ON THE NONEXISTENCE OF POSITIVE SOLUTION OF SOME SINGULAR NONLINEAR INTEGRAL EQUATIONS

NGUYEN THANH LONG

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We consider the singular nonlinear integral equation $u(x) = \int_{\mathbb{R}^N} g(x,y,u(y))dy/|y-x|^\sigma$ for all $x \in \mathbb{R}^N$, where σ is a given positive constant and the given function $g(x,y,u)$ is continuous and $g(x,y,u) \geq M|x|^{\beta_1}|y|^{\beta}(1+|x|)^{-\gamma_1}(1+|y|)^{-\gamma}u^\alpha$ for all $x,y \in \mathbb{R}^N$, $u \geq 0$, with some constants $\alpha,\beta,\beta_1,\gamma,\gamma_1 \geq 0$ and $M > 0$. We prove in an elementary way that if $0 \leq \alpha \leq (N+\beta-\gamma)/(\sigma+\gamma_1-\beta_1)$, $(1/2)(N+\beta+\beta_1-\gamma-\gamma_1) < \sigma < \min\{N, N+\beta+\beta_1-\gamma-\gamma_1\}$, $\sigma+\gamma_1-\beta_1 > 0$, $N \geq 2$, the above nonlinear integral equation has no positive solution.

1. Introduction

We consider the nonexistence of positive solutions of the following singular nonlinear integral equation

$$u(x) = b_N \int_{\mathbb{R}^N} \frac{g(x,y,u(y))dy}{|y-x|^\sigma} \quad \forall x \in \mathbb{R}^N, \tag{1.1}$$

where $b_N = 2((N-1)\omega_{N+1})^{-1}$ with ω_{N+1} being the area of unit sphere in $\mathbb{R}^{N+1}$, $N \geq 2$, σ is a given positive constant with $0 < \sigma < N$, and $g : \mathbb{R}^{2N} \times \mathbb{R}_+ \to \mathbb{R}$ is given continuous function satisfying the following.

There exist the constants $\alpha,\beta,\beta_1,\gamma,\gamma_1 \geq 0$ and $M > 0$ such that

$$g(x,y,u) \geq M|x|^{\beta_1}|y|^{\beta}(1+|x|)^{-\gamma_1}(1+|y|)^{-\gamma}u^\alpha \quad \forall x,y \in \mathbb{R}^N,\ u \geq 0, \tag{1.2}$$

and some auxiliary conditions below.

In the case of $\sigma = N-1$, $g(x,y,u(y)) = g(y,u(y))$, the integral equation (1.1) is a consequence of the following nonlinear Neumann problem

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$$\Delta v = \sum_{i=1}^{N+1} v_{x_i x_i} = 0, \quad x \in \mathbb{R}^N,\ x_{N+1} > 0, \tag{1.3}$$

$$-v_{x_{N+1}}(x,0) = g(x, v(x,0)) = 0, \quad x \in \mathbb{R}^N, \tag{1.4}$$

of which the boundary value $u(x) = v(x,0)$ together with some auxiliary conditions will be a solution of the equation

$$u(x) = b_N \int_{\mathbb{R}^N} \frac{g(y,u(y))\,dy}{|y-x|^\sigma} \quad \forall x \in \mathbb{R}^N. \tag{1.5}$$

In [3] the authors have studied a problem (1.3), (1.4) for $N = 2$ with the Laplace equation (1.3) having the axial symmetry

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0 \quad \forall r > 0,\ \forall z > 0, \tag{1.6}$$

and with the nonlinear boundary condition of the form

$$-u_z(r,0) = I_0 \exp\left(-r^2/r_0^2\right) + u^\alpha(r,0) \quad \forall r > 0, \tag{1.7}$$

where I_0, r_0, α are given positive constants. The problem (1.6), (1.7) is the stationary case of the problem associated with ignition by radiation. In the case of $0 < \alpha \leq 2$ the authors in [3] have proved that the following nonlinear integral equation

$$u(r,0) = \frac{1}{2\pi}\int_0^{+\infty}\left[I_0 \exp\left(-s^2/r_0^2\right) + u^\alpha(s,0)\right] s\,ds \int_0^{2\pi} \frac{d\theta}{\sqrt{r^2+s^2-2rs\cos\theta}} \quad \forall r > 0, \tag{1.8}$$

associated to the problem (1.6), (1.7) has no positive solution. Afterwards, this result has been extended in [8] to the general nonlinear boundary condition

$$-u_z(r,0) = g(r, u(r,0)) \quad \forall r > 0. \tag{1.9}$$

In [7] the problem (1.3), (1.4) is considered for $N = 2$ and for a function g continuous, nondecreasing and bounded below by the power function of order α with respect to the third variable and it is proved that for $0 < \alpha \leq 2$ such a problem has no positive solution.

