SCHUR-CONVEXITY OF THE COMPLETE ELEMENTARY SYMMETRIC FUNCTION

KAIZHONG GUAN

Received 2 October 2004; Revised 15 January 2005; Accepted 27 January 2005

We prove that the complete elementary symmetric function $c_r = c_r(x) = C_n^{[r]}(x) = \sum_{i_1+\cdots+i_n=r} x_1^{i_1}\cdots x_n^{i_n}$ and the function $\phi_r(x) = c_r(x)/c_{r-1}(x)$ are Schur-convex functions in $R_+^n = \{(x_1,x_2,\ldots,x_n) \mid x_i > 0\}$, where $i_1,i_2,\ldots,i_n$ are nonnegative integers, $r \in N = \{1, 2,\ldots\}$, $i = 1,2,\ldots,n$. For which, some inequalities are established by use of the theory of majorization. A problem given by K. V. Menon (Duke Mathematical Journal **35** (1968), 37–45) is also solved.

1. Introduction

Consider the complete elementary symmetric function

$$c_r = c_r(x) = C_n^{[r]}(x) = \sum_{i_1+\cdots+i_n=r} x_1^{i_1}\cdots x_n^{i_n}, \tag{1.1}$$

where $i_1,i_2,\ldots,i_n$ are nonnegative integers, $r \in N$. Define $c_0(x) = 1$. Correspondingly, the generalized r-order symmetric mean is

$$D_r(x) = D_n^{[r]}(x) = \binom{r+n-1}{n-1}^{-1} C_n^{[r]}(x), \tag{1.2}$$

where $\binom{r+n-1}{n-1} = (n+r-1)!/(n-1)!r!$.

For (1.1) and (1.2), Menon [7] mainly obtained the following results

$$\left(C_n^{[r]}(a+b)\right)^{1/r} \le \left(C_n^{[r]}(a)\right)^{1/r} + \left(C_n^{[r]}(b)\right)^{1/r}; \tag{1.3}$$

$$c_r(a)c_{s-1}(a) \ge c_{r-1}(a)c_s(a), \quad 0 < r < s; \tag{1.4}$$

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 67624, Pages 1–9
DOI 10.1155/JIA/2006/67624

$$\left(c_r(a)\right)^{1/r} \ge \left(c_s(a)\right)^{1/s}, \quad 0 < r < s; \tag{1.5}$$

$$D_{r-2}(a)D_{r+2}(a) - D_{r-1}(a)D_{r+1}(a) \ge 0, \quad n = 2. \tag{1.6}$$

When $n > 2$, is inequality (1.6) true? This problem was given out by Menon in [7].

Detemple and Robertson [2] derived

$$D_{r-1}(a)D_{r+1}(a) - D_r^2(a) \ge 0, \quad r = 1,2,3. \tag{1.7}$$

Whether inequality (1.7) is still valid for $r \ge 4$ was given in [5], and this problem was solved in [3].

The Schur-convex functions were introduced by I. Schur in 1923 [6], and has many important applications in analytic inequalities. Hardy et al. were also interested in some inequalities that are related to Schur-convex functions [4], the following definitions can be found in many references such as [5, 6, 8, 9].

Definition 1.1. Suppose that $x_i, y_i \in R$, $i = 1,2,\ldots,n$, $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$. Rearrange the components of x and y such that $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}, y_{[1]} \ge y_{[2]} \ge \cdots \ge y_{[n]}$. If $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$ $(1 \le k \le n-1)$, and $\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$, then x is said to be majorized by y, denote it by $x \prec y$.

Definition 1.2. $A \subseteq R^n$ is called symmetric set, if $x \in A$ implies $Px \in A$ for $n \times n$ permutation matrix P.

Definition 1.3. $f : A \to R(A \subset R^n)$ is called Schur-convex if $x \prec y$, then

$$f(x) \le f(y). \tag{1.8}$$

It is called strictly Schur-convex if the inequality is strict; $f(x)$ is called Schur-concave (resp., strictly Schur-concave) if the inequality (1.8) is reversed.

