,s) = \begin{cases} 0 & \text{if } u_0(x) \leq s, \\ -(u_0(x) - s)^{p-1} & \text{if } s < u_0(x), \end{cases} \tag{3.21}$$

with $u_0 = \max(u_1, u_2)$. The function b is easily seen to be a Carathéodory function satisfying a growth condition of order $p-1$ and thus $B : V_0 \to V_0^*$ defines a compact and bounded operator. This allows to apply the same arguments as in the proof of Lemma 3.2 to show the existence of solutions of problem (3.20) provided we are able to verify that the corresponding multivalued operator related with (3.20) is coercive, that is, we only need to verify the coercivity of $\mathcal{A}(v) = -\Delta_p v + \lambda B(v) + \partial(J|_{V_0})(v)$, $v \in V_0$. This, however, readily follows from the proof of the coercivity of the operator $-\Delta_p + \partial(J|_{V_0})$ and the following estimate of $\langle B(v), v\rangle$. In view of the definition (3.21) the function $s \mapsto b(x,s)$ monotone nondecreasing and $b(\cdot, u_0) = 0$. Therefore we get by applying Young's inequality for any $\eta > 0$ the estimate

$$\langle B(v), v\rangle = \int_\Omega b(\cdot, v)(v - u_0 + u_0)\,dx \geq \int_\Omega b(\cdot, v)u_0\,dx \geq -\eta\|v\|_p^p - c(\eta), \tag{3.22}$$

which implies the coercivity of $-\Delta_p + \lambda B + \partial(J|_{V_0})$ when η is chosen sufficiently small, and hence the existence of solutions of the auxiliary problem (3.20). Now the set $\mathcal{S}$ is shown to be upward directed provided that any solution u of (3.20) satisfies $u_k \leq u$, $k = 1,2$, because then $Bu = 0$ and thus $u \in \mathcal{S}$ exceeding u_k.

By assumption $u_k \in \mathcal{S}$ which means u_k satisfies

$$u_k \in V_0 : \langle -\Delta_p u_k - f, v - u_k\rangle + \int_\Omega j^o(u_k; v - u_k)\,dx \geq 0, \quad \forall\, v \in V_0. \tag{3.23}$$

Taking the special functions $v = u + (u_k - u)^+$ in (3.20) and $v = u_k - (u_k - u)^+$ in (3.23) and adding the resulting inequalities we obtain

$$\begin{aligned}&\langle -\Delta_p u_k - (-\Delta_p u), (u_k - u)^+\rangle - \lambda\langle B(u), (u_k - u)^+\rangle \\ &\quad \le \int_\Omega \left(j^o(u;(u_k - u)^+) + j^o(u_k; -(u_k - u)^+)\right) dx.\end{aligned} \tag{3.24}$$

Next we estimate the right-hand side of (3.24) by using the following facts from nonsmooth analysis, (cf. [3, Chapter 2]): The function $r \mapsto j^o(s;r)$ is finite and positively homogeneous, $\partial j(s)$ is a nonempty, convex, and compact subset of $\mathbb{R}$, and one has

$$j^o(s;r) = \max\{\xi r \mid \xi \in \partial j(s)\}. \tag{3.25}$$

Denote $\{w > v\} = \{x \in \Omega \mid w(x) > v(x)\}$, then by using (H1)(i) and the properties on j^o and ∂j, we get for certain $\xi_k(x) \in \partial j(u_k(x))$ and $\xi(x) \in \partial j(u(x))$ the following estimate:

$$\begin{aligned}&\int_\Omega \left(j^o(u;(u_k - u)^+) + j^o(u_k; -(u_k - u)^+)\right) dx \\ &\quad = \int_{\{u_k > u\}} \left(j^o(u; u_k - u) + j^o(u_k; -(u_k - u))\right) dx \\ &\quad = \int_{\{u_k > u\}} \left(\xi(x)(u_k(x) - u(x)) + \xi_k(x)(-(u_k(x) - u(x)))\right) dx \\ &\quad = \int_{\{u_k > u\}} (\xi(x) - \xi_k(x))(u_k(x) - u(x))\, dx \le \int_{\{u_k > u\}} c_1 (u_k(x) - u(x))^p dx.\end{aligned} \tag{3.26}$$

For the terms on the left-hand side of (3.24) we have

$$\langle -\Delta_p u_k - (-\Delta_p u), (u_k - u)^+\rangle \ge 0 \tag{3.27}$$

and in view of (3.21) yields

$$\begin{aligned}\langle B(u), (u_k - u)^+\rangle &= -\int_{\{u_k > u\}} (u_0(x) - u(x))^{p-1}(u_k(x) - u(x))\, dx \\ &\le -\int_{\{u_k > u\}} (u_k(x) - u(x))^p\, dx.\end{aligned} \tag{3.28}$$

By means of (3.26), (3.27), (3.28), we get the inequality

$$(\lambda - c_1)\int_{\{u_k > u\}} (u_k(x) - u(x))^p dx \le 0. \tag{3.29}$$

Selecting λ such that $\lambda > c_1$ from (3.29) it follows $u_k \le u$, $k = 1, 2$, which proves the upward directedness. By obvious modifications of the auxiliary problem one can show analogously that $\mathcal{S}$ is also downward directed.