In [1, 2] we have considered the problem (1.3), (1.4) for $N \geq 3$. The function $g : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty)$ is continuous, nondecreasing with respect to variable u, satisfies the condition (1.2) with $\gamma = 0$ and some auxiliary conditions. In the case of $0 \leq \alpha \leq N/(N-1)$, $N \geq 2$ we have proved that the problem (1.3), (1.4) has no positive solution [1, 2].

In [5, 6] the authors have proved the nonexistence of a positive solution of the problem (1.3), (1.4) with

$$g(x,u) = u^\alpha. \tag{1.10}$$

In [6] it is proved with $1 \leq \alpha < N/(N-1)$, $N \geq 2$, and in [5] with $1 < \alpha < (N+1)/(N-1)$, $N \geq 2$. We also note that the function $g(x,u) = u^\alpha$ does not satisfy the conditions in the papers [1, 7, 8].

In this paper, we consider the nonlinear integral equation (1.1) for $(1/2)(N+\beta+\beta_1-\gamma-\gamma_1)<\sigma<\min\{N,N+\beta+\beta_1-\gamma-\gamma_1\}$, $\sigma+\gamma_1-\beta_1>0$, $N\geq 2$. The function $g(x,y,u)$ is continuous, satisfies the condition (1.2) of which (1.10) is a special case. By proving elementarily we generalize the results from [1–10] that for $0\leq\alpha\leq(N+\beta-\gamma)/(\sigma+\gamma_1-\beta_1)$ (1.1) has no continuous positive solution.

2. The theorem of nonexistence of positive solution

Without loss of generality, we can suppose that $b_N=1$ with a change of the constant M in the assumption (1.2) of g. We rewrite the integral equation (1.1):

$$u(x)=Tu(x)\equiv\int_{\mathbb{R}^N}\frac{g(x,y,u(y))\,dy}{|y-x|^{\sigma}}\quad\forall x\in\mathbb{R}^N. \tag{2.1}$$

Then we have the main result as follows.

Theorem 2.1. *Let $g:\mathbb{R}^{2N}\times[0,+\infty)\to\mathbb{R}$ be a continuous function satisfying the following hypothesis. There exist constants $M>0$, $\alpha,\beta,\beta_1,\gamma,\gamma_1\geq 0$ with*

$$\begin{gathered}\frac{1}{2}(N+\beta+\beta_1-\gamma-\gamma_1)<\sigma<\min\{N,N+\beta+\beta_1-\gamma-\gamma_1\},\\ \sigma+\gamma_1-\beta_1>0,\quad N\geq 2,\end{gathered} \tag{2.2}$$

such that

$$g(x,y,u)\geq M|x|^{\beta_1}|y|^{\beta}(1+|x|)^{-\gamma_1}(1+|y|)^{-\gamma}u^{\alpha}\quad\forall x,y\in\mathbb{R}^N,\ u\geq 0. \tag{2.3}$$

If $0\leq\alpha\leq(N+\beta-\gamma)/(\sigma+\gamma_1-\beta_1)$ then, the integral equation (2.1) has no continuous positive solution.

Remark 2.2. The result of theorem is stronger than that in [1, 7]. Indeed, corresponding to the same equation (1.5), the following assumptions which were made in [1, 7] are not needed here.

(G_1) $g(y,u)$ is nondecreasing with respect to variable u, that is,

$$(g(y,u)-g(y,v))(u-v)\geq 0\quad\forall u,v\geq 0,\ y\in\mathbb{R}^N. \tag{2.4}$$

(G_2) The integral $\int_{\mathbb{R}^N}(g(y,0)dy/(1+|y|)^{N-1})$ exists and is positive.

Remark 2.3. In the case of $N\geq 2$, we have also obtained some results concerning in the papers [2, 7, 9] in the cases as follows:

(a) $\beta=\beta_1=\gamma=\beta=0$, $\sigma=N-1$, $0\leq\alpha\leq N/(N-1)$ (see [2]).
(b) $\beta=\beta_1=\gamma=\beta=0$, $0\leq\alpha\leq N/\sigma$ (see [7].
(c) $\beta_1=\gamma=0$, $0<\sigma<\min\{N,N+\beta-\gamma_1\}$, $0\leq\alpha\leq(N+\beta)/(\sigma+\gamma_1)$ (see [9]).

First, we need the following lemma.

