ON EXTRAPOLATION BLOWUPS IN THE L_p SCALE

CLAUDIA CAPONE, ALBERTO FIORENZA, AND MIROSLAV KRBEC

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Yano's extrapolation theorem dated back to 1951 establishes boundedness properties of a subadditive operator T acting continuously in L_p for p close to 1 and/or taking L_∞ into L_p as $p \to 1_+$ and/or $p \to \infty$ with norms blowing up at speed $(p-1)^{-\alpha}$ and/or p^β, $\alpha,\beta > 0$. Here we give answers in terms of Zygmund, Lorentz-Zygmund and small Lebesgue spaces to what happens if $\|Tf\|_p \le c(p-r)^{-\alpha}\|f\|_p$ as $p \to r_+$ $(1 < r < \infty)$. The study has been motivated by current investigations of convolution maximal functions in stochastic analysis, where the problem occurs for $r = 2$. We also touch the problem of comparison of results in various scales of spaces.

1. Preliminaries

Throughout the paper $\Omega \subset \mathbb{R}^N$ will be measurable and with Lebesgue measure $|\Omega| = 1$. The latter is purely technical, any $|\Omega| < \infty$ can be considered. If f is a (real) measurable function on Ω, we will use the standard symbol f^* for its nonincreasing rearrangement—see, for example, [2, 10]. The usual Lebesgue space of functions integrable with the pth power will be denoted by $L_p = L_p(\Omega)$; we will use the averaging norm

$$\|f\|_p = \left(\frac{1}{|\Omega|}\int_\Omega |f(x)|^p dx\right)^{1/p} \tag{1.1}$$

(since we assume that $|\Omega| = 1$ the fraction will be omitted throughout the paper, of course).

The symbol $\sim$ will denote equivalence between functions or expressions containing functions, that is, $f \sim g$ (and/or $A \sim B$) if $c_1 f(x) \le g(x) \le c_2 f(x)$ a.e. in Ω (and/or $c_1 A \le B \le c_2 A$), where c_1 and c_2 are independent of functions and variables involved. Should no misunderstanding occur we will sometimes denote various constants in formulas by the same symbol.

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If X and Y are quasinormed linear spaces, then we write $X \subset Y$ for the ordinary inclusion and $X \hookrightarrow Y$ for the imbedding. (By the word *imbedding* we always mean *continuous imbedding*.) Throughout the paper we will tacitly use the well-known fact that if X and Y are Banach Function Spaces, then the inclusion $X \subset Y$ implies the imbedding $X \hookrightarrow Y$ (cf., e.g., [1]). This particularly applies to all the spaces in the following—they are all Banach Function Spaces.

Some special Orlicz spaces will be needed in the sequel. A *Young function* will be an even function $\Phi : \mathbb{R}^1 \to [0,\infty)$, convex on $[0,\infty)$, $\Phi(0) = 0$, $\Phi(\infty) = \infty$. The monographs [12–14] can be listed among basic references for the theory of Orlicz spaces.

The Zygmund space $L_p(\log L)^\beta = L_p(\log L)^\beta(\Omega)$, $1 \le p < \infty$, $\beta > 0$, is the Orlicz space generated by the Young function $\Phi(t) \sim t^p(\log t)^\beta$ for large t (as $|\Omega| < \infty$ only values of Φ near infinity matter). In $L_p(\log L)^\beta$ we will consider the norm $(\int_0^1 f^*(t)^p[\log(1/t)]^\beta dt)^{1/p}$, equivalent to the usual Luxemburg norm, where f^* is the nonincreasing rearrangement of f (see, e.g., [2]). More generally, if $\alpha \in \mathbb{R}^1$, $1 \le p$, $q < \infty$, then the Lorentz-Zygmund space $L^{p,q;\alpha}$ is equipped with the quasinorm $(\int_0^1 [t^{1/p} f^*(t)[\log(1/t)]^\alpha]^q (1/t)dt)^{1/q}$ (see [1]). With this notation we have $L_p(\log L)^\beta = L^{p,p;\beta/p}$. Recall (see [1, Theorem 9.3]) that $L^{p,q;\alpha} \subset L^{p,r;\beta}$ provided either (i) $q \le r$ and $\alpha \ge \beta$ or (ii) $q > r$ and $\alpha + 1/q > \beta + 1/r$.

Our last main tool are the small Lebesgue spaces $(sL)_{p,\lambda} = (sL)_{p,\lambda}(\Omega)$. Their formal definition is rather complicated, however, quite natural in the light of work on duality properties of grand Lebesgue spaces and extrapolation of Lebesgue spaces. We will say that a function f belongs to $(sL)_{p,\lambda}$ $(p > 1, \lambda > 0)$ if

$$\|f\|_{(sL)_{p,\lambda}} = \inf_{f=\sum f_j} \left(\sum_j \inf_{0<\varepsilon<p'-1} \varepsilon^{-\lambda/(p'-\varepsilon)} \|f_j\|_{(p'-\varepsilon)'} \right) < \infty, \tag{1.2}$$

where $(p'-\varepsilon)'$ denotes the index conjugate to $(p'-\varepsilon)$, that is,

$$(p'-\varepsilon)' = \frac{p'-\varepsilon}{p'-\varepsilon-1} = \frac{p-\varepsilon(p-1)}{1-\varepsilon(p-1)}. \tag{1.3}$$

Observe that in the definition of the norm in (1.2) one can consider $0 < \varepsilon < \varepsilon_0$ for any $0 < \varepsilon_0 < p'-1$ to arrive at the same space (up to an equivalence of norms), see [9].

