# ESSENTIAL SPECTRA OF QUASISIMILAR $(p,k)$-QUASIHYPONORMAL OPERATORS

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It is shown that if $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is an $2 \times 2$ upper-triangular operator matrix acting on the Hilbert space $\mathcal{H} \oplus \mathcal{K}$ and if $\sigma_e(\cdot)$ denotes the essential spectrum, then the passage from $\sigma_e(A) \cup \sigma_e(B)$ to $\sigma_e(M_C)$ is accomplished by removing certain open subsets of $\sigma_e(A) \cap \sigma_e(B)$ from the former. Using this result we establish that quasisimilar $(p,k)$-quasihyponormal operators have equal spectra and essential spectra.



## 1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be infinite-dimensional separable complex Hilbert spaces and let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We abbreviate $\mathcal{L}(\mathcal{H},\mathcal{H})$ by $\mathcal{L}(\mathcal{H})$. If $T \in \mathcal{L}(\mathcal{H})$ write $\sigma(T)$ for the spectrum of $T$. An operator $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is called *left-Fredholm* if it has closed range with finite-dimensional null space and *right-Fredholm* if it has closed range with its range of finite codimension. If $A$ is both left- and right-Fredholm, we call it *Fredholm*: in this case, we define the *index* of $A$ by

$$\operatorname{index}(A) = \frac{\dim A^{-1}(0) - \dim \mathcal{H}}{A(\mathcal{H})}. \tag{1.1}$$

An operator $A \in \mathcal{L}(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero. If $A \in \mathcal{L}(\mathcal{H})$, then the left essential spectrum $\sigma_e^+(A)$, the right essential spectrum $\sigma_e^-(A)$, the essential spectrum $\sigma_e(A)$, and the Weyl spectrum $w(A)$ are defined by

$$\begin{aligned} \sigma_e^+(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not left-Fredholm}\}; \\ \sigma_e^-(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not right-Fredholm}\}; \\ \sigma_e(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}; \\ w(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Weyl}\}. \end{aligned} \tag{1.2}$$

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When $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are given we denote by $M_C$ an operator acting on $\mathcal{H} \oplus \mathcal{K}$ of the form

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \tag{1.3}$$

where $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. For bounded linear operators $A$, $B$, and $C$, the equality

$$\sigma(M_C) = \sigma(A) \cup \sigma(B) \tag{1.4}$$

and the equality

$$w(M_C) = w(A) \cup w(B) \tag{1.5}$$

were studied by numerous authors. In [5, 10], it was shown that if $\sigma(A) \cap \sigma(B)$ (or $w(A) \cap w(B)$) has no interior points, then (1.4) (or (1.5)) is satisfied for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$.

Recall [9] that an operator $T \in \mathcal{L}(\mathcal{H})$ is called $(p,k)$-*quasihyponormal* if $T^{*k}(|T|^{2p} - |T^*|^{2p})T^k \geq 0$, where $0 < p \leq 1$ and $k$ is a positive integer. This includes $p$-hyponormal operators ($k = 0$), $k$-quasihyponormal operators ($p = 1$), and $p$-quasihyponormal operators ($k = 1$). The followings are well known:

$$\begin{aligned} \{\text{hyponormal operators}\} &\subseteq \{p\text{-hyponormal operators}\} \\ &\subseteq \{p\text{-quasihyponormal operators}\} \\ &\subseteq \{(p,k)\text{-quasihyponormal operators}\}, \\ \{\text{hyponormal operators}\} &\subseteq \{k\text{-quasihyponormal operators}\} \\ &\subseteq \{(p,k)\text{-quasihyponormal operators}\}. \end{aligned} \tag{1.6}$$

Recall that an operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called *regular* if there is an operator $A' \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ for which $A = AA'A$; then $A'$ is called a *generalized inverse* for $A$. In this case, $\mathcal{H}$ and $\mathcal{K}$ can be decomposed as follows (cf. [6, 7]):

$$A^{-1}(0) \oplus A'A(\mathcal{H}) = \mathcal{H}, \qquad A(\mathcal{H}) \oplus (AA')^{-1}(0) = \mathcal{K}. \tag{1.7}$$

It is familiar [3, 7] that $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is regular if and only if $A$ has closed range.