Definition 1.4. $f : A \to R$ is called symmetric if for every permutation matrix P,

$$f(Px) = f(x) \tag{1.9}$$

for all $x \in A$.

Let the mark "$x \le y$" stand for $x_i \le y_i$, $i = 1,2,\ldots,n$.

Definition 1.5. $f : A(\subseteq R^n) \to R$ is called monotonic increasing function if $x \le y$, then $f(x) \le f(y)$.

In this paper, we prove the functions $c_r(x)$ and $c_r(x)/c_{r-1}(x)$ to be Schur-convex functions in $R_+^n = \{(x_1,x_2,\ldots,x_n) \mid x_i > 0, i = 1,2,\ldots,n\}$. Some inequalities for them are established by using of the theory of majorization. "Ky Fan" inequality is generalized. We show that inequality (1.6) is true for $n > 2$, and thus the problem in [7] is solved.

2. Lemma

In this section, We give the following lemmas for the proofs of our main results. Every Schur-convex function is a symmetric function [11]. It is not hard to see that not every symmetric function can be a Schur-convex function [9, page 258]. However, we have the following so-called Schur's condition.

Lemma 2.1 [9, page 259]. *Let $f(x) = f(x_1, x_2, \ldots, x_n)$ be symmetric and have continuous partial derivative on $I^n = I \times I \times \cdots \times I$ (n copies), where I is an open interval. Then $f : I^n \to R$ is Schur-convex if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0 \tag{2.1}$$

on I^n. It is strictly Schur-convex if (2.1) is a strict inequality for $x_i \neq x_j$, $1 \leq i, j \leq n$.

In Schur's condition, the domain of $f(x)$ does not have to be a Cartesian product I^n. Lemma 2.1 remains true if we replace I^n by a set $A \subseteq R^n$ with the following properties ([6, page 57]):

(i) *A is convex and has a nonempty interior,*
(ii) *A is symmetric.*

Lemma 2.2 [10]. *Suppose that $x_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^n x_i = s$, $c \geq s$, then*

$$\frac{c - x}{nc/s - 1} = \left(\frac{c - x_1}{nc/s - 1}, \ldots, \frac{c - x_n}{nc/s - 1}\right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{2.2}$$

Lemma 2.3 [10]. *Suppose that $x_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^n x_i = s$, $c \geq s$, then*

$$\frac{c + x}{s + nc} = \left(\frac{c + x_1}{s + nc}, \frac{c + x_2}{s + nc}, \ldots, \frac{c + x_n}{s + nc}\right) \prec \left(\frac{x_1}{s}, \frac{x_2}{s}, \ldots, \frac{x_n}{s}\right) = \frac{x}{s}. \tag{2.3}$$

Lemma 2.4 [6]. *Suppose that $x_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^n x_i = s$, then*

$$\frac{s}{n} = \left(\frac{s}{n}, \frac{s}{n}, \ldots, \frac{s}{n}\right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{2.4}$$

Lemma 2.5. *Suppose that $x_i > 0$, $i = 1, 2, \ldots, n$. Let*

$$\overline{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n). \tag{2.5}$$

Then we have

$$c_r(x) = x_i c_{r-1}(x) + c_r(\overline{x}_i). \tag{2.6}$$

Proof. It is easy to see that

$$\begin{gathered} c_r(x) = \sum_{i_1 + i_2 + \cdots + i_n = r} x_i^{i_1} \cdots x_n^{i_n} = x_i^r + x_i^{r-1} c_1(\overline{x}_i) + \cdots + c_r(\overline{x}_i), \\ c_{r-1}(x) = x_i^{r-1} + x_i^{r-2} c_1(\overline{x}_i) + \cdots + c_{r-1}(\overline{x}_i). \end{gathered} \tag{2.7}$$

Hence

$$c_r(x) = x_i c_{r-1}(x) + c_r(\overline{x}_i). \tag{2.8}$$

□

Lemma 2.6 [3]. *Suppose that $a = (a_1, a_2, \dots, a_n)$, $a_i \geq 0$, $i = 1, 2, \dots, n$, and that $r \geq 1$ is an integer, then*

$$D_r^2(a) \leq D_{r-1}(a) D_{r+1}(a). \tag{2.9}$$

3. Main results

In this section we give our main results. Some Schur-convex functions of the complete elementary symmetric function are given here. Some analytic inequalities are established.