The small Lebesgue spaces have been introduced in [6]. (Observe that the notation here is different—corresponding to $L^{(p',\lambda}$ in the preceding papers.) They are Banach function spaces; for this we refer to [3].

The small Lebesgue spaces turn out to be a natural counterpart of the grand Lebesgue spaces defined by Iwaniec and Sbordone in [11]. Observe that both scales of spaces have found important applications in Analysis, particularly in differential equations in the last years (see [3]).

For reader's convenience we state several claims that will be used in the sequel.

Proposition 1.1 [3]. *Let $1 < p < \infty$ and $\theta > 0$. Then*

$$\bigcup_{\beta>1} L_p(\log L)^{\beta\theta/(p'-1)} \subset (sL)_{p,\theta} \subset L_p(\log L)^{\theta/(p'-1)} \tag{1.4}$$

with continuous imbeddings. Moreover, both inclusions in (1.4) are proper.

Observe that in terms of Lorentz-Zygmund spaces the relations (1.4) read $L^{p,p;\beta\theta/p'} \subset (sL)_{p,\theta} \subset L^{p,p;\theta/p'}$.

With help of Stirling's formula it is not difficult to establish the following estimate for the norms in $L_1(\log L)^\gamma$ (see, e.g., [8] for details).

Proposition 1.2. *Let $\gamma > 0$. Then*

$$\|g\|_{L_1(\log L)^\gamma} \le c(q')^\gamma \|g\|_q, \quad q \longrightarrow 1_+ \tag{1.5}$$

with $c > 0$ independent of g and q.

An easy consequence is the following assertion.

Proposition 1.3. *Let $\gamma > 0$ and $1 \le r < \infty$. Then*

$$\|g\|_{L_r(\log L)^\gamma} \le c\left(\frac{p}{p-r}\right)^{\gamma/r} \|g\|_p, \quad p \longrightarrow r_+ \tag{1.6}$$

with $c > 0$ independent of g and p.

Proof. Let $p > r$ and $q = p/r$. We have

$$\begin{aligned} \|g\|_{L_r(\log L)^\gamma} &\le c\big\||g|^r\big\|^{1/r}_{L_1(\log L)^\gamma} \le c(q')^{\gamma/r}\big\||g|^r\big\|^{1/r}_q = c\left(\frac{q}{q-1}\right)^{\gamma/r}\left(\int_\Omega |g(x)|^{rq}dx\right)^{1/(rq)} \\ &= c\left(\frac{rq}{rq-r}\right)^{\gamma/r}\|g\|_{rq} = c\left(\frac{p}{p-r}\right)^{\gamma/r}\|g\|_p. \end{aligned} \tag{1.7}$$

□

We will also need an estimate similar to that in Proposition 1.2 for the small Lebesgue spaces.

Lemma 1.4. *Let $r > 1$, $\lambda > 0$. Then*

$$\|g\|_{(sL)_{r,\lambda}} \le \left[\frac{p-1}{(r'-1)(p-r)}\right]^{\lambda(p-1)/p} \|g\|_p, \quad p > r. \tag{1.8}$$

Proof. Consider trivial decomposition of g of the form

$$(g_1, g_2, \dots, g_j, \dots) = (0, 0, \dots, 0, g, 0, \dots). \tag{1.9}$$

Then

$$\sum_j \inf_{0<\varepsilon<r'-1} \varepsilon^{-\lambda/(r'-\varepsilon)}\|g_j\|_{(r'-\varepsilon)'} = \inf_{0<\varepsilon<r'-1} \varepsilon^{-\lambda/(r'-\varepsilon)}\|g\|_{(r'-\varepsilon)'}. \tag{1.10}$$

We have $r' - \varepsilon < r'$ hence $(r' - \varepsilon)' > r$. Put $p = (r' - \varepsilon)'$. Then $\varepsilon = r' - p' < r' - 1$. This yields

$$\left[\frac{(r'-1)(p-r)}{p-1}\right]^{-\lambda/[r'-(r'-1)(p-r)/(p-1)]} = \left[\frac{(r'-1)(p-r)}{p-1}\right]^{-\lambda(p-1)/p} = \left[\frac{p-1}{(r'-1)(p-r)}\right]^{\lambda(p-1)/p}. \tag{1.11}$$

□

2. Statement of main results

If $T : L_p \to L_p$ for $1 < p < p_0$, $p_0 \in (1, \infty)$ some fixed number, $\alpha > 0$, and if T is subadditive and such that

$$\|Tf\|_p \le \frac{c}{(p-1)^\alpha}\|f\|_p \quad \text{as } p \longrightarrow 1_+, \tag{2.1}$$

then the celebrated Yano's theorem [16] gives the consequence

$$\|Tf\|_1 \le c\|f\|_{L_1(\log L)^\alpha}, \quad f \in L_1(\log L)^\alpha, \tag{2.2}$$

with c independent of f. The blowup of the norms in (2.1) is often met in Analysis when studying properties of various integral operators. It includes the problem of what is going on in the Marcinkiewicz interpolation theorem if the resulting power tends to the left end point of the interpolation interval. Let us point out that (2.2) holds true if the underlying set Ω has a finite Lebesgue measure; it is more complicated to consider operators, for example, in the whole of $\mathbb{R}^N$.