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $X : \mathcal{H} \to \mathcal{K}$ is a bounded linear transformation having trivial kernel and dense range, then $X$ is called *quasiaffinity*. If $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{K})$, and there exist quasiaffinities $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $Y \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ satisfying $XA = BX$, $AY = YB$, then $A$ and $B$ are said to be *quasisimilar*. Quasisimilarity is an equivalent relation weaker than similarity. Similarity preserves the spectrum and essential spectrum of an operator, but this fails to be true for quasisimilarity. Therefore it is natural to ask that for operators $A$ and $B$ such that $A$ and $B$ are quasisimilar, what condition should be imposed on $A$ and $B$ to insure the equality relation $\sigma_e(A) = \sigma_e(B)$ ($\sigma(A) = \sigma(B)$)?

It is known that quasisimilar normal operators are unitarily equivalent [2, Lemma 4.1]. Thus quasisimilar normal operators have equal spectra and essential spectra. Clary [1, Theorem 2] proved that quasisimilar hyponormal operators have equal spectra and asked whether quasisimilar hyponormal operators also have essential spectra. Later Williams (see [11, Theorem 1], [12, Theorem 3]) showed that two quasisimilar quasinormal operators and under certain conditions two quasisimilar hyponormal operators have equal essential spectra. Gupta [4, Theorem 4] showed that biquasitriangular and quasisimilar $k$-quasihyponormal operators have equal essential spectra. On the other hand, Yang [13, Theorem 2.10] proved that quasisimilar $M$-hyponormal operators have equal essential spectra, and Yingbin and Zikun [14, Corollary 12] showed that quasisimilar $p$-hyponormal operators have also equal spectra and essential spectra. Very recently, Jeon et al. [8, Theorem 5] showed that quasisimilar injective $p$-quasihyponormal operators have equal spectra and essential spectra. In this paper we give some conditions for operators $A$ and $B$ ($A$ is left-Fredholm and $B$ is right-Fredholm) to exist an operator $C$ such that $M_C$ is Fredholm, and describe the essential spectra of $M_C$. Using this result we establish that quasisimilar $(p,k)$-quasihyponormal operators have equal spectra and essential spectra.

## 2. Main results

We need auxiliary lemmas to prove the main result.

Lemma 2.1. *For a given pair $(A,B)$ of operators if $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is Fredholm, then $M_C$ is Fredholm for every $C \in \mathcal{L}(\mathcal{K},\mathcal{H})$. Hence, in particular,*

$$\sigma_e(M_C) \subseteq \sigma_e \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_e(A) \cup \sigma_e(B). \tag{2.1}$$

*Proof.* This follows at once from the observation that $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$. □

Lemma 2.2 [10, Corollary 2]. *Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are Hilbert spaces. If $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, $S \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$, and $ST \in \mathcal{L}(\mathcal{X},\mathcal{Z})$ have closed ranges, then there is isomorphism*

$$T^{-1}(0) \oplus S^{-1}(0) \oplus (ST\mathcal{X})^{\perp} \cong (ST)^{-1}(0) \oplus (T\mathcal{X})^{\perp} \oplus (S\mathcal{Y})^{\perp}. \tag{2.2}$$

The following lemma gives a necessary and sufficient condition for $M_C$ to be Fredholm. This is a Fredholm version of [10, Lemma 4].