Theorem 3.1. *The complete elementary symmetric function*

$$c_r = c_r(x) = C_n^{[r]}(x) = \sum_{i_1 + \cdots + i_n = r} x_1^{i_1} \cdots x_n^{i_n} \tag{3.1}$$

is a Schur-convex function in R_+^n, and is increasing in x_i, $i = 1, 2, \dots, n$.

Proof. In the first, we prove that $c_r(x)$ is an increasing function with respect to x_i. In fact, by Lemma 2.5, we have

$$\frac{\partial c_r(x)}{\partial x_i} = c_{r-1}(x) + x_i \frac{\partial c_{r-1}(x)}{\partial x_i}. \tag{3.2}$$

We can inductively conclude that

$$\frac{\partial c_r(x)}{\partial x_i} \geq 0, \quad i = 1, 2, \dots, n. \tag{3.3}$$

Hence, $c_r(x)$ is an increasing function in x_i.

Next, we prove that $c_r(x)$ is a Schur-convex function in R_+^n. It is clear that $c_r(x)$ is symmetric and have continuous partial derivatives in R_+^n. By Lemma 2.1, we only need prove that

$$(x_i - x_j)\left(\frac{\partial c_r(x)}{\partial x_i} - \frac{\partial c_r(x)}{\partial x_j}\right) \geq 0, \quad i \neq j. \tag{3.4}$$

This can be obtained by induction.

(i) When $r = 2$, differentiating $c_r(x)$ with respect to x_i, we obtain

$$\frac{\partial c_r(x)}{\partial x_i} = c_{r-1}(x) + x_i \frac{\partial c_{r-1}(x)}{\partial x_i} = \sum_{k=1}^{n} x_k + x_i. \tag{3.5}$$

And so

$$(x_i - x_j)\left(\frac{\partial c_r(x)}{\partial x_i} - \frac{\partial c_r(x)}{\partial x_j}\right) = (x_i - x_j)^2 \geq 0. \tag{3.6}$$

(ii) Assume that (3.4) is true for $r - 1$. Then, still by Lemma 2.5, it follows that

$$\frac{\partial c_r(x)}{\partial x_i} = c_{r-1}(x) + x_i \frac{\partial c_{r-1}(x)}{\partial x_i}, \qquad \frac{\partial c_r(x)}{\partial x_j} = c_{r-1}(x) + x_j \frac{\partial c_{r-1}(x)}{\partial x_j}. \tag{3.7}$$

Noticing

$$\begin{aligned}\frac{\partial c_r(x)}{\partial x_i}-\frac{\partial c_r(x)}{\partial x_j}&=x_i\frac{\partial c_{r-1}(x)}{\partial x_i}-x_j\frac{\partial c_{r-1}(x)}{\partial x_j}\\&=x_i\frac{\partial c_{r-1}(x)}{\partial x_i}-x_j\frac{\partial c_{r-1}(x)}{\partial x_i}+x_j\frac{\partial c_{r-1}(x)}{\partial x_i}-x_j\frac{\partial c_{r-1}(x)}{\partial x_j}\\&=(x_i-x_j)\frac{\partial c_{r-1}(x)}{\partial x_i}+x_j\left(\frac{\partial c_{r-1}(x)}{\partial x_i}-\frac{\partial c_{r-1}(x)}{\partial x_j}\right),\end{aligned}\tag{3.8}$$

we get

$$\begin{aligned}&(x_i-x_j)\left(\frac{\partial c_r(x)}{\partial x_i}-\frac{\partial c_r(x)}{\partial x_j}\right)\\&\quad=(x_i-x_j)^2\frac{\partial c_{r-1}(x)}{\partial x_i}+x_j(x_i-x_j)\left(\frac{\partial c_{r-1}(x)}{\partial x_i}-\frac{\partial c_{r-1}(x)}{\partial x_j}\right)\geq 0.\end{aligned}\tag{3.9}$$

