ON THE CONSTANT IN MEŃSHOV-RADEMACHER INEQUALITY

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The goal of the paper is twofold: (1) to show that the exact value D_2 in the Meńshov-Rademacher inequality equals 4/3, and (2) to give a new proof of the Meńshov-Rademacher inequality by use of a recurrence relation. The latter gives the asymptotic estimate $\limsup_n D_n/\log_2^2 n \leq 1/4$.

1. Introduction

The Meńshov-Rademacher inequality deals with the estimation of

$$D_n = \sup \mathbf{E} \max_{1\leq k\leq n} \left(\sum_{l=1}^{k} \alpha_l \varphi_l \right)^2, \tag{1.1}$$

where sup is taken over all probability spaces $(\Omega, \mathcal{F}, P)$, all real orthonormal systems $(\varphi_1,\dots,\varphi_n)$ on them, and all real coefficient collections $(\alpha_1,\dots,\alpha_n)$ with $\sum_1^n \alpha_i^2 = 1$.

Rademacher [9] and Meńshov [7] independently proved that there exists an absolute constant $C > 0$ such that for each $n \geq 2$,

$$D_n \leq C \log_2^2 n. \tag{1.2}$$

A traditional proof using a bisection method (see, e.g., Doob [2] and Loève [6]) leads to the inequality

$$D_n \leq \left(\log_2 n + 2\right)^2, \quad n \geq 2. \tag{1.3}$$

Kounias [4] used a trisection method to get a finer inequality:

$$D_n \leq \left(\frac{\log_2 n}{\log_2 3} + 2 \right)^2, \quad n \geq 2. \tag{1.4}$$

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The aim of this paper is twofold: to show that the exact starting value $D_2 = 4/3$ and to establish a recurrence relation which leads to a refinement of (1.4) and an asymptotic constant $\leq 1/4$. Note that there are several other proofs of the Meńshov-Rademacher inequality and its generalizations, see, for example, Somogyi [10] and Móricz and Tandori [8].

Section 2 deals with the proof of $D_2 = 4/3$, while Section 3 is devoted to the proof of the Meńshov-Rademacher inequality with the asymptotic constant $\leq 1/4$. Section 4 contains alternative proofs to those results using the concept of main triangle projection, a subject which was studied in depth in Gohberg and Kreĭn [3] and Kwapień and "Pełczyński" [5].

2. The value of D_2

Theorem 2.1. $D_2 = 4/3$.

The proof of the theorem is based on the following lemma which may be of independent interest.

Lemma 2.2. *Let* $c > 0$, $p_c \equiv c^2/(1+c^2)$, *and define*

$$f(p,c) = \sup_{X \in \mathcal{A}(p,c)} \mathbf{E}(X\mathbf{1}_{X>-c}), \quad p_c \leq p < 1, \tag{2.1}$$

where

$$\mathcal{A}(p,c) = \{X \in L_0(\Omega,\mathcal{F},P) : \mathbf{E}(X) = 0,\ \mathbf{E}(X^2) = 1,\ P(X > -c) = p\}. \tag{2.2}$$

Then

$$f(p,c) = \sqrt{p(1-p)}. \tag{2.3}$$

Proof of Lemma 2.2. To show that the left-hand side is greater than or equal to right-hand side, we observe that $\mathbf{E}(X_p\mathbf{1}_{X_p>-c}) = \sqrt{p(1-p)}$, where the distribution of $X_p \in \mathcal{A}(p,c)$ is given by

$$p = P\left(X_p = \sqrt{\frac{(1-p)}{p}}\right) = 1 - P\left(X_p = -\sqrt{\frac{p}{(1-p)}}\right). \tag{2.4}$$

To see that the left-hand side is less than or equal to right-hand side, we define

$$h_p(x) = x \cdot 1_{x>-c} - p \cdot x - \sqrt{\frac{p(1-p)}{4}} \cdot x^2. \tag{2.5}$$

The maximum of $h_p(x)$ is achieved at $x = \sqrt{(1-p)/p}$ and at $-\sqrt{p/(1-p)}$ for the regions $x > -c$ and $x \leq -c$, respectively. We conclude that for any $X \in \mathcal{A}(p,c)$,

$$0 \leq \mathbf{E}(h_p(X_p)) - \mathbf{E}(h_p(X)) = \mathbf{E}(X_p \cdot \mathbf{1}_{X_p>-c}) - \mathbf{E}(X \cdot \mathbf{1}_{X>-c}). \tag{2.6}$$

This completes the proof of the lemma. □

Let us note also that $\mathcal{A}(p,c)$ is empty for $p < p_c$. Indeed, by the Chebyshev inequality, $\mathbf{E}(X) = 0$ and $\mathbf{E}(X^2) = 1$ imply $P(X \le -c) \le 1/(1+c^2) = 1 - p_c$.