Lemma 2.4. *For every $p \geq 0$, $q \geq 0$, $0 < \sigma < N$, $x \in \mathbb{R}^N$. Put*

$$A[p,q](x) = \int_{\mathbb{R}^N} \frac{|y|^p (1+|y|)^{-q} dy}{|y-x|^\sigma}, \tag{2.5}$$

we have

$$A[p,q](x) = +\infty, \quad \text{if } q - p \leq N - \sigma, \tag{2.6}$$

$$\begin{aligned} &A[p,q](x) \text{ convergent and } A[p,q](x) \\ &\quad \geq \left(\frac{1}{N+p} + \frac{1}{q}\right) \frac{\omega_N}{2^\sigma} |x|^{p+N-\sigma} (1+|x|)^{-q}, \quad \text{if } q - p > N - \sigma, \end{aligned} \tag{2.7}$$

where ω_N is the area of unit sphere in $\mathbb{R}^N$.

The proof of lemma can be found in [9].

Proof of Theorem 2.1. We prove by contradiction. Suppose that there exists a continuous positive solution $u(x)$ of the integral equation (2.1). We suppose that there exists $x_0 \in \mathbb{R}^N$, such that $u(x_0) > 0$. Since u is continuous, then there exists $r_0 > 0$ such that

$$u(x) > \frac{1}{2} u(x_0) \equiv L \quad \forall x \in \mathbb{R}^N,\ |x - x_0| \leq r_0. \tag{2.8}$$

It follows from (2.1), (2.3), (2.8) and the monotonicity of the integral operator

$$\begin{aligned} u(x) = Tu(x) &\geq M|x|^{\beta_1} (1+|x|)^{-\gamma_1} \int_{\mathbb{R}^N} |y|^\beta (1+|y|)^{-\gamma} \frac{u^\alpha(y) dy}{|y-x|^\sigma} \\ &\geq M|x|^{\beta_1} (1+|x|)^{-\gamma_1} L^\alpha \int_{|y-x_0| \leq r_0} |y|^\beta (1+|y|)^{-\gamma} \frac{dy}{|y-x|^\sigma} \\ &\geq M L^\alpha (1+|x_0|+r_0)^{-\sigma} |x|^{\beta_1} (1+|x|)^{-\sigma-\gamma_1} \int_{|y-x_0| \leq r_0} |y|^\beta (1+|y|)^{-\gamma} dy, \end{aligned} \tag{2.9}$$

for all $x \in \mathbb{R}^N$.

Using the inequality

$$|y-x| \leq |y| + |x| \leq (1+|x_0|+r_0)(1+|x|) \quad \forall x, y \in \mathbb{R}^N,\ |y-x_0| \leq r_0, \tag{2.10}$$

we obtain from (2.9), (2.10) that

$$u(x) \geq u_1(x) = m_1 |x|^{p_1} (1+|x|)^{-q_1} \quad \forall x \in \mathbb{R}^N, \tag{2.11}$$

where

$$
\begin{gathered}
p_1=\beta_1, \qquad q_1=\sigma+\gamma_1, \\
m_1=ML^\alpha\left(1+|x_0|+r_0\right)^{-\sigma}\int_{|y-x_0|\leq r_0}|y|^\beta(1+|y|)^{-\gamma}dy.
\end{gathered} \tag{2.12}
$$

Using again the equality (2.1), it follows from (2.3), (2.11) that

$$
\begin{aligned}
u(x)=Tu(x) & \geq M|x|^{\beta_1}(1+|x|)^{-\gamma_1}\int_{\mathbb{R}^N}|y|^\beta(1+|y|)^{-\gamma}\frac{u_1^\alpha(y)dy}{|y-x|^\sigma} \\
& \geq M|x|^{\beta_1}(1+|x|)^{-\gamma_1}\int_{\mathbb{R}^N}|y|^\beta(1+|y|)^{-\gamma}\left(m_1|y|^{p_1}(1+|y|)^{-q_1}\right)^\alpha\frac{dy}{|y-x|^\sigma} \\
& = Mm_1^\alpha|x|^{\beta_1}(1+|x|)^{-\gamma_1}\int_{\mathbb{R}^N}|y|^{\beta+\alpha p_1}(1+|y|)^{-\gamma-\alpha q_1}\frac{dy}{|y-x|^\sigma} \\
& = Mm_1^\alpha|x|^{\beta_1}(1+|x|)^{-\gamma_1}A[\beta+\alpha p_1,\gamma+\alpha q_1](x) \quad \forall x\in\mathbb{R}^N.
\end{aligned} \tag{2.13}
$$

Now, we consider separately the cases of different values of α.