From (i) and (ii), by mathematical induction method, inequality (3.4) is true. Thus, the proof is complete. □

Theorem 3.2. *The function $\phi_r(x)=c_r(x)/c_{r-1}(x)$ is a Schur-convex function in R^n_+, and is increasing in x_i, $i=1,2,\dots,n$, where $r\geq 1$ is a positive integer.*

Proof. It is clear that $\phi_r(x)$ is symmetric and have continuous partial derivatives in R^n_+. Differentiating $\phi_r(x)$ with respect to x_i, we have

$$\frac{\partial\phi_r(x)}{\partial x_i}=\frac{1}{(c_{r-1}(x))^2}\left[c_{r-1}(x)\frac{\partial c_r(x)}{\partial x_i}-c_r(x)\frac{\partial c_{r-1}(x)}{\partial x_i}\right].\tag{3.10}$$

By Lemma 2.5 and computing, we derive

$$\frac{\partial\phi_r(x)}{\partial x_i}-\frac{\partial\phi_r(x)}{\partial x_j}=\frac{1}{(c_{r-1}(x))^2}\left[c_r(\overline{x}_j)\frac{\partial c_{r-1}(x)}{\partial x_j}-c_r(\overline{x}_i)\frac{\partial c_{r-1}(x)}{\partial x_i}\right].\tag{3.11}$$

Notice

$$\begin{aligned}\frac{\partial c_r(x)}{\partial x_i}&=c_{r-1}(x)+x_i\frac{\partial c_{r-1}(x)}{\partial x_i}=c_{r-1}(x)+x_i\left[c_{r-2}(x)+x_i\frac{\partial c_{r-2}(x)}{\partial x_i}\right]\\&=c_{r-1}(x)+x_ic_{r-2}(x)+x_i^2\frac{\partial c_{r-2}(x)}{\partial x_i}=\cdots\\&=c_{r-1}(x)+x_ic_{r-2}(x)+x_i^2c_{r-3}(x)+\cdots+x_i^{r-2}c_1(x)+x_i^{r-1}.\end{aligned}\tag{3.12}$$

By Lemma 2.5 and using (3.12), we have

$$\begin{aligned}\frac{\partial\phi_r(x)}{\partial x_i} &= \left(c_{r-1}(x)c_{r-1}(x) - c_r(x)c_{r-2}(x)\right) + x_i\left(c_{r-1}(x)c_{r-2}(x) - c_r(x)c_{r-3}(x)\right)\\ &\quad + \cdots + x_i^{r-2}\left(c_{r-1}(x)c_1(x) - c_r(x)c_0(x)\right) + c_{r-1}(x)x_i^{r-1},\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\frac{\partial\phi_r(x)}{\partial x_i} - \frac{\partial\phi_r(x)}{\partial x_j}\\ &\quad = \frac{1}{\left(c_{r-1}(x)\right)^2}\Big\{\left[c_r(x) - x_j c_{r-1}(x)\right]\\ &\qquad\qquad \times\left[c_{r-2}(x) + x_j c_{r-3}(x) + x_j^2 c_{r-4}(x) + \cdots + x_j^{r-3}c_1(x) + x_i^{r-2}\right]\\ &\qquad\qquad - \left[c_r(x) - x_i c_{r-1}(x)\right]\left[c_{r-2}(x) + x_i c_{r-3}(x) + x_i^2 c_{r-4}(x)\right.\\ &\qquad\qquad \left. + \cdots + x_i^{r-3}c_1(x) + x_i^{r-2}\right]\Big\}\\ &\quad = \frac{1}{\left(c_{r-1}(x)\right)^2}\Big\{\left[c_{r-1}(x)c_{r-2}(x) - c_r(x)c_{r-3}(x)\right]\left(x_i - x_j\right)\\ &\qquad\qquad + \left[c_{r-1}(x)c_{r-3}(x) - c_r(x)c_{r-4}(x)\right]\left(x_i^2 - x_j^2\right) + \cdots\\ &\qquad\qquad + \left[c_{r-1}(x)c_1(x) - c_r(x)c_0(x)\right]\left(x_i^{r-2} - x_j^{r-2}\right)\\ &\qquad\qquad + c_{r-1}(x)\left(x_i^{r-1} - x_j^{r-1}\right)\Big\}.\end{aligned} \tag{3.14}$$