(c) Claim: $\mathcal{S}$ possesses extremal solutions

The proof of this assertion is based on steps (a) and (b). We will show the existence of the greatest element of $\mathcal{S}$. Since V_0 is separable we have that $\mathcal{S} \subset V_0$ is separable too, so there exists a countable, dense subset $Z = \{z_n \mid n \in \mathbb{N}\}$ of $\mathcal{S}$. By step (b), $\mathcal{S}$ is upward directed, so we can construct an increasing sequence $(u_n) \subset \mathcal{S}$ as follows. Let $u_1 = z_1$. Select $u_{n+1} \in \mathcal{S}$ such that

$$\max\{z_n, u_n\} \leq u_{n+1}. \tag{3.30}$$

The existence of u_{n+1} is due step (b). By the compactness of $\mathcal{S}$, we find a subsequence of (u_n), denoted again by (u_n), and an element $u \in \mathcal{S}$ such that $u_n \to u$ in V_0, and $u_n(x) \to u(x)$ a.e. in Ω. This last property of (u_n) combined with its increasing monotonicity implies that the entire sequence is convergent in V_0 and, moreover, $u = \sup_n u_n$. By construction, we see that

$$\max\{z_1, z_2, \ldots, z_n\} \leq u_{n+1} \leq u, \quad \forall n, \tag{3.31}$$

thus $Z \subset V_0^{\leq u} := \{w \in V_0 \mid w \leq u\}$. Since $V_0^{\leq u}$ is closed in V_0, we infer

$$\mathcal{S} \subset \overline{Z} \subset V_0^{\leq u}, \tag{3.32}$$

which in conjunction with $u \in \mathcal{S}$ ensures that u is the greatest solution of (3.1).

The existence of the least solution of (3.1) can be proved in a similar way. This completes the proof of Lemma 3.3. □

3.3. Monotonicity of the extremal solutions of (3.1). From Lemma 3.3, we know that for given $f \in V_0^*$ the hemivariational inequality (3.1) has a least solution u_* and a greatest solution u^*. The purpose of this subsection is to show that these extremal solutions depend monotonously on f. Let the dual order be defined by

$$f_1, f_2 \in V_0^* : f_1 \leq f_2 \Longleftrightarrow \langle f_1, \varphi \rangle \leq \langle f_2, \varphi \rangle, \quad \forall\, \varphi \in V_0 \cap L_+^p(\Omega). \tag{3.33}$$

Lemma 3.4. *Let u_k^* be the greatest and $u_{k,*}$ the least solutions of the hemivariational inequality (3.1) with right-hand sides $f_k \in V_0^*$, $k = 1,2$, respectively. If $f_1 \leq f_2$, then it follows that $u_1^* \leq u_2^*$ and $u_{1,*} \leq u_{2,*}$.*

Proof. We are going to prove $u_1^* \leq u_2^*$. To this end, we consider the following auxiliary hemivariational inequality:

$$u \in V_0 : \langle -\Delta_p u + \lambda B(u), v - u \rangle + \int_\Omega j^o(u; v - u)\, dx \geq \langle f_2, v - u \rangle, \quad \forall\, v \in V_0, \tag{3.34}$$

where $\lambda \geq 0$ is a free parameter to be chosen later, and B is the Nemytskij operator given by the following cutoff function $b : \Omega \times \mathbb{R} \to \mathbb{R}$:

$$b(x,s) = \begin{cases} 0 & \text{if } u_1^*(x) \leq s, \\ -(u_1^*(x) - s)^{p-1} & \text{if } s < u_1^*(x), \end{cases} \tag{3.35}$$

which can be written as $b(x,s) = -[(u_1^*(x) - s)^+]^{p-1}$. The existence of solutions of (3.34) can be proved in just the same way as for the auxiliary problem (3.20). By definition, u_1^* satisfies

$$u_1^* \in V_0 : \langle -\Delta_p u_1^*, v - u_1^* \rangle + \int_\Omega j^o(u_1^*; v - u_1^*)\, dx \geq \langle f_1, v - u_1^* \rangle, \quad \forall\, v \in V_0. \tag{3.36}$$