Case 1. $0\leq\alpha\leq(N-\sigma+\beta-\gamma)/(\sigma+\gamma_1-\beta_1)$. We obtain from (2.6), (2.13) with $p=\beta+\alpha p_1$, $q=\gamma+\alpha q_1$, $q-p=\gamma-\beta+\alpha(q_1-p_1)=\gamma-\beta+\alpha(\sigma+\gamma_1-\beta_1)\leq N-\sigma$, that

$$
u(x)=+\infty \quad \forall x\in\mathbb{R}^N. \tag{2.14}
$$

It is a contradiction.

Case 2. $(N-\sigma+\beta-\gamma)/(\sigma+\gamma_1-\beta_1)<\alpha<(N+\beta-\gamma)/(\sigma+\gamma_1-\beta_1)$. Using (2.7) with $p=\beta+\alpha p_1$, $q=\gamma+\alpha q_1$, $q-p=\gamma-\beta+\alpha(q_1-p_1)=\gamma-\beta+\alpha(\sigma+\gamma_1-\beta_1)>N-\sigma$, we deduce from (2.13) that

$$
u(x)\geq u_2(x)=m_2|x|^{p_2}(1+|x|)^{-q_2} \quad \forall x\in\mathbb{R}^N, \tag{2.15}
$$

where

$$
\begin{aligned}
p_2 &= \alpha p_1+\beta+\beta_1+N-\sigma, \\
q_2 &= \alpha q_1+\gamma+\gamma_1, \\
m_2 &= Mm_1^\alpha\left(\frac{1}{N+\beta+\alpha p_1}+\frac{1}{\gamma+\alpha q_1}\right)\frac{\omega_N}{2^\sigma}.
\end{aligned} \tag{2.16}
$$

Suppose that

$$
u(x)\geq u_{k-1}(x)=m_{k-1}|x|^{p_{k-1}}(1+|x|)^{-q_{k-1}} \quad \forall x\in\mathbb{R}^N, \tag{2.17}
$$

If $\gamma+\alpha q_{k-1}-\beta-\alpha p_{k-1}>N-\sigma$, then, using (2.1), (2.3), (2.7), and (2.17), we obtain

$$\begin{aligned}
u(x)=Tu(x)&\geq M|x|^{\beta_1}(1+|x|)^{-\gamma_1}\int_{\mathbb{R}^N}|y|^{\beta}(1+|y|)^{-\gamma}\frac{u^{\alpha}(y)dy}{|y-x|^{\sigma}}\\
&\geq M|x|^{\beta_1}(1+|x|)^{-\gamma_1}\int_{\mathbb{R}^N}|y|^{\beta}(1+|y|)^{-\gamma}\frac{u_{k-1}^{\alpha}(y)dy}{|y-x|^{\sigma}}\\
&\geq Mm_{k-1}^{\alpha}|x|^{\beta_1}(1+|x|)^{-\gamma_1}\int_{\mathbb{R}^N}|y|^{\beta}(1+|y|)^{-\gamma}\frac{|y|^{\alpha p_{k-1}}(1+|y|)^{-\alpha q_{k-1}}(y)dy}{|y-x|^{\sigma}}\\
&= Mm_{k-1}^{\alpha}|x|^{\beta_1}(1+|x|)^{-\gamma_1}A[\beta+\alpha p_{k-1},\gamma+\alpha q_{k-1}](x)\\
&\geq Mm_{k-1}^{\alpha}\left(\frac{1}{N+\beta+\alpha p_{k-1}}+\frac{1}{\gamma+\alpha q_{k-1}}\right)\frac{\omega_N}{2^{\sigma}}|x|^{\beta_1+\beta+\alpha p_{k-1}+N-\sigma}(1+|x|)^{-\gamma_1-\alpha q_{k-1}-\gamma}.
\end{aligned} \tag{2.18}$$

Hence

$$u(x)\geq u_k(x)=m_k|x|^{p_k}(1+|x|)^{-q_k}\quad\forall x\in\mathbb{R}^N, \tag{2.19}$$

where the sequences $\{p_{k-1}\}$, $\{q_{k-1}\}$ and $\{m_{k-1}\}$ are defined by the recurrence formulas

$$\begin{aligned}
p_k&=\alpha p_{k-1}+\beta+\beta_1+N-\sigma,\\
q_k&=\alpha q_{k-1}+\gamma+\gamma_1,\\
m_k&=Mm_{k-1}^{\alpha}\left(\frac{1}{N+\beta+\alpha p_{k-1}}+\frac{1}{\gamma+\alpha q_{k-1}}\right)\frac{\omega_N}{2^{\sigma}},\qquad k\geq 2.
\end{aligned} \tag{2.20}$$