There are, however, subadditive operators for which the blowup of norms occurs if $p \to 2_+$. In [5] Da Prato and Zabczyk investigated the stochastic convolution maximal function and established an inequality of type (2.1) where $p - 2$ appears instead of $p - 1$. In [15], the authors investigate behaviour of this maximal function near L_2 and they restrict themselves for $p \ge 2 + \delta$ with some positive δ. Rather surprisingly Yano's theorem does not permit a straightforward "shift" of the situation from $p \to 1_+$ to $p \to 2_+$. A major problem is the subadditivity, which fails for $T(g^2)$ with $g \in L_p$, $p > 2$. Even though one can decompose g^2 into a sum of g_j^2, where $g_j \in L_{(2^j)'}$, $j = 1, \dots$ $((2^j)' = 2^j/(2^j - 1))$, have disjoint support (cf. [8]), the latter property need not be inherited by the functions $T(g_j^2)$.

In [8] a generalization of Yano's theorem was established for operators $T : L_p \to L_{r(p)}$, where $r : [1, p_0) \to [1, \infty)$, $p \le r(1)p \le r(p)$ for every $p \in [1, p_0]$ and $p \to 1_+$. Here we will give an answer to the extrapolation problem for $p \to r_+$ $(1 < r < \infty)$ in terms of three "limiting" spaces: Zygmund, small Lebesgue, and Lorentz-Zygmund spaces. A special case of this situation for $p \to 2_+$ and the blowup $(p - 2)^{-1}$ has been considered by Carro and Martín [4] by means of the abstract extrapolation theory. In the last section we will consider the problem of comparison of these various results. In our knowledge a complete picture is not available at the moment. We illustrate the situation with several examples.

Our approach in this paper does not require any special background, in particular, the abstract extrapolation method, which has been used for the small Lebesgue spaces, for example, in [7]. The basic idea follows the classical Titschmarsch proof of the $L\log L$ theorem (e.g., in [17]), namely, to estimate the quasinorms in terms of suitable decompositions, permitting a satisfactory analysis of the rather delicate situation near the left end point of the extrapolation interval.

First we will tackle the problem of what is going on if we try to estimate the Zygmund norm of Tf.

Theorem 2.1. *Let $1 < r < p_0 < \infty$ and let T be a subadditive and homogeneous operator such that*

$$\|Tf\|_p \le \frac{c}{(p-r)^\alpha}\|f\|_p, \quad r < p < p_0, \tag{2.3}$$

with some $\alpha > 0$. Then, for any $\gamma > 0$,

$$\|Tf\|_{L_r(\log L)^\gamma} \le c \inf_{f=\sum f_j} \sum_j \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\delta/(r'-\varepsilon)} \|f_j\|_{(r'-\varepsilon)'}, \tag{2.4}$$

where $\delta = r'(\alpha+\gamma/r)$ and c is independent of f, that is,

$$\|Tf\|_{L_r(\log L)^\gamma} \le c\|f\|_{(sL)_{r,\delta}}, \quad f \in (sL)_{r,\delta}. \tag{2.5}$$

Observe that γ is *positive* in the above theorem. To get the limiting estimate in L_r we will need the small Lebesgue spaces. We get actually estimates in a scale with L_r as the "left end point" in next two theorems. Indeed, going along the lines of the proof of Theorem 2.2 in the next section one can easily check that the proof works also for $\lambda = 0$ (the estimate (2.7) below). At the same time it is not difficult to see that $(sL)_{r,0} = L_r$.

Theorem 2.2. *Let $1 < r < p_0 < \infty$ and let T be a subadditive operator satisfying (2.3) and $\lambda > 0$. Then*

$$\|Tf\|_{(sL)_{r,\lambda}} \le c \inf_{f=\sum f_j} \sum_j \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\mu/(r'-\varepsilon)} \|f_j\|_{(r'-\varepsilon)'}, \tag{2.6}$$

with $\mu = r'\alpha+\lambda$ and c independent of f, that is,

$$\|Tf\|_{(sL)_{r,\lambda}} \le c\|f\|_{(sL)_{r,\mu}}, \quad f \in (sL)_{r,\mu}. \tag{2.7}$$

The "left end point" variant of this is the following theorem.

Theorem 2.3. *Let $1 < r < p_0 < \infty$ and let T be a subadditive operator satisfying (2.3). Then*

$$\|Tf\|_r \le c \inf_{f=\sum f_j} \sum_j \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-r'\alpha/(r'-\varepsilon)} \|f_j\|_{(r'-\varepsilon)'}, \tag{2.8}$$

with c independent of f, that is,

$$\|Tf\|_r \le c\|f\|_{(sL)_{r,r'\alpha}}, \quad f \in (sL)_{r,r'\alpha}. \tag{2.9}$$

In the next theorem we use the scale of Lorentz-Zygmund spaces.