Lemma 2.3. *Let $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$. Then $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is Fredholm for some $C \in \mathcal{L}(\mathcal{K},\mathcal{H})$ if and only if $A$ and $B$ satisfy the following conditions:*
(i) *$A$ is left-Fredholm,*
(ii) *$B$ is right-Fredholm,*
(iii) *($A$ Fredholm $\Leftrightarrow$ $B$ Fredholm).*

*Proof.* Since $M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$, we can see that if $M_C$ is Fredholm, then $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ is left-Fredholm and $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$ is right-Fredholm, so that $A$ is left-Fredholm and $B$ is right-Fredholm. On the other hand, since, evidently, $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$ $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ have closed ranges, it follows from Lemma 2.2 that

$$A^{-1}(0) \oplus B^{-1}(0) \oplus \left(\operatorname{ran}(M_C)\right)^{\perp} \cong \ker(M_C) \oplus A(\mathcal{H})^{\perp} \oplus B(\mathcal{K})^{\perp}. \tag{2.3}$$

Since by assumption $M_C$ is Fredholm, we have

$$\dim B^{-1}(0) < \infty \Longleftrightarrow \dim A(\mathcal{H})^{\perp} < \infty, \tag{2.4}$$

which together with the fact that $A$ is left-Fredholm and $B$ is right-Fredholm gives the condition (iii).

For the converse we asssume that conditions (i), (ii), and (iii) hold. First observe that if $A$ and $B$ are both Fredholm, then by Lemma 2.1, $M_C$ is Fredholm for every $C$. Thus we suppose that $A$ and $B$ are not Fredholm. But since $A$ is left-Fredholm and $B$ is right-Fredholm, it follows that

$$B^{-1}(0) \cong A(\mathcal{H})^{\perp}. \tag{2.5}$$

Note that $A$ and $B$ are both regular, and so we suppose $A = AA'A$ and $B = BB'B$. Then as in (1.7), $\mathcal{H}$ and $\mathcal{K}$ can be decomposed as

$$A(\mathcal{H}) \oplus (AA')^{-1}(0) = \mathcal{H}, \qquad B^{-1}(0) \oplus B'B(\mathcal{K}) = \mathcal{K}. \tag{2.6}$$

By (2.5) we have $(AA')^{-1}(0) \cong B^{-1}(0)$. So there exists an isomorphism $J : B^{-1}(0) \to (AA')^{-1}(0)$. Define an operator $C : \mathcal{K} \to \mathcal{H}$ by

$$C := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} : B^{-1}(0) \oplus B'B(\mathcal{K}) \longrightarrow (AA')^{-1}(0) \oplus A(\mathcal{H}). \tag{2.7}$$

Then we have that $C \in \mathcal{L}(\mathcal{K},\mathcal{H})$, $C(\mathcal{K}) = (AA')^{-1}(0)$, and $C^{-1}(0) = B'B(\mathcal{K})$. We now claim that $M_C$ is Fredholm. Indeed,

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = 0 \Longrightarrow Ah = Ck = Bk = 0 \quad (\text{because } A(\mathcal{H}) \cap C(\mathcal{K}) = \{0\}), \tag{2.8}$$

which implies $k = 0$, and hence

$$\ker \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = A^{-1}(0) \oplus 0_{\mathcal{K}}, \tag{2.9}$$

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} = \begin{pmatrix} A(\mathcal{H}) + (AA')^{-1}(0) \\ B(\mathcal{K}) \end{pmatrix} = \begin{pmatrix} \mathcal{H} \\ B(\mathcal{K}) \end{pmatrix}, \tag{2.10}$$

and hence

$$\left( \operatorname{ran} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right)^{\perp} \cong 0_{\mathcal{H}} \oplus B(\mathcal{K})^{\perp}. \tag{2.11}$$

The spaces in (2.9) and (2.11) are both finite dimensional. Thus $M_C$ is Fredholm. This completes the proof. □

Corollary 2.4. *For a given pair $(A,B)$ of operators the following holds*

$$\bigcap_{C \in \mathcal{L}(\mathcal{K},\mathcal{H})} \sigma_e(M_C) = \sigma_e^+(A) \cup \sigma_e^-(B) \cup \left[ (\sigma_e(A) \cup \sigma_e(B)) \setminus (\sigma_e(A) \cap \sigma_e(B)) \right]. \tag{2.12}$$