Note that $(N-\sigma+\beta-\gamma)/(\sigma+\gamma_1-\beta_1)<1<(N+\beta-\gamma)/(\sigma+\gamma_1-\beta_1)$, hence we obtain from (2.16), (2.20) that

$$p_k=\begin{cases}(\beta+\beta_1+N-\sigma)(k-1)+\beta_1, & \text{if } \alpha=1,\\[4pt] (\beta+\beta_1+N-\sigma)\left(\dfrac{1-\alpha^{k-1}}{1-\alpha}\right)+\beta_1\alpha^{k-1}, & \\ \qquad\text{if } \dfrac{N-\sigma+\beta-\gamma}{\sigma+\gamma_1-\beta_1}<\alpha<\dfrac{N+\beta-\gamma}{\sigma+\gamma_1-\beta_1},\ \alpha\neq 1, & \end{cases} \tag{2.21}$$

$$q_k=\begin{cases}(k-1)(\gamma+\gamma_1)+\sigma+\gamma_1, & \text{if } \alpha=1,\\[4pt] (\gamma+\gamma_1)\left(\dfrac{1-\alpha^{k-1}}{1-\alpha}\right)+(\sigma+\gamma_1)\alpha^{k-1}, & \\ \qquad\text{if } \dfrac{N-\sigma+\beta-\gamma}{\sigma+\gamma_1-\beta_1}<\alpha<\dfrac{N+\beta-\gamma}{\sigma+\gamma_1-\beta_1},\ \alpha\neq 1. & \end{cases} \tag{2.22}$$

It follows from (2.1), (2.3), and (2.18) that

$$u(x)\geq Mm_k^{\alpha}|x|^{\beta_1}(1+|x|)^{-\gamma_1}A[\beta+\alpha p_k,\gamma+\alpha q_k](x)\quad\forall x\in\mathbb{R}^N. \tag{2.23}$$

So, from (2.22), (2.23), we only need to choose the natural number $k \geq 2$ such that

$$\gamma + \alpha q_k - \beta - \alpha p_k \leq N - \sigma < \gamma + \alpha q_{k-1} - \beta - \alpha p_{k-1}, \tag{2.24}$$

since $A[\beta + \alpha p_k, \gamma + \alpha q_k](x) = +\infty$.

On the other hand, by (2.21), (2.22) the inequalities (2.24) equivalent to

$$k - 1 < \frac{\sigma}{N - \sigma + \beta + \beta_1 - \gamma - \gamma_1} \leq k, \quad \text{if } \alpha = 1, \tag{2.25}$$

or

$$k - 1 < \frac{1}{\ln \alpha} \ln \left(\frac{\alpha(\gamma_1 - \beta_1) - (N - \sigma + \beta - \gamma)}{\alpha(\sigma + \gamma_1 - \beta_1) - (N + \beta - \gamma)} \right) \leq k, \tag{2.26}$$

if

$$\frac{N - \sigma + \beta - \gamma}{\sigma + \gamma_1 - \beta_1} < \alpha < \frac{N + \beta - \gamma}{\sigma + \gamma_1 - \beta_1}, \quad \alpha \neq 1. \tag{2.27}$$

By (2.23)–(2.26) we choose k as follows.

(i) If $\alpha = 1$, we choose k satisfying $\sigma/(N - \sigma + \beta + \beta_1 - \gamma - \gamma_1) \leq k < 1 + \sigma/(N - \sigma + \beta + \beta_1 - \gamma - \gamma_1)$.

(ii) If $(N - \sigma + \beta - \gamma)/(\sigma + \gamma_1 - \beta_1) < \alpha < (N + \beta - \gamma)/(\sigma + \gamma_1 - \beta_1)$ and $\alpha \neq 1$, we choose k satisfying $k_0 \leq k < k_0 + 1$, where

$$k_0 = \frac{1}{\ln \alpha} \ln \left(\frac{(\gamma_1 - \beta_1)\alpha - (N - \sigma + \beta - \gamma)}{(\sigma + \gamma_1 - \beta_1)\alpha - (N + \beta - \gamma)} \right). \tag{2.28}$$

Case 3. $\alpha = (N + \beta - \gamma)/(\sigma + \gamma_1 - \beta_1)$. Note that by $\beta + \alpha p_1 = \beta + \alpha\beta_1$ and $\gamma + \alpha q_1 = N + \beta + \alpha\beta_1$, we rewrite (2.13) as follows

$$\begin{aligned} u(x) &\geq M m_1^{\alpha} |x|^{\beta_1} (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} \frac{|y|^{\beta + \alpha p_1} (1 + |y|)^{-\gamma - \alpha q_1} dy}{|y - x|^{\sigma}} \\ &= M m_1^{\alpha} |x|^{\beta_1} (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} \frac{|y|^{\beta + \alpha\beta_1} (1 + |y|)^{-N - \beta - \alpha\beta_1} dy}{|y - x|^{\sigma}} \\ &= M m_k^{\alpha} |x|^{\beta_1} (1 + |x|)^{-\gamma_1} A[\beta + \alpha\beta_1, N + \beta + \alpha\beta_1](x) \end{aligned} \tag{2.29}$$

for all $x \in \mathbb{R}^N$.