Let u be any solution of (3.34). Applying the special test functions $v = u + (u_1^* - u)^+$ and $v = u_1^* - (u_1^* - u)^+$ in (3.34) and (3.36), respectively, and adding the resulting inequalities, we obtain

$$\begin{aligned}&\langle -\Delta_p u_1^* - (-\Delta_p u), (u_1^* - u)^+\rangle - \lambda\langle B(u), (u_1^* - u)^+\rangle + \langle f_2 - f_1, (u_1^* - u)^+\rangle \\ &\quad \le \int_\Omega \big(j^o(u; (u_1^* - u)^+) + j^o(u_1^*; -(u_1^* - u)^+)\big)\,dx.\end{aligned} \tag{3.37}$$

Since $\langle -\Delta_p u_1^* - (-\Delta_p u), (u_1^* - u)^+\rangle \ge 0$ and $\langle f_2 - f_1, (u_1^* - u)^+\rangle \ge 0$ (note $f_2 \ge f_1$), the left hand-side of (3.37) can be estimated below by the term

$$-\lambda\langle B(u), (u_1^* - u)^+\rangle = \lambda \int_\Omega [(u_1^* - u)^+]^p\,dx. \tag{3.38}$$

Similar as in the proof of Lemma 3.3, the right-hand side of (3.37) can be estimated above as follows:

$$\int_\Omega \big(j^o(u; (u_1^* - u)^+) + j^o(u_1^*; -(u_1^* - u)^+)\big)\,dx \le c_1 \int_\Omega [(u_1^* - u)^+]^p\,dx, \tag{3.39}$$

which yields

$$(\lambda - c_1)\int_\Omega [(u_1^* - u)^+]^p\,dx \le 0. \tag{3.40}$$

Selecting λ such that $\lambda > c_1$ from (3.40), it follows that $u_1^* \le u$. Thus $Bu = 0$ and any solution u of (3.34) is in fact a solution of the hemivariational inequality (3.1) with right-hand side f_2 which exceeds u_1^*. Because u_2^* is the greatest solution of (3.1) with right-hand side f_2, it follows that $u_1^* \le u_2^*$.

The proof for the monotonicity of the least solutions follows by similar arguments and can be omitted. □

4. Proof of the main result

In this section, we will prove our main result, Theorem 2.1. Its proof is based on the results of Section 3 and on an existence result for an abstract operator equation in ordered Banach spaces which has been obtained recently in [4] and which we recall here for convenience.

4.1. Abstract operator equation. Consider the operator equation

$$u \in E : Lu = Nu, \tag{4.1}$$

where $L, N : W \to E$ are mappings defined on a partially ordered set W whose images are in a lattice-ordered Banach space $E = (E, \|\cdot\|, \le)$ that possesses the following properties.

(E0) Bounded and monotone sequences of E have weak or strong limits.

(E1) $\|u^+\| \le \|u\|$ for each $u \in E$, where $u^+ = \sup(0, u)$.

Then the following theorem holds, (cf. [4]).

THEOREM 4.1. *Let E be a lattice-ordered Banach space satisfying (E0) and (E1), and let W be some partially ordered set. Assume that the mappings $L,N : W \to E$ satisfy the following hypotheses.*

(A1) *The equation $Lu = f$ has for each $f \in E$ least and greatest solutions $u_*, u^* \in W$, and these extremal solutions depend monotonously on f.*

(A2) *$N : W \to E$ is increasing, that is, $u < v$ implies that $Nu \le Nv$ for all $u,v \in W$.*

(A3) *An estimate in the form*

$$\|Nu\| \le h(\|Lu\|), \quad u \in W, \tag{4.2}$$

holds, where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function having the property that there exists an $R > 0$ such that $R = h(R)$, and if $s \le h(s)$ then $s \le R$.

Then the operator equation (4.1) admits minimal and maximal solutions.

Note that according to hypotheses (A1)–(A3), no continuity or compactness conditions are imposed on the operators L,N. The notions *greatest* and *least*, and *minimal* and *maximal* have to be understood in the usual set-theoretical sense.

As for examples of ordered Banach spaces E satisfying (E0) and (E1) we refer to [4] and note that in particular, the following two spaces have these properties.

(i) $L^p(\Omega)$, $1 \le p \le \infty$, ordered a.e. pointwise, where Ω is a σ-finite measure space.

(ii) The Sobolev spaces $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$, $1 < p < \infty$, ordered a.e. pointwise with Ω being a bounded Lipschitz domain in $\mathbb{R}^N$.