From (1.4), we obtain

$$\frac{c_{r-1}(x)}{c_r(x)} > \frac{c_{r-3}(x)}{c_{r-2}(x)}, \frac{c_{r-1}(x)}{c_r(x)} > \frac{c_{r-4}(x)}{c_{r-3}(x)}, \ldots, \frac{c_{r-1}(x)}{c_r(x)} > \frac{c_0(x)}{c_1(x)}. \tag{3.15}$$

Therefore

$$\frac{\partial\phi_r(x)}{\partial x_i} \geq 0, \tag{3.16}$$

which means that $\phi_r(x)$ is increasing with respect to x_i.

Notice

$$\left(x_i - x_j\right)\left(x_i^k - x_j^k\right) \geq 0\left(1 \leq k \leq r-1\right). \tag{3.17}$$

From (3.15) and (3.17), we get

$$\left(x_i - x_j\right)\left(\frac{\partial\phi_r(x)}{\partial\phi_{x_i}} - \frac{\partial\phi_r(x)}{\partial\phi_{x_j}}\right) \geq 0. \tag{3.18}$$

By Lemma 2.1, $\phi_r(x)$ is Schur-convex in R_+^n. □

THEOREM 3.3. *Suppose that $x_i > 0$, $i = 1,2,\dots,n$, $\sum_{i=1}^n x_i = s$, $c \geq s$. Then the following statements are valid:*

(i)

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leq \left(D_r(x)\right)^{1/r}. \tag{3.19}$$

(ii)

$$\frac{c_r(c-x)}{c_r(x)} \leq \left(\frac{nc}{s} - 1\right)\frac{c_{r-1}(c-x)}{c_{r-1}(x)}. \tag{3.20}$$

Proof. (i) By Theorem 3.1 and Lemma 2.4, we have $c_r(s/n) \leq c_r(x)$. From this, we obtain (3.19).

(ii) By Theorem 3.2 and Lemma 2.2, we have $\phi_r((c-x)/(nc/s-1)) \leq \phi_r(x)$, which shows that (3.20) is true. □

THEOREM 3.4. *Suppose that $x_i > 0$, $i = 1,2,\dots,n$, and $\sum_{i=1}^n x_i = s$, $c > 0$, then*

$$\frac{c_r(c+x)}{c_r(x)} \leq \left(\frac{nc}{s} + 1\right)\frac{c_{r-1}(c+x)}{c_{r-1}(x)}. \tag{3.21}$$

Proof. By Theorem 3.2 and Lemma 2.3, we have $\phi_r((c+x)/(s+nc)) \leq \phi_r(x/s)$, from which we obtain (3.21). □

Using Theorems 3.3 and 3.4, we can immediately get the following consequences.