On the other hand, for every $x \in \mathbb{R}^N$, $|x| \geq 1$, we have

$$\begin{aligned} A[\beta + \alpha\beta_1, N + \beta + \alpha\beta_1](x) &\geq \int_{\mathbb{R}^N} \frac{|y|^{\beta + \alpha\beta_1} (1 + |y|)^{-N - \beta - \alpha\beta_1} dy}{(|y| + |x|)^{\sigma}} \\ &= \omega_N \int_0^{+\infty} \frac{r^{\beta + \alpha\beta_1 + N - 1} dr}{(1 + r)^{N + \beta + \alpha\beta_1} (r + |x|)^{\sigma}} \\ &\geq \omega_N \int_1^{|x|} \frac{r^{\beta + \alpha\beta_1 + N - 1} dr}{(1 + r)^{N + \beta + \alpha\beta_1} (r + |x|)^{\sigma}} = \omega_N B(x). \end{aligned} \tag{2.30}$$

Notice that for every r such that $1 \le r \le |x|$ we have

$$\left(\frac{r}{1+r}\right)^{\beta+\alpha\beta_1+N} \ge \frac{1}{2^{\beta+\alpha\beta_1+N}}, \qquad \frac{1}{(r+|x|)^{\sigma-1}} \ge \frac{\min\{1,2^{1-\sigma}\}}{|x|^{\sigma-1}}. \tag{2.31}$$

Then

$$\begin{aligned} B(x) &= \int_1^{|x|}\left(\frac{r}{1+r}\right)^{\beta+\alpha\beta_1+N}\frac{1}{(r+|x|)^{\sigma-1}}\frac{dr}{r(r+|x|)} \\ &\ge \frac{1}{2^{\beta+\alpha\beta_1+N}}\frac{\min\{1,2^{1-\sigma}\}}{|x|^{\sigma-1}}\int_1^{|x|}\frac{dr}{r(r+|x|)} \\ &= \frac{1}{2^{\beta+\alpha\beta_1+N}}\frac{\min\{1,2^{1-\sigma}\}}{|x|^{\sigma}}\ln\left(\frac{1+|x|}{2}\right). \end{aligned} \tag{2.32}$$

It follows from (2.29), (2.30), (2.32) that

$$u(x) \ge v_2(x) = \begin{cases} 0, & \text{if } |x| \le 1, \\ C_2|x|^{\beta_1-\sigma}(1+|x|)^{-\gamma_1}\left(\ln\left(\frac{1+|x|}{2}\right)\right)^{s_2}, & \text{if } |x| \ge 1, \end{cases} \tag{2.33}$$

with

$$s_2 = 1, \qquad C_2 = Mm_1^{\alpha}\omega_N\frac{1}{2^{\beta+\alpha\beta_1+N}}\min\{1,2^{1-\sigma}\}. \tag{2.34}$$

Suppose that

$$u(x) \ge v_{k-1}(x) = \begin{cases} 0, & \text{if } |x| \le 1, \\ C_{k-1}|x|^{\beta_1-\sigma}(1+|x|)^{-\gamma_1}\left(\ln\left(\frac{1+|x|}{2}\right)\right)^{s_{k-1}}, & \text{if } |x| \ge 1, \end{cases} \tag{2.35}$$

and C_{k-1}, s_{k-1}, are positive constants.