Theorem 2.4. *Let $1 < r < \infty$ and let T be a subadditive operator satisfying*

$$\|Tf\|_p \le \frac{c}{(p-r)^\alpha}\|f\|_p, \quad p > r, \tag{2.10}$$

Then

$$\|Tf\|_r \le c\int_0^1 t^{1/r} f^*(t)\left[\log(1/t)\right]^\alpha \frac{dt}{t}, \tag{2.11}$$

that is,

$$\|Tf\|_r \le c\|f\|_{L^{r,1;\alpha}}. \tag{2.12}$$

The space L_r in the norm on the left-hand side of (2.11) can be viewed as another "left end point" of another suitable scale of function spaces, namely of the logarithmic Lebesgue (i.e., Zygmund) spaces $L_r(\log L)^\beta = L^{r,r;\beta/r}$, $\beta > 0$. The proof of the corresponding estimate repeats the basic idea of the proof of Theorem 2.4. For completeness we state the claim as a separate theorem and in the next section we describe the appropriate modification of the proof.

Theorem 2.5. *Let $1 < r < \infty$ and let T be a subadditive operator satisfying (2.3) and α, $\beta > 0$. Then*

$$\|Tf\|_{L_r(\log L)^\beta} \le c\int_0^1 t^{1/r} f^*(t)\left[\log(1/t)\right]^{\alpha+\beta/r} \frac{dt}{t}, \tag{2.13}$$

that is,

$$\|Tf\|_{L_r(\log L)^\beta} \le c\|f\|_{L^{r,1;\alpha+\beta/r}}. \tag{2.14}$$

3. Proofs

Proof of Theorem 2.1. Let $p > r$ and $\gamma > 0$. Then by virtue of Proposition 1.3 we have

$$\|Tf\|_{L_r(\log L)^\gamma} \le c\left(\frac{p}{p-r}\right)^{\gamma/r}\|Tf\|_p. \tag{3.1}$$

Hence, if we plug in the assumption (2.3), we have

$$\|Tf\|_{L_r(\log L)^\gamma} \le \left(\frac{c}{p-r}\right)^{\alpha+\gamma/r}\|f\|_p, \quad r < p < p_0. \tag{3.2}$$

Put $p - r = \varepsilon$. Then this becomes

$$\|Tf\|_{L_r(\log L)^\gamma} \le \frac{c}{\varepsilon^{\alpha+\gamma/r}}\|f\|_{r+\varepsilon}, \quad r < p < p_0. \tag{3.3}$$

Let us choose $\tilde{\varepsilon}$ such that $(r' - \tilde{\varepsilon})' = r + \varepsilon$, that is,

$$\varepsilon = (r' - \tilde{\varepsilon})' - r = \frac{(r-1)\tilde{\varepsilon}}{r' - 1 - \tilde{\varepsilon}}. \tag{3.4}$$

Then

$$\begin{aligned}\|Tf\|_{L_r(\log L)^\gamma} &\le c\left[\frac{(r-1)\tilde{\varepsilon}}{r'-1-\tilde{\varepsilon}}\right]^{-\alpha-\gamma/r}\|f\|_{(r'-\tilde{\varepsilon})'}\\ &\le c\left(\frac{1}{r'-1-\tilde{\varepsilon}}\right)^{-\alpha-\gamma/r}(\tilde{\varepsilon})^{-\alpha-\gamma/r}\|f\|_{(r'-\tilde{\varepsilon})'}.\end{aligned} \tag{3.5}$$

Write again ε instead of $\tilde{\varepsilon}$; we have thus

$$\|Tf\|_{L_r(\log L)^\gamma} \le c\left(\frac{1}{r'-1-\varepsilon}\right)^{-\alpha-\gamma/r}\varepsilon^{-\alpha-\gamma/r}\|f\|_{(r'-\varepsilon)'}. \tag{3.6}$$

But

$$\varepsilon^{-\alpha-\gamma/r} \le c\varepsilon^{-r'(\alpha+\gamma/r)/(r'-\varepsilon)} \tag{3.7}$$

therefore, if $f=\sum f_j$, we have

$$\|Tf\|_{L_r(\log L)^\gamma} \le \sum_j \|Tf_j\|_{L_r(\log L)^\gamma} \le c\sum_j \inf_{0<\varepsilon<\varepsilon_0}\varepsilon^{-r'(\alpha+\gamma/r)/(r'-\varepsilon)}\|f_j\|_{(r'-\varepsilon)'}, \tag{3.8}$$

and passing to the infimum over all decompositions we finally obtain

$$\|Tf\|_{L_r(\log L)^\gamma} \le c\|f\|_{(sL)_{r,r'(\alpha+\gamma/r)}}. \tag{3.9}$$

□

Proof of Theorem 2.2. Applying Lemma 1.4 we have (for $1<r<p<p_0$)

$$\begin{aligned}\|Tf\|_{(sL)_{r,\lambda}} &\le c\left(\frac{p-1}{p-r}\right)^{\lambda(p-1)/p}\|Tf\|_p \le c\left(\frac{p-1}{p-r}\right)^{\lambda(p-1)/p}\frac{c}{(p-r)^\alpha}\|f\|_p\\ &= c(p-1)^{\lambda(p-1)/p}\frac{1}{(p-r)^{\alpha+\lambda(p-1)/p}}\|f\|_p = c\frac{1}{(p-r)^{\alpha+\lambda(p-1)/p}}\|f\|_p.\end{aligned} \tag{3.10}$$