4.2. Proof of Theorem 2.1. We are going to relate our original hemivariational inequality (1.1) to the abstract setting (4.1) and apply Theorem 4.1. For this purpose, we choose $E = L^q(\Omega)$, and denote by $\mathcal{S}_f \subset V_0$ the solution set of the hemivariational inequality (3.1) with right-hand side $f \in E$. Apparently, we then have $\mathcal{S}_{f_1} \cap \mathcal{S}_{f_2} = \emptyset$ if $f_1 \ne f_2$. We introduce the subset $W \subset V_0$ given by

$$W = \bigcup_{f \in E} \mathcal{S}_f, \tag{4.3}$$

and let $L : W \to E$ be the mapping which assigns $u \mapsto f$ whenever $u \in \mathcal{S}_f$. Further, let us define the operator N by

$$u \in W : Nu := \mathcal{F}u. \tag{4.4}$$

With the mappings L, $N : W \to E$ introduced this way, we readily can see that $u \in V_0$ is a solution of the original problem (1.1) if and only if u satisfies (4.1). Since $E = L^q(\Omega)$ possesses the properties (E0) and (E1), we only need to verify hypotheses (A1)–(A3) of Theorem 4.1 for the operators L and N specified above. First, due to Lemma 3.3 for each $f \in E$ (note $E \subset V_0^*$), there exist extremal solutions of

$$u \in W : Lu = f, \tag{4.5}$$

and these extremal solutions depend monotonously on f in view of Lemma 3.4. Thus (A1) of Theorem 4.1 is valid. Due to (H2)(i), the operator $N : W \to E$ defined by (4.4) is increasing which verifies (A2). To show (A3), let $u \in \mathcal{S}_f$, that is, $Lu = f$ (which is (3.1) with $f \in E$). From (3.1), we obtain by taking $v = 0$ and applying (H1)(ii) the following estimate:

$$\lambda_1 \|u\|_p^p \leq \|u\|_{V_0}^p \leq \|f\|_q \|u\|_p + c\|u\|_p + (\lambda_1 - \varepsilon)\|u\|_p^p, \tag{4.6}$$

for some constant $c \geq 0$, and thus (note that $Lu = f$)

$$\|u\|_p \leq \left(\frac{1}{\varepsilon}\right)^{(1)/(p-1)} \left(\|Lu\|_q + c\right)^{(1)/(p-1)}. \tag{4.7}$$

Finally, applying (H2)(ii) we obtain by means of (4.7)

$$\|Nu\|_q = \|\mathcal{F}u\|_q \leq c_3 + \mu\left(\frac{1}{\varepsilon}\right)^{\alpha/(p-1)} \left(\|Lu\|_q + c\right)^{\alpha/(p-1)}, \tag{4.8}$$

which means that

$$\|Nu\|_q \leq h(\|Lu\|_q), \tag{4.9}$$

where

$$h(s) = c_3 + \mu\left(\frac{1}{\varepsilon}\right)^{\alpha/(p-1)} (s + c)^{\alpha/(p-1)}. \tag{4.10}$$

With α specified in (H2)(ii), one can show by elementary calculations that the function $s \mapsto h(s)$ obtained in (4.10) has all the properties supposed in (A3). Therefore, Theorem 4.1 can be applied which completes the proof of our main result.

4.3. Example. Consider problem (1.1) with the nonlocal term $\mathcal{F}$ generated by the following F of the introduction:

$$F(x,u) = [|x|] + \gamma \int_\Omega [u(x)]\,dx, \tag{4.11}$$

where $[\cdot] : \mathbb{R} \to \mathbb{Z}$ is the integer function and γ is some positive constant. Let $|\Omega|$ denote the Lebesgue measure of $\Omega \subset \mathbb{R}^N$, and $c > 0$ some generic constant not depending on u. Then for $u \in L^p(\Omega)$, we get

$$|(\mathcal{F}u)(x)| \leq c + \gamma \int_\Omega (|u(y)| + 1)\,dy \leq c + \gamma|\Omega|^{1/q}\|u\|_p, \tag{4.12}$$

which yields the estimate

$$\|\mathcal{F}u\|_q \leq c + \gamma|\Omega|^{2/q}\|u\|_p. \tag{4.13}$$

According to hypothesis (H2)(ii), we have the following correspondences: $c_3 \cong c$, $\mu \cong \gamma|\Omega|^{2/q}$, and $\alpha \cong 1$. Hence, under the assumption (H1) on j by Theorem 2.1, the existence of solutions of (1.1) follows provided either $p > 2$ or $p = 2$ and $\gamma|\Omega|^{2/q} < \varepsilon$.

Remark 4.2. We note that the results obtained in the preceding sections can be extended to more general problems in the form (1.1) by replacing the p-Laplacian by a general quasilinear elliptic and coercive operator of Leray-Lions type.

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