Then, using (2.1), (2.3), (2.35), we have

$$\begin{aligned} u(x) &\ge M|x|^{\beta_1}(1+|x|)^{-\gamma_1}\int_{\mathbb{R}^N}\frac{|y|^{\beta}(1+|y|)^{-\gamma}v_{k-1}^{\alpha}(y)dy}{|y-x|^{\sigma}} \\ &\ge M|x|^{\beta_1}(1+|x|)^{-\gamma_1}\int_{|y|\ge 1}\frac{|y|^{\beta}(1+|y|)^{-\gamma}v_{k-1}^{\alpha}(y)dy}{(|y|+|x|)^{\sigma}} \\ &= M|x|^{\beta_1}(1+|x|)^{-\gamma_1}C_{k-1}^{\alpha} \\ &\quad\times\int_{|y|\ge 1}\frac{|y|^{\beta}(1+|y|)^{-\gamma}|y|^{\alpha(\beta_1-\sigma)}(1+|y|)^{-\alpha\gamma_1}(\ln((1+|y|)/2))^{\alpha s_{k-1}}dy}{(|y|+|x|)^{\sigma}} \\ &= MC_{k-1}^{\alpha}|x|^{\beta_1}(1+|x|)^{-\gamma_1}\int_{|y|\ge 1}\frac{|y|^{\beta+\alpha(\beta_1-\sigma)}(\ln((1+|y|)/2))^{\alpha s_{k-1}}dy}{(1+|y|)^{\gamma+\alpha\gamma_1}(|y|+|x|)^{\sigma}} \\ &= M\omega_N C_{k-1}^{\alpha}|x|^{\beta_1}(1+|x|)^{-\gamma_1}\int_1^{+\infty}\frac{r^{\beta+\alpha(\beta_1-\sigma)+N-1}(\ln((1+r)/2))^{\alpha s_{k-1}}dr}{(1+r)^{\gamma+\alpha\gamma_1}(r+|x|)^{\sigma}}. \end{aligned} \tag{2.36}$$

Considering $|x| \geq 1$, we have

$$\begin{aligned}
\int_1^{+\infty} & \frac{r^{\beta+\alpha(\beta_1-\sigma)+N-1}\left(\ln((1+r)/2)\right)^{\alpha s_{k-1}} dr}{(1+r)^{\gamma+\alpha\gamma_1}\left(r+|x|\right)^{\sigma}} \\
&\geq \left(\ln\left(\frac{1+|x|}{2}\right)\right)^{\alpha s_{k-1}} \int_{|x|}^{+\infty} \frac{r^{\beta+\alpha(\beta_1-\sigma)+N-1} dr}{(r+r)^{\gamma+\alpha\gamma_1}(r+r)^{\sigma}} \\
&= \frac{1}{2^{\gamma+\alpha\gamma_1+\sigma}}\left(\ln\left(\frac{1+|x|}{2}\right)\right)^{\alpha s_{k-1}} \int_{|x|}^{+\infty} r^{-1-\sigma} dr \\
&= \frac{1}{\sigma 2^{\gamma+\alpha\gamma_1+\sigma}} \times \frac{1}{|x|^{\sigma}} \times \left(\ln\left(\frac{1+|x|}{2}\right)\right)^{\alpha s_{k-1}}.
\end{aligned} \tag{2.37}$$

We deduce from (2.36), (2.37) that

$$u(x) \geq v_k(x) = \begin{cases} 0, & \text{if } |x| \leq 1, \\ C_k |x|^{\beta_1-\sigma}(1+|x|)^{-\gamma_1}\left(\ln\left(\frac{1+|x|}{2}\right)\right)^{s_k}, & \text{if } |x| \geq 1, \end{cases} \tag{2.38}$$

where

$$s_k = \alpha s_{k-1}, \quad C_{k-1} = \frac{1}{\sigma 2^{\gamma+\alpha\gamma_1+\sigma}} M\omega_N C_{k-1}^{\alpha}, \quad k \geq 3. \tag{2.39}$$

From (2.34), (2.39) we obtain

$$s_k = s_2\alpha^{k-2} = \alpha^{k-2} = \left(\frac{N+\beta-\gamma}{\sigma+\gamma_1-\beta_1}\right)^{k-2}, \qquad C_k = \frac{1}{d}\left(dC_2\right)^{\alpha^{k-2}}, \tag{2.40}$$

where

$$d = \left(\frac{1}{\sigma 2^{\gamma+\alpha\gamma_1+\sigma}} M\omega_N\right)^{1/(\alpha-1)}, \qquad \alpha = \frac{(N+\beta-\gamma)}{(\sigma+\gamma_1-\beta_1)} > 1. \tag{2.41}$$

Then, with $|x| \geq 1$, we rewrite (2.38) in the form

$$u(x) \geq v_k(x) = \frac{1}{d}|x|^{\beta_1-\sigma}(1+|x|)^{-\gamma_1}\left(dC_2 \ln\left(\frac{1+|x|}{2}\right)\right)^{\alpha^{k-2}}. \tag{2.42}$$

Choosing x_1 such that $dC_2 \ln((1+|x_1|)/2) > 1$. By (2.42), we deduce that $u(x_1) = +\infty$. It is a contradiction.