Put $p-r=\varepsilon$. Then we can rewrite the last estimate as

$$\|Tf\|_{(sL)_{r,\lambda}} \le \frac{c}{\varepsilon^{\alpha+\lambda(r-1+\varepsilon)/(r+\varepsilon)}}\|f\|_{r+\varepsilon} \tag{3.11}$$

so that (with $\tilde{\varepsilon}$ as in the proof of Theorem 2.1)

$$\|Tf\|_{(sL)_{r,\lambda}} \le c(\tilde{\varepsilon})^{-\alpha-\lambda[r-1+(r-1)\tilde{\varepsilon}/(r'-1-\tilde{\varepsilon})]/[r+(r-1)\tilde{\varepsilon}/(r'-1-\tilde{\varepsilon})]}\|f\|_{(r'-\tilde{\varepsilon})'}. \tag{3.12}$$

Writing ε again we have

$$\|Tf\|_{(sL)_{r,\lambda}} \le c\varepsilon^{-(r'\alpha+\lambda)/(r'-\varepsilon)}\|f\|_{(r'-\varepsilon)'}. \tag{3.13}$$

Now we put $\mu=r'\alpha+\lambda$. Considering any decomposition $f=\sum f_j$ we proceed similarly as in the proof of the previous theorem to get our claim. □

Proof of Theorem 2.4. Let $p_k = r(1+1/k) > r$, $k = 1,2,\ldots$, and define $f_k^*(t) = f^*(t)$ for $t \in I_k = (e^{-k}, e^{-k+1})$. Since f^* and f are equimeasurable, in particular, $|\{f^*(t) = f^*(e^{-k})\}| = |\{|f(x)| = f^*(e^{-k})\}|$, we can choose f_k so that $f = \sum f_k$ a.e. and $(f_k)^* = f_k^*$ a.e. By subadditivity, Hölder's inequality, and the blowup assumption,

$$\|Tf\|_r \le \sum_{k=1}^{\infty} \|Tf_k\|_{p_k} \le \sum_{k=1}^{\infty} \frac{1}{(p_k - r)^{\alpha}} \|f_k\|_{p_k}. \tag{3.14}$$

Discretising the right-hand side we get

$$\begin{aligned} \|Tf\|_r &\le c \sum_{k=1}^{\infty} k^{\alpha} \left(\int_0^{e^{-k}(e-1)} \left[f^*(e^{-k}+s) \right]^{p_k} ds \right)^{1/p_k} \\ &\le c \sum_{k=1}^{\infty} k^{\alpha} e^{-k} f^*(e^{-k}) e^{k-k/p_k} \sim c \sum_{k=1}^{\infty} \int_{I_k} k^{\alpha} e^{k/p_k'} f^*(e^{-k}) ds \end{aligned} \tag{3.15}$$

and since $e^{k/p_k'} \sim e^{k/r'}$,

$$\|Tf\|_r \le c \sum_{k=1}^{\infty} \int_{I_k} \left[\log(1/t) \right]^{\alpha} t^{-1/r'} f^*(t) dt = c \int_0^1 t^{1/r} \left[\log(1/t) \right]^{\alpha} f^*(t) \frac{dt}{t} = c\|f\|_{L^{r,1;\alpha}}. \tag{3.16}$$

□

It remains to prove Theorem 2.5. Since it goes along the same lines as that of Theorem 2.4 we will proceed briefly.

Sketch of the proof of Theorem 2.5. Let decompose f as before. By Hölder's inequality,

$$\begin{aligned} &\left(\int_0^1 (Tf)^*(t)^r \left[\log(1/t) \right]^{\beta} dt \right)^{1/r} \\ &\quad \le \sum_{k=1}^{\infty} \left(\int_0^1 (Tf_k)^*(t)^{r(k+1)/k} dt \right)^{(1/r)(k/(k+1))} \left(\int_0^1 [\log(1/t)]^{\beta(k+1)} dt \right)^{1/[r(k+1)]}. \end{aligned} \tag{3.17}$$

According to Stirling's formula for the Gamma function (we omit details of the easy integration),

$$\begin{aligned} \left(\int_0^1 \left[\log(1/t) \right]^{\beta(k+1)} dt \right)^{1/[r(k+1)]} &\sim \left[\beta(k+1)\Gamma(\beta(k+1)) \right]^{1/[r(k+1)]} \\ &\sim \left[(\beta(k+1))^{\beta(k+1)-1/2} e^{-\beta(k+1)} \right]^{1/[r(k+1)]} \\ &\sim k^{\beta/r} \end{aligned} \tag{3.18}$$

as $k \to \infty$. Hence the $L_r(\log L)^{\beta}$ norm of Tf is estimated by

$$\sum_{k=1}^{\infty} k^{\beta/r} \left(\int_0^1 (Tf_k)^*(t)^{p_k} dt \right)^{1/p_k} \tag{3.19}$$

and we see that there is just the extra term $k^{\beta/r}$ in comparison with (3.16), resulting in additional $[\log(1/t)]^{\beta/r}$ in the final estimate. □

4. Miscellanea

For the sake of simplicity, particularly because of special examples of functions when comparing various spaces, we restrict ourselves to the case $r = 2$ here.

Remark 4.1. Since

$$(sL)_{2,\lambda} \hookrightarrow L_2(\log L)^{\lambda}, \qquad \lambda > 0, \tag{4.1}$$

(see [3, (1.2)]) Theorem 2.1 follows from Theorem 2.2. Nevertheless, we preferred to give an independent proof of Theorem 2.1 since it throws some more light on the problems considered.

A simple direct proof of (4.1) for $\lambda = 1$ can be given, different from that from [3]. We will give it for completeness.