Proof of Theorem 2.1. The result follows by standard calculations from the representation

$$D_2 = \sup_{a^2+b^2=1,\, b^2/(1+3a^2)<p<1} \left\{ a^2 + b^2 p + 2ab \cdot \sqrt{p(1-p)} \right\}. \tag{2.7}$$

To prove (2.7) convert an orthonormal pair (φ_1, φ_2) defined on $(\Omega, \mathcal{F}, P)$ into $(X \equiv \varphi_1/\varphi_2, 1)$. The new pair is orthonormal with respect to the measure $dP' = \varphi_2^2 dP$. Also

$$\begin{aligned} \mathbf{E}_P \max\left\{ (a\varphi_1)^2, (a\varphi_1 + b\varphi_2)^2 \right\} &= \mathbf{E}_{P'} \max\left\{ (aX)^2, (aX+b)^2 \right\} \\ &= a^2 + b^2 P'(X > -b/2a) + 2\mathrm{ab} \cdot \mathbf{E}_{P'}\left(X \cdot \mathbf{1}_{X > -b/2a} \right) \\ &\le a^2 + b^2 p + 2ab \cdot f\left(p, \frac{b}{2a} \right), \end{aligned} \tag{2.8}$$

where $p = P'(X > -b/2a)$. Now (2.7) follows from Lemma 2.2 with $c = b/2a$. □

3. An induction proof of the Meńshov-Rademacher inequality

Theorem 3.1. (i)

$$D_m \le \frac{1}{4}(3 + \log_2 m)^2, \quad m \ge 2. \tag{3.1}$$

In particular, (ii)

$$\limsup_m \frac{D_m}{\log_2^2 m} \le \frac{1}{4}. \tag{3.2}$$

Lemma 3.2. *The following recurrence relation holds true for any $n \in \mathbb{N}$:*

$$D_{2n} \le D_n + D_n^{1/2}. \tag{3.3}$$

Proof of Lemma 3.2. We have for any $n \in \mathbb{N}$,

$$\begin{aligned} \max_{k \le 2n} \left| \sum_1^k \alpha_i \varphi_i \right|^2 &\le \max\left(\max_{k\le n} \left| \sum_1^k \alpha_i \varphi_i \right|^2, \left(\left| \sum_1^n \alpha_i \varphi_i \right| + \max_{n<k\le 2n} \left| \sum_{n+1}^k \alpha_i \varphi_i \right| \right)^2 \right) \\ &\le \max_{k\le n} \left| \sum_1^k \alpha_i \varphi_i \right|^2 + 2 \left| \sum_1^n \alpha_i \varphi_i \right| \max_{n<k\le 2n} \left| \sum_{n+1}^k \alpha_i \varphi_i \right| + \max_{n<k\le 2n} \left| \sum_{n+1}^k \alpha_i \varphi_i \right|^2. \end{aligned} \tag{3.4}$$

Taking expectations in (3.4) and using the Cauchy-Schwartz inequality, we come to the desired recurrence relation:

$$D_{2n} \le pD_n + 2\sqrt{p(1-p)D_n} + (1-p)D_n = D_n + \sqrt{D_n}, \tag{3.5}$$

where $p = \sum_1^n \alpha_i^2$.

The lemma is proved. □

Proof of Theorem 3.1. Lemma 3.2 implies that for any $n \in \mathbb{N}$,

$$D_{2n}^{1/2} \le D_n^{1/2} + \frac{1}{2}. \tag{3.6}$$

Since $D_1 = 1$, this implies that for each $n \in \mathbb{N}$,

$$D_{2^n}^{1/2} \le 1 + \frac{n}{2}. \tag{3.7}$$

Let us take now $2^n \le m < 2^{n+1}$. Then

$$D_m \le D_{2^{n+1}} \le \left(1 + \frac{n+1}{2}\right)^2 \le \left(1 + \frac{\log_2 m + 1}{2}\right)^2. \tag{3.8}$$

This implies the validity of Theorem 3.1. □

Remark 3.3. (1) The proof of Theorem 3.1 is a refinement of that appeared in Chobanyan [1].