*Hence in particular, for every* $C \in \mathcal{L}(\mathcal{K},\mathcal{H})$,

$$(\sigma_e(A) \cup \sigma_e(B)) \setminus (\sigma_e(A) \cap \sigma_e(B)) \subset \sigma_e(M_C) \subset \sigma_e(A) \cup \sigma_e(B). \tag{2.13}$$

The proof is immediate from Lemma 2.3, Corollary 2.4, and Lemma 2.1.

From Corollary 2.4 we see that $\sigma_e(M_C)$ shrinks from $\sigma_e\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \sigma_e(A) \cup \sigma_e(B)$. How much of $\sigma_e(A) \cup \sigma_e(B)$ survives? The following says that the passage from $\sigma_e(A) \cup \sigma_e(B)$ to $\sigma_e(M_C)$ is accomplished by removing certain open subsets of $\sigma_e(A) \cap \sigma_e(B)$ from the former.

Theorem 2.5. *For operators* $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{K})$, *and* $C \in \mathcal{L}(\mathcal{K},\mathcal{H})$, *there is equality*

$$\sigma_e(A) \cup \sigma_e(B) = \sigma_e(M_C) \cup \mathfrak{S}, \tag{2.14}$$

*where* $\mathfrak{S}$ *is the union of certain of the holes in* $\sigma_e(M_C)$ *which happen to be subsets of* $\sigma_e(A) \cap \sigma_e(B)$.

*Proof.* We first claim that, for every $C \in \mathcal{L}(\mathcal{K},\mathcal{H})$,

$$\eta(\sigma_e(M_C)) = \eta(\sigma_e(A) \cup \sigma_e(B)), \tag{2.15}$$

where $\eta\mathfrak{C}$ denotes the "polynomially convex hull," which is also the "connected hull" obtained [6, 7] by "filling in the holes" of a compact subset. Since by (2.15), $\sigma_e(M_C) \subseteq \sigma_e(A) \cup \sigma_e(B)$ for every $C \in \mathcal{L}(\mathcal{K},\mathcal{H})$, we need to show that $\partial(\sigma_e(A) \cup \sigma_e(B)) \subseteq \partial\sigma_e(M_C)$, where $\partial\mathfrak{C}$ denotes the topological boundary of the compact set $\mathfrak{C} \subseteq \mathbb{C}$. But since $\operatorname{int}\sigma_e(M_C) \subseteq \operatorname{int}(\sigma_e(A) \cup \sigma_e(B))$, it suffices to show that $\partial(\sigma_e(A) \cup \sigma_e(B)) \subseteq \sigma_e(M_C)$. Indeed we have

$$\partial(\sigma_e(A) \cup \sigma_e(B)) \subseteq \partial\sigma_e(A) \cup \partial\sigma_e(B) \subseteq \sigma_e^+(A) \cup \sigma_e^-(B) \subseteq \sigma_e(M_C), \tag{2.16}$$

where the last inclusion follows from (2.13) and the second inclusion follows from the punctured neighborhood theorem (cf. [7]): for every operator $T$,

$$\partial\sigma_e(T) \subseteq \sigma_e^+(T) \cap \sigma_e^-(T). \tag{2.17}$$

This proves (2.15). Consequently, (2.15) says that the passage from $\sigma_e(M_C)$ to $\sigma_e(A) \cup \sigma_e(B)$ is the filling in certain of the holes in $\sigma_e(M_C)$. But since, by (2.12), $(\sigma_e(A) \cup \sigma_e(B)) \setminus \sigma_e(M_C)$ is contained in $\sigma_e(A) \cap \sigma_e(B)$, it follows that any holes in $\sigma_e(M_C)$ which are filled in should occur in $\sigma_e(A) \cap \sigma_e(B)$. This completes the proof. □

Corollary 2.6. *If* $\sigma_e(A) \cap \sigma_e(B)$ *has no interior points, then, for every* $C \in \mathcal{L}(\mathcal{K},\mathcal{H})$,

$$\sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B). \tag{2.18}$$

*Proof.* This follows at once from Theorem 2.5. □

The following lemma is used for proof of the main theorem.