Let $f \in (sL)_{2,1}$ and let $f = \sum_j f_j$ be any decomposition. According to Proposition 1.3 we have (one can write $\varepsilon^{1/2}$ instead of $\varepsilon^{1/(2-\varepsilon)}$ for sufficiently small ε)

$$\|f\|_{L_2(\log L)^1} \le \sum_j \|f_j\|_{L_2(\log L)^1} \le c \sum_j \inf_{\varepsilon} \varepsilon^{-1/2} \|f_j\|_{2+\varepsilon}. \tag{4.2}$$

Suppose that $\|f\|_{(sL)_{2,1}} < 1$. Then there exists a decomposition $f = \sum \tilde{f}_j$ such that (for small $\varepsilon > 0$)

$$\sum_j \inf_{\varepsilon} \varepsilon^{-1/2} \|\tilde{f}_j\|_{2+\varepsilon} < 1. \tag{4.3}$$

Hence $\inf_{\varepsilon} \varepsilon^{-1/2} \|\tilde{f}_j\|_{2+\varepsilon} < 1$ for all j and we can consider only such decompositions $f = \sum \tilde{f}_j$ such that (4.3) holds. Moreover, for every j we can find ε_j such that

$$\varepsilon_j^{-1/2} \|\tilde{f}_j\|_{2+\varepsilon_j} < 1 \tag{4.4}$$

and when taking the $\inf_{\varepsilon}$ in the $(sL)_{2,1}$ norm one can consider only such ε's for which (4.4) is true. Thus for each j let E_j consists of those $\varepsilon \in (0, 1/2)$ for which (4.4) holds. Then

$$\sum_j \inf_{\varepsilon \in E_j} \varepsilon^{-1/2} \|\tilde{f}_j\|_{2+\varepsilon} \sim \|f\|_{(sL)_{2,1}} \tag{4.5}$$

and we are done.

Remark 4.2. It will be of interest to compare various estimates we arrived at. We will give various examples and prove several imbeddings. As observed earlier at the moment there is no complete imbedding picture available for all the spaces involved. Two papers should be mentioned in this connection: first the recent paper [7], where the norm in $(sL)_{p,1}$ is shown to be equivalent to the norm in a sort of limiting interpolation space, namely, to

$$\int_0^1 [\log(e/t)]^{-1/p} \left(\int_0^t [f^*(s)]^p ds \right)^{1/p} \frac{dt}{t}. \tag{4.6}$$

The second reference of interest is the above mentioned estimate by Carro and Martín in [4]. They consider the special case $r = 2$ and $\alpha = 1$ in (2.3) and show that the L_2 norm of Tf can be estimated by a sort of an averaging norm given by

$$\int_0^1 \left(\int_0^t f^*(s)^2 ds \right)^{1/2} \frac{dt}{t}. \tag{4.7}$$

For the moment let us denote the space of all f with the finite norm (4.7) by AV_2. It is easy to see that if (4.7) is finite, then $f \in L^{2,1;0}$. Indeed, by monotonicity of f^*,

$$\int_0^1 t^{1/2} f^*(t) \frac{dt}{t} \leq \int_0^1 \left(\int_0^t f^*(s)^2 ds \right)^{1/2} \frac{dt}{t}. \tag{4.8}$$

In Theorem 2.4 (for $r = 2$) we have $L^{2,1;1}$ as the limiting domain for T. Clearly also the latter space is smaller than $L^{2,1;0}$.

Further, it is easy to show that

$$(sL)_{2,2} \hookrightarrow (sL)_{2,1} . \tag{4.9}$$

Using the characterization of $(sL)_{2,1}$ from [7], namely,

$$\int_0^1 [\log(e/t)]^{-1/2} \left(\int_0^t f^*(s)^2 ds \right)^{1/2} \frac{dt}{t} < \infty, \tag{4.10}$$

and comparing it with the AV_2 norm from (4.7), we see that

$$\mathrm{AV}_2 \hookrightarrow (sL)_{2,1}. \tag{4.11}$$

Plainly, this inclusion is proper. Hence both spaces AV_2 and $(sL)_{2,2}$ are subspaces of $(sL)_{2,1}$. By virtue of (1.4) we have $(sL)_{2,1} \subset L^{2,2;1/2}$ and according to imbedding properties of Lorentz-Zygmund spaces we have $L^{2,1;1} \subset L^{2,2;1/2}$ (see Section 1).

The following examples show that even $L^{2,1;1} \setminus \mathrm{AV}_2 \neq \emptyset$ and that $(sL)_{2,2} \setminus \mathrm{AV}_2 \neq \emptyset$.

Example 4.3. There is a function that belongs to $L^{2,1;1}$ $(\subset L_2(\log L)^2)$ and such that, at the same time,

$$\int_0^1 \frac{1}{t} \left(\int_0^t f^*(s)^2 ds \right)^{1/2} dt = \infty. \tag{4.12}$$

To this end choose any f such that

$$\frac{1}{t}\left(\int_0^t f^*(s)^2 ds\right)^{1/2} = \frac{1}{t}\frac{1}{[\log(e^2/t)][\log\log(e^2/t)]}. \tag{4.13}$$

Then (4.12) holds and

$$\int_0^t f^*(s)^2 ds = \frac{1}{[\log(e^2/t)]^2[\log\log(e^2/t)]^2}. \tag{4.14}$$

After differentiation we get

$$f^*(t)^2 \sim \frac{1}{[\log(e^2/t)]^3}\cdot\frac{1}{t}\cdot\frac{1}{[\log\log(e^2/t)]^2} + \frac{1}{[\log(e^2/t)]^3}\cdot\frac{1}{t}\cdot\frac{1}{[\log\log(e^2/t)]^3}. \tag{4.15}$$

Then plainly

$$\int_0^1 f^*(t)^2\left(\log\frac{e^2}{t}\right)^2 dt < \infty. \tag{4.16}$$

A direct calculation shows that the function f such that

$$f^*(t)^2 = \frac{1}{t}\frac{1}{[\log e^2/t]^3}\frac{1}{[\log\log e^2/t]^3} \tag{4.17}$$

belongs to $L^{2,1;1}$ and the above construction shows that its AV_2 norm is infinite.