COROLLARY 3.5. *Suppose that $x_i > 0$, $\sum_{i=1}^n x_i = s$, $c \geq s$, then*

$$\begin{aligned}\frac{c_r(c-x)}{c_r(x)} &\leq \left(\frac{nc}{s} - 1\right)\frac{c_{r-1}(c-x)}{c_{r-1}(x)} \leq \left(\frac{nc}{s} - 1\right)^2\frac{c_{r-2}(c-x)}{c_{r-2}(x)} \\ &\leq \cdots \leq \left(\frac{nc}{s} - 1\right)^r\frac{c_0(c-x)}{c_0(x)} = \left(\frac{nc}{s} - 1\right)^r.\end{aligned} \tag{3.22}$$

Remark 3.6. Let $c = 1$, we can establish the converse inequality of "Ky Fan" inequality [1], that is

$$\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i)} \leq \left(\frac{c_r(x)}{c_r(1-x)}\right)^{1/r}. \tag{3.23}$$

COROLLARY 3.7. *Suppose that $x_i > 0$, $\sum_{i=1}^n x_i = s$, $c \geq 0$, then*

$$\begin{aligned}\frac{c_r(c+x)}{c_r(x)} &\leq \left(\frac{nc}{s} + 1\right)\frac{c_{r-1}(c+x)}{c_{r-1}(x)} \leq \left(\frac{nc}{s} + 1\right)^2\frac{c_{r-2}(c+x)}{c_{r-2}(x)} \\ &\leq \cdots \leq \left(\frac{nc}{s} + 1\right)^r\frac{c_0(c-x)}{c_0(x)} = \left(\frac{nc}{s} + 1\right)^r.\end{aligned} \tag{3.24}$$

Theorem 3.8. *Suppose that* $0 < x_i \leq 1/2$, $i = 1,2,\dots,n$, *let* $1 - x = (1 - x_1, 1 - x_2, \dots, 1 - x_n)$, *then*

$$\frac{c_n(1-x)}{c_n(x)} \geq \cdots \geq \frac{c_r(1-x)}{c_r(x)} \geq \frac{c_{r-1}(1-x)}{c_{r-1}(x)} \geq \cdots \geq \frac{c_1(1-x)}{c_1(x)} = \frac{A_n(1-x)}{A_n(x)}, \tag{3.25}$$

where $A_n(x)$ *is arithmetic mean of real numbers* $x_1, x_2, \dots, x_n$.

Proof. By Theorem 3.2, $\phi_r(x) = c_r(x)/c_{r-1}(x)$ is an increasing function in $A = \{(x_1, x_2, \dots, x_n) \mid 0 < x_i < 1\}$, and $1 - x \geq x$. Therefore

$$\phi_r(1-x) \geq \phi_r(x). \tag{3.26}$$

Or

$$\frac{c_r(1-x)}{c_{r-1}(1-x)} \geq \frac{c_r(x)}{c_{r-1}(x)}. \tag{3.27}$$

It means (3.25) is valid. □

Remark 3.9. The inequality (3.25) is of the type of the "Ky Fan" inequality [1]:

$$\frac{G_n(1-x)}{G_n(x)} \geq \frac{A_n(1-x)}{A_n(x)}. \tag{3.28}$$

Theorem 3.10. *Suppose that* $x_i > 0$, $i = 1,2,\dots,n$, $n \geq 2$, *then*

$$D_{r-2}(x)D_{r+2}(x) - D_{r-1}(x)D_{r+1}(x) \geq 0. \tag{3.29}$$

Proof. By Lemma 2.6, we can obtain that

$$D_r^2(x) \leq D_{r-1}(x)D_{r+1}(x); \qquad D_{r-1}^2(x) \leq D_{r-2}(x)D_r(x); \qquad D_{r+1}^2(x) \leq D_r(x)D_{r+2}(x). \tag{3.30}$$

From them, it follows that

$$D_{r-2}(x)D_{r+2}(x) - D_{r-1}(x)D_{r+1}(x) \geq 0. \tag{3.31}$$

□

Remark 3.11. Theorem 3.10 shows the inequality (1.6) is true for $n > 2$. So, our result solve the problem given by Menon in [7].

Acknowledgments

The author is greatly indebted to the referees for their valuable suggestions and comments. A project supported by Scientific Research Fund of Hunan Provincial Education Department (China) (granted 03C427).

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Kaizhong Guan: Department of Mathematics and Physics, Nanhua University, Hengyang, Hunan 421001, China
E-mail address: kaizhongguan@yahoo.com.cn