Theorem is proved completely. □

Remark 2.5. In the case of $g(x,u)$ we have not a conclusion about $\alpha > N/(N-1)$ and $N \geq 2$, yet. However, when $g(x,u) = u^{\alpha}, N/(N-1) \leq \alpha < (N+1)/(N-1)$, $N \geq 2$, Hu in [5] have proved that the problem (1.3), (1.4) has no positive solution. In the *limiting case* $\alpha = (N+1)/(N-1)$, positive solutions do exist (see [4–6]). In particular, for this value of α, the authors of [4] gave explicit forms for all nontrivial nonnegative solutions $u \in C^2(\mathbb{R}_+^{N+1}) \cap C^1(\overline{\mathbb{R}_+^{N+1}})$ of the problem

$$\begin{aligned} & -\Delta u = au^{\alpha+(2/N-1)} \quad \text{in } x' \in \mathbb{R}^N,\ x_{N+1} > 0, \\ & -u_{x_{N+1}}(x',0) = bu^{\alpha}(x',0) \quad \text{on } x_{N+1} = 0. \end{aligned} \tag{2.43}$$

They proved the following results:

(i) if $a > 0$ or $a \le 0$, $b > B = \sqrt{a(1-N)/(N+1)}$, then $u(x) = C(|x - x^0|^2 + \beta)^{(1-N)/2}$ for some $C > 0$, $\beta \in \mathbb{R}$ and $x^0 = (x_1^0, \dots, x_{N+1}^0) \in \mathbb{R}^{N+1}$, where $x_1^0 = (b/(N-1))C^{2/(N-1)}$ and $\beta = (a/(N+1)(N-1))C^{4/(N-1)}$;

(ii) if $a = 0$ and $b = 0$, then $u(x) = C$ for some $C > 0$;

(iii) if $a = 0$ and $b < 0$, then $u(x) = Cx_1 + (-C/b)^{(N-1)/(N+1)}$ for some $C > 0$;

(iv) if $a < 0$ and $b = B$, then $u(x) = ((2B/N-1)x_1 + C)^{(1-N)/2}$ for some $C > 0$;

(v) if $a < 0$ and $b < B$, then there is no nontrivial nonnegative solution of the problem.

References

[1] D. T. T. Binh, T. N. Diem, D. V. Ruy, and N. T. Long, *On nonexistence of positive solution of a nonlinear Neumann problem in half-space* $\mathbf{R}_+^n$, Demonstratio Mathematica **31** (1998), no. 4, 773–782.

[2] D. T. T. Binh and N. T. Long, *On the nonexistence of positive solution of Laplace equation in half-space* $\mathbf{R}_+^N$ *with a nonlinear Neumann boundary condition*, Demonstratio Mathematica **33** (2000), no. 2, 365–372.

[3] F. V. Bunkin, V. A. Galaktionov, N. A. Kirichenko, S. P. Kurdyumov, and A. A. Samarskiĭ, *A nonlinear boundary value problem of ignition by radiation*, Akademiya Nauk SSSR. Zhurnal Vychislitel'noĭ Matematiki i Matematicheskoĭ Fiziki **28** (1988), no. 4, 549–559, 623 (Russian), translated in U.S.S.R. Comput. Math. and Math. Phys. **28** (1988), no. 2, 157–164 (1989).

[4] M. Chipot, I. Shafrir, and M. Fila, *On the solutions to some elliptic equations with nonlinear Neumann boundary conditions*, Advances in Differential Equations **1** (1996), no. 1, 91–110.

[5] B. Hu, *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition*, Differential Integral Equations **7** (1994), no. 2, 301–313.

[6] B. Hu and H.-M. Yin, *The profile near blowup time for solution of the heat equation with a nonlinear boundary condition*, Transactions of the American Mathematical Society **346** (1994), no. 1, 117–135.

[7] N. T. Long and D. T. T. Binh, *On the nonexistence of positive solution of a nonlinear integral equation*, Demonstratio Mathematica **34** (2001), no. 4, 837–845.

[8] N. T. Long and D. V. Ruy, *On a nonexistence of positive solution of Laplace equation in upper half-space with Cauchy data*, Demonstratio Mathematica **28** (1995), no. 4, 921–927.

[9] ———, *On the nonexistence of positive solution of some nonlinear integral equation*, Demonstratio Mathematica **36** (2003), no. 2, 393–404.

[10] D. V. Ruy, N. T. Long, and D. T. T. Binh, *On a nonexistence of positive solution of Laplace equation in upper half-space*, Demonstratio Mathematica **30** (1997), no. 1, 7–14.

Nguyen Thanh Long: Department of Mathematics and Computer Science,
University of Natural Science, Vietnam National University-Ho Chi Minh City,
227 Nguyen Van Cu Street, District 5, Ho Chi Minh City, Vietnam
E-mail address: longnt@hcmc.netnam.vn