Remark 4.4. Observe that by integration by parts,

$$\begin{aligned}\int_0^1\left(\int_0^t f^*(s)^2 ds\right)^{1/2}\frac{dt}{t} &= \left[-\log(1/t)\left(\int_0^t f^*(s)^2 ds\right)^{1/2}\right]_0^1 \\ &\quad + \frac{1}{2}\int_0^1 \frac{f^*(t)^2(\log(1/t))^2}{\log(1/t)\left(\int_0^t f^*(s)^2 ds\right)^{1/2}}dt.\end{aligned} \tag{4.18}$$

Further,

$$\begin{aligned}\int_0^1\left(\int_0^t f^*(s)^2 ds\right)^{1/2}\frac{dt}{t} &\sim \sum_{k=1}^{\infty}\left(\sum_{m=k}^{\infty} e^{-m} f^*(e^{-m})^2\right)^{1/2} \geq c\left(\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m} e^{-m/2} f^*(e^{-m})\right)^2\right)^{1/2} \\ &\geq c\left(\sum_{m=1}^{\infty} m^4 e^{-m} f^*(e^{-m})^2\right)^{1/2}.\end{aligned} \tag{4.19}$$

Hence $f^*(e^{-m})^2 \leq ce^m m^{-4}$ and

$$\begin{aligned}\left(\int_0^{e^{-m}} f^*(s)^2 ds\right)^{1/2} &\leq \left(\sum_{j=m}^{\infty}\int_{e^{-j-1}}^{e^{-j}} f^*(s)^2 ds\right)^{1/2} \leq c\left(\sum_{j=m}^{\infty} e^{-j} f^*(e^{-j})^2\right)^{1/2} \\ &\leq c\left(\sum_{j=m}^{\infty} e^{-j}e^{j} j^{-4}\right)^{1/2} \sim \left(\int_m^{\infty}\frac{dt}{t^4}\right)^{1/2} \sim m^{-3/2}.\end{aligned} \tag{4.20}$$

From this

$$\left(\int_0^t f^*(s)^2 ds\right)^{1/2} \leq c\big(\log(1/t)\big)^{-3/2} \tag{4.21}$$

and therefore

$$\lim_{t\longrightarrow 0}\big(\log(1/t)\big)\left(\int_0^t f^*(s)^2 ds\right)^{1/2} = 0. \tag{4.22}$$

Plugging this into (4.18) we get

$$\|f\|_{\mathrm{AV}_2} \leq c\int_0^1 \frac{f^*(t)^2\big(\log(1/t)^2\big)}{\big(\log(1/t)\big)\big(\int_0^t f^*(s)^2 ds\big)^{1/2}} dt. \tag{4.23}$$

On the other hand,

$$\begin{aligned}&\int_0^1 \frac{f^*(t)^2\big(\log(1/t)\big)^2}{\big(\log(1/t)\big)\big(\int_0^t f^*(s)^2 ds\big)^{1/2}} dt \\ &\quad\leq c\int_0^1 \frac{\big(\log(1/t)\big)^2\big(\int_0^t f^*(s)^2 ds\big)^{1/2}}{\log(1/t)} dt \\ &\quad\leq c\int_0^1 \frac{1}{t}\left(\int_0^t f^*(s)^2 ds\right)^{1/2} t\log(1/t) dt \leq c\|f\|_{\mathrm{AV}_2}.\end{aligned} \tag{4.24}$$

Inequalities (4.23) and (4.24) show that if $f \in \mathrm{AV}_2$, then

$$\int_0^1 \frac{f^*(t)^2\big(\log(1/t)\big)^2}{\log(1/t)\big(\int_0^t f^*(s)^2 ds\big)^{1/2}} dt < \infty \tag{4.25}$$

and the expression on the left is equivalent to the norm in AV_2. This also gives some idea about the integrability properties of functions in $L_2(\log L)^2$ in comparison of those in AV_2.

Of interest is of course the question whether $(sL)_{2,2} \setminus \mathrm{AV}_2 \neq \emptyset$. The next example shows that this is indeed the case.