(2) Kounias's result mentioned in the introduction leads to $\limsup(D_n/\log_2^2 n) \le (\log 2/\log 3)^2$ which is larger than 1/4 of Theorem 3.1.

4. An alternative approach: the main triangle projection

Consider the space $\mathbf{L}(\mathbb{R}^n)$ of all linear operators (matrices) acting in $\mathbb{R}^n$. The correspondence between the operators and matrices is given by $a_{ij} = (Ae_j, e_i)$, $i, j = 1, \dots, n$. The *main triangle projection* $T_n : \mathbf{L}(\mathbb{R}^n) \to \mathbf{L}(\mathbb{R}^n)$ is a linear operator introduced as follows. For an $A \in \mathbf{L}(\mathbb{R}^n)$, the matrix of the operator $B = T_n A$ has the form $b_{ij} = a_{ij}$ if $i + j \le n + 1$ and $b_{ij} = 0$ otherwise.

We assume that $\mathbb{R}^n$ is endowed with the Euclidean norm, and the norm in $\mathbf{L}(\mathbb{R}^n)$ is the usual operator norm.

Theorem 4.1. $D_n = \|T_n\|^2$, $n \in \mathbb{N}$.

Proof. Let us prove first that $\|T_n\|^2 \equiv \sup_{\|A\| \le 1} \|T_n A\|^2 \le D_n$. Since the orthogonal operators (and only them) are the extreme points of the unit ball of $\mathbf{L}(\mathbb{R}^n)$, it suffices to show that for any orthogonal operator $u \in \mathbf{L}(\mathbb{R}^n)$, $\|T_n u\|^2 \le D_n$. Let us relate with u the orthonormal system $\varphi_1, \dots, \varphi_n$ defined on (Ω, P), where $\Omega = \{1, \dots, n\}$, $P(j) = 1/n$, $j = 1, \dots, n$, as follows:

$$\varphi_k(j) = \sqrt{n}(ue_k, e_j), \quad k, j = 1, \dots, n. \tag{4.1}$$

We have for any vector $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$ with $|\alpha| = 1$,

$$\begin{aligned} D_n &\geq \mathbf{E}\max_{k\leq n}\left|\sum_{i=1}^{k}\alpha_i\varphi_i\right|^2 = \sum_{j=1}^{n}\max_{k\leq n}\left|\sum_{i=1}^{k}\alpha_i(ue_i,e_j)\right|^2 \\ &\geq \sum_{j=1}^{n}\left|\sum_{i=1}^{n-j+1}\alpha_i(ue_i,e_j)\right|^2 = \left|\left|(T_n u)\alpha\right|\right|^2. \end{aligned} \tag{4.2}$$

Taking supremum over all orthogonal u's and α's from the unit ball of $\mathbb{R}^n$, we get $D_n \geq \|T_n\|^2$. To prove the inverse inequality, consider an orthonormal system $(\varphi_1,\dots,\varphi_n) \subset L_2(\Omega,\mathcal{F},P)$ and any vector $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$ with $|\alpha| = 1$.

$$I(\alpha,\varphi) \equiv \mathbf{E}\max_{k\leq n}\left|\sum_{i=1}^{k}\alpha_i\varphi_i\right|^2 = \sum_{k=1}^{n}\mathbf{E}\mathbf{1}_{S_k}\left|\sum_{i=1}^{k}\alpha_i\varphi_i\right|^2, \tag{4.3}$$

where $S_k = \{\omega \in \Omega :$ the minimum of l's at which $|\sum_{i=1}^{l}\alpha_i\varphi_i(\omega)|$ attains its maximum equals $k\}$. Then we have

$$I(\alpha,\varphi) = \sup_g \sum_{k=1}^{n}\left[\mathbf{E}g_k\mathbf{1}_{S_k}\left|\sum_{i=1}^{k}\alpha_i\varphi_i\right|\right]^2, \tag{4.4}$$

where supremum is taken over all collections $g = (g_1,\dots,g_n)$ such that g_k's vanish outside of S_k and $\|g_k\|_2 = 1$, $k = 1,\dots,n$. We have further

$$\begin{aligned} I(\alpha,\varphi) &= \sup_g \sum_{k=1}^{n}\sum_{i,j=1}^{k}\alpha_i\alpha_j\mathbf{E}g_k\varphi_i\varphi_j \\ &= \sup_g \sum_{i,j=1}^{n}\sum_{k=\max(i,j)}^{n}\alpha_i\alpha_j\mathbf{E}g_k\varphi_i\varphi_j = \sup_g \left|\left|T_nA\alpha\right|\right|^2, \end{aligned} \tag{4.5}$$