Lemma 2.7 [9, Lemma 1]. *If $A$ is $(p,k)$-quasihyponormal operator and the range of $A^k$ is not dense, then $A$ has the following matrix representation:*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \quad on\ \overline{\operatorname{ran}(A^k)} \oplus \ker(A^{*k}), \tag{2.19}$$

*where $A_1$ is $p$-hyponormal on $\overline{\operatorname{ran}(A^k)}$ and ${A_3}^k = 0$. Furthermore, $\sigma(A) = \sigma(A_1) \cup \{0\}$.*

We are ready for proving the main theorem.

Theorem 2.8. *If $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are quasisimilar $(p,k)$-quasihyponormal operators, then $\sigma(A) = \sigma(B)$ and $\sigma_e(A) = \sigma_e(B)$.*

*Proof.* Suppose that $X \in \mathcal{L}(\mathcal{H},\mathcal{K})$ and $Y \in \mathcal{L}(\mathcal{K},\mathcal{H})$ are injective operators with dense range such that $XA = BX$ and $AY = YB$. If the range of $A^k$ is dense, then $B^kX = XA^k$ implies that the range of $B^k$ is also dense. Therefore $A$ and $B$ are quasisimilar $p$-hyponormal operators, and hence the result follows from [14, Corollary 12]. If instead the range of $A^k$ is not dense, then $A^kY = YB^k$ implies that the range of $B^k$ is not dense. Therefore by Lemma 2.7, $A$ and $B$ have the following matrix representations:

$$\begin{aligned} A &= \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \quad \text{on } \overline{\operatorname{ran}(A^k)} \oplus \ker(A^{*k}), \\ B &= \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} \quad \text{on } \overline{\operatorname{ran}(B^k)} \oplus \ker(B^{*k}), \end{aligned} \tag{2.20}$$

where $A_1$ and $B_1$ are $p$-hyponormal and ${A_3}^k = {B_3}^k = 0$. Since quasisimilar $p$-hyponormal operators have equal spectra and essential spectra, in view of Corollary 2.6 and Lemma 2.7, it suffices to show that

(i) $A_1$ and $B_1$ are quasisimilar;

(ii) domain $(A_3) = \{0\} \Leftrightarrow$ domain $(B_3) = \{0\}$.

Towards the statement (i), observe that

$$XA^k = BXA^{k-1} = \cdots = B^kX, \qquad YB^k = AYB^{k-1} = \cdots = A^kY. \tag{2.21}$$

If we denote the $X_1 : \overline{\operatorname{ran}(A^k)} \to \overline{\operatorname{ran}(B^k)}$ and $Y_1 : \overline{\operatorname{ran}(B^k)} \to \overline{\operatorname{ran}(A^k)}$, then $X_1$ and $Y_1$ are injective and have dense range. Now for any $x \in \overline{\operatorname{ran}(A^k)}$, $X_1A_1x = XAx = BXx = B_1X_1x$ and for any $y \in \overline{\operatorname{ran}(B^k)}$, $Y_1B_1y = YBy = AYy = A_1Y_1y$. Hence $A_1$ and $B_1$ are quasisimilar.

For the statement (ii), assume that $A^{*k}x = 0$ for nonzero $x$ in $\mathcal{H}$. Then by (2.21) we have that $B^{*k}Y^*x = 0$. Since $Y^*$ is one to one, we have that domain$(B_3) = \{0\}$ implies domain$(A_3) = \{0\}$, and similarly, domain$(A_3) = \{0\}$ implies domain$(B_3) = \{0\}$, which completes the proof. □

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