Example 4.5. Here we give an example of a function that belongs to $(sL)_{2,2}$ and such that

$$\int_0^1 \frac{1}{t}\left(\int_0^t f^*(s)^2 ds\right)^{1/2} dt = \infty. \tag{4.26}$$

Let

$$\varepsilon_k = \frac{1}{k^{1/2-\eta}}, \quad b_k = \frac{1}{k^{1+\eta}\log k}, \quad \text{for some } \eta \in (0,1/2). \tag{4.27}$$

It is easy to see that

$$\inf_{f=\sum f_j} \sum_j \inf_{0<\varepsilon<1} \varepsilon^{-1} \|f_j\|_{2+\varepsilon} \tag{4.28}$$

is an equivalent norm in $(sL)_{2,2}$. Let us split $(0,1)$ into mutually disjoint intervals $I_j = (e^{-j}, e^{-j+1})$, $j = 1,2,\ldots$. If $\{\varepsilon_j\} \subset (0,1)$ is any sequence converging to 0, then

$$\|f\|_{(sL)_{2,2}} \le c \sum_j \varepsilon_j^{-1} \|f_j\|_{2+\varepsilon_j}, \tag{4.29}$$

where $f_j = f^*|_{I_j}$. Put $f^*(e^{-j}) = e^{j/2}\varepsilon_j b_j$, $j = 1,2,\ldots$, and define f^* on the whole of $(0,1]$ so that it is continuous in $(0,1]$ and linear on all I_j. Since f is decreasing at least in some neighbourhood of the origin we can consider it as a nonincreasing rearrangement of some function f on Ω. We have

$$\|f\|_{(sL)_{2,2}} \le c \sum_j \frac{1}{\varepsilon_j} e^{-j/2} f^*(e^{-j}) \le c \sum_j \frac{1}{\varepsilon_j} e^{-j/2} e^{j/2} \varepsilon_j b_j \le c \sum_j b_j < \infty. \tag{4.30}$$

On the other hand, since $\int_{I_j} dt/t$ is constant independent of j we have,

$$\begin{aligned}\int_0^1 \frac{1}{t}\left(\int_0^t f^*(s)^2 ds\right)^{1/2} dt &\sim \sum_{j=1}^{\infty}\left(\sum_{k=j}^{\infty} e^{-k} e^{k} \varepsilon_k^2 b_k^2\right)^{1/2} = \sum_{j=1}^{\infty}\left(\sum_{k=j}^{\infty} \varepsilon_k^2 b_k^2\right)^{1/2} \\ &= \sum_{j=1}^{\infty}\left(\sum_{k=j}^{\infty} \frac{1}{k^3(\log k)^2}\right)^{1/2}.\end{aligned} \tag{4.31}$$

But

$$\sum_{k=j}^{\infty} \frac{1}{k^3(\log k)^2} \sim \frac{1}{j^2(\log j)^2} \tag{4.32}$$

(compare $\int_A^\infty dt/t^3(\log t)^2$ with $1/A^2(\log A)^2$ as $A \to \infty$, e.g., using l'Hôpital's rule) so that the last series in (4.31) is divergent.

We finish the paper with two more examples, throwing some more light on the spaces involved.

Example 4.6. We use the last example to construct a function f such that $f \in (sL)_{2,2}$ and $f \notin L^{2,2;1}$ (hence also $f \notin \mathrm{AV}_2$ since $\mathrm{AV}_2 \hookrightarrow L^{2,2;1}$ in view of (4.25) and (4.22)). Consider any f such that

$$f^*(t) = \frac{1}{\sqrt{t}} \frac{1}{[\log(e/t)]^{3/2} \log(\log(e^2/t))}, \tag{4.33}$$

that is,

$$f^*(t) \sim \frac{e^{k/2}}{k^{3/2} \log k} \quad \text{on } I_k, \tag{4.34}$$

then according to Example 4.5 we have $f \notin \mathrm{AV}_2$ and $f \in (sL)_{2,2}$. An easy direct calculation shows that $f \notin L^{2,2;1}$.

Example 4.7. Any function f such that

$$f^*(t) = \frac{1}{\sqrt{t}[\log(1/t)]^2} \tag{4.35}$$

belongs to AV_2 and hence also to $L^{2,2;1}$ (simple direct calculations). At the same time it belongs to $(sL)_{2,2}$. To see that consider the decomposition $f = \sum f_k = \sum f\chi_{I_k}$ with I_k as in the proof of Theorem 2.4. In the series

$$\sum_k \inf_\varepsilon \varepsilon^{-1/2} \|f_k\|_{2+\varepsilon} \tag{4.36}$$

choose $\varepsilon = 1/k$ in the kth term. The infimum in (4.36) is then estimated from above by

$$k^{1/2}\left(\int_{I_k} \frac{dt}{t[\log(1/t)]^{2(2+1/k)}}\right)^{1/(2+1/k)} \sim k^{1/2}\left(e^{-k} e^{k(1+1/2k)} \frac{1}{k^{2(2+1/k)}}\right)^{1/(2+1/k)} \sim \frac{1}{k^{3/2}} \tag{4.37}$$

so that in (4.36) we get a convergent series.

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Claudia Capone: CNR Istituto per le Applicazioni del Calcolo "Mauro Picone," Via P. Castellino 111, 80131 Napoli, Italy
E-mail address: c.capone@na.iac.cnr.it

Alberto Fiorenza: Dipartimento di Costruzioni e Metodi Matematici in Architettura,
Università degli Studi di Napoli "Federico II," via Monteoliveto 3, 80134 Napoli, Italy;
CNR Istituto per le Applicazioni del Calcolo "Mauro Picone,"
Via P. Castellino 111, 80131 Napoli, Italy
E-mail address: fiorenza@unina.it

Miroslav Krbec: Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25,
CZ-115 67 Prague 1, Czech Republic
E-mail address: krbecm@matsrv.math.cas.cz