where $(Ae_j,e_i) = \mathbf{E}g_{n-j+1}\cdot\varphi_i$, $i,j = 1,\dots,n$. We have

$$\|A\| = \sup_{|\alpha|=1}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\mathbf{E}\alpha_j g_{n-j+1}\varphi_i\right)^2 = \sup_{|\alpha|=1}\sum_{i=1}^{n}\left(\mathbf{E}f\varphi_i\right)^2 = \sup_{|\alpha|=1}\mathbf{E}f^2 = 1, \tag{4.6}$$

where $f = \alpha_j g_j$, if $\omega \in S_j$, $j = 1,\dots,n$. Therefore, (4.5) implies $D_n \leq \|T_n\|^2$. The theorem is proved. □

The following corollary is our Theorem 2.1.

Corollary 4.2. $D_2 = 4/3$.

Proof. We have according to Theorem 4.1,

$$D_2 = \left|\left|T_2\right|\right|^2 = \sup_u \left|\left|T_2u\right|\right|^2 = \sup\left\{\left|\left|\begin{pmatrix} a & b \\ b & 0 \end{pmatrix}\right|\right|^2 : a^2+b^2 = 1\right\} = \frac{4}{3}. \tag{4.7}$$

□

Remark 4.3. It follows from the proof of Theorem 4.1 that $D_n = \sup \mathbf{E}[\max_j(\sum_{l=1}^{j} a_l\varphi_l)^2]$, where the supremum is over all real orthonormal systems $\varphi_1,\dots,\varphi_n$, where each φ_j, $j = 1,\dots,n$ takes at most n values, and all reals $\alpha_1,\dots,\alpha_n$ with $|\alpha| = 1$.

The following lemma establishes a finer recurrence relation than Lemma 3.2. However, the two lemmas are asymptotically equivalent.

Lemma 4.4.

$$D_{2n} \le \frac{4}{3}D_n \quad \text{if } D_n \le 3, \qquad D_{2n} \le D_n - \frac{1}{2} + \sqrt{D_n - \frac{3}{4}} \quad \text{if } D_n \ge 3. \tag{4.8}$$

Proof. We have for any $n \in \mathbb{N}$:

$$||T_{2n}|| = \sup\left\{\left\|\begin{pmatrix} A & T_nB \\ T_nC & 0 \end{pmatrix}\right\|\right\}, \tag{4.9}$$

where the supremum runs over all matrices A, B, C, and D in $\mathbf{L}(\mathbb{R}^n)$ such that $\|(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})\| \le 1$. For such matrices A, B, C, and D we check that $|uA|^2 + |uT_nB|^2 \le \|T_n\|^2|u|^2$ and $|Ax|^2 + |T_nCx|^2 \le \|T_n\|^2|x|^2$ for all $u,x \in \mathbb{R}^n$. Therefore, $\|T_{2n}\| \le \sup\{(u,Ax) + (u,Fy) + (v,Gy) : u,v,x,y \in \mathbb{R}^n,\ |u|^2 + |v|^2 \le 1,\ |x|^2 + |y|^2 \le 1,\ A,F,G \in \mathbf{L}(\mathbb{R}^n),\ \|A\| \le 1,\ |wA|^2 + |wF|^2 \le D_n|w|^2,\ |Az|^2 + |Gz|^2 \le D_n|z|^2 \text{ for all } w,z \in \mathbb{R}^n\}$. The last supremum can easily be computed and its square equals $\sup_{a\in[0,1]}(D_n - a/2 + \sqrt{D_na - 3a^2/4})$. Hence, $D_{2n} \le 4/3D_n$ if $D_n \le 3$ and $D_{2n} \le D_n - 1/2 + \sqrt{D_n - 3/4}$ if $D_n \ge 3$. This completes the proof of Lemma 4.4. □

Finally, it is known that for the Hilbert matrix ($H_n(i,j) = 1/(i-j)$, if $i \ne j$ and $H_n(i,i) = 0$, $i,j = 1,\dots,n$, $n \ge 2$),

$$\frac{||T_nH_n||}{||H_n||} \ge \frac{\ln n}{\pi}. \tag{4.10}$$

This along with Theorem 3.1 implies the following bilateral estimate:

$$\frac{1}{\pi^2\log_2^2 e} \le \liminf \frac{D_n}{\log_2^2 n} \le \limsup \frac{D_n}{\log_2^2 n} \le \frac{1}{4}. \tag{4.11}$$

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