A NOTE ON EULER NUMBER AND POLYNOMIALS

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We investigate some properties of non-Archimedean integration which is defined by Kim. By using our results in this paper, we can give an answer to the problem which is introduced by I.-C. Huang and S.-Y. Huang in 1999.

1. Introduction

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$.

Let p be a fixed prime number and let l be a fixed integer with $(p,l) = 1$. We set

$$\begin{aligned} X &= \varprojlim_{N} (\mathbb{Z}/lp^N\mathbb{Z}), \\ X^* &= \bigcup_{\substack{0<a<lp \\ (a,p)=1}} (a + lp\mathbb{Z}_p), \\ a + lp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{lp^N}\}, \end{aligned} \tag{1.1}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < lp^N$ (cf. [3, 4]).

For any positive integer N, we set

$$\mu_1(a + lp^N\mathbb{Z}_p) = \frac{1}{lp^N} \tag{1.2}$$

and this can be extended to a distribution on X (see [3, 9]).

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This distribution yields an integral for nonnegative integer m:

$$\int_X x^m d\mu_1(x) = B_m, \tag{1.3}$$

where B_m are called usual Bernoulli numbers (cf. [8]).

The Euler numbers E_m are defined by the generating function in the complex number field as follows:

$$\frac{2}{e^t+1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!} \quad (|t| < \pi) \tag{1.4}$$

where we use the technique method notation by replacing E^m by E_m $(m \geq 0)$, symbollically (cf. [3, 5, 7, 9, 10]).

The Bernoulli numbers with order k, $B_n^{(k)}$, were defined by

$$\left(\frac{t}{e^t-1}\right)^k = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} \quad \text{(cf. [5, 10])}. \tag{1.5}$$

Let u be algebraic in complex number field. Then Frobenius-Euler numbers were defined by

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!} \quad \text{(cf. [5])}. \tag{1.6}$$

By (1.4) and (1.6), note that $H_n(-1) = E_n$.

In this paper, we will give the interesting formulae for sums of products of Euler numbers (= Frobenius-Euler numbers) by using p-adic Euler integration which is defined in [3, 5, 8–10]. Our result is an answer to the problem which is introduced by I.-C. Huang and S.-Y. Huang in [2, page 179].

2. Sums of products of Euler numbers

Let $u \in \mathbb{C}_p$ with $|1-u^f|_p \geq 1$ for each positive integer f. Then the p-adic Euler measure was defined by

$$E_u(x) = E_u(x+dp^N\mathbb{Z}_p) = \frac{u^{dp^N-x}}{1-u^{dp^N}}, \quad \text{(cf. [3, 5])}. \tag{2.1}$$

Now, we define Euler polynomials with order n by

$$\left(\frac{u}{1-u}\right)^m H_n^{(m)}(u,x) = \underbrace{\int_X \cdots \int_X}_{m \text{ times}} (x+x_1+\cdots+x_m)^n dE_u(x_1)\cdots dE_u(x_m). \tag{2.2}$$

In the case $x = 0$, we use the following notations:

$$H_n^{(k)}(u,0) = H_n^{(k)}(u), \qquad H_n^{(1)}(u) = H_n(u) \quad \text{(cf. [3, 9])}. \tag{2.3}$$

In [3], the following formula can be found:

$$\int_{\mathbb{Z}_p} x^n dE_u(x) = \frac{u}{1-u} H_n(u). \tag{2.4}$$

By (2.2) and (2.4), we easily see that $\lim_{k\to 1} H_n^{(k)}(u) = H_n(u)$.

For any positive integer m, $H_n^{(m)}(u,x)$ can be written by

$$H_n^{(m)}(u,x) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} H_j^{(m)}(u). \tag{2.5}$$

We may now mention the following formulae which are easy to prove:

$$\left(\frac{u}{1-u}\right)^m H_n^{(m)}(u,x) = l^n \sum_{l_1,\ldots,l_m=0}^{l-1} \frac{u^{ml-\sum_{i=1}^m l_i}}{(1-u^l)^m} H_n^{(m)}\left(u^l, \frac{x+l_1+\cdots+l_m}{l}\right), \tag{2.6}$$

where

$$\sum_{l_1,\ldots,l_m=0}^{l-1} = \sum_{l_1=0}^{l-1} \sum_{l_2=0}^{l-1} \cdots \sum_{l_m=0}^{l-1}. \tag{2.7}$$

By using (2.2) and multinomial coefficients, We obtain the following theorem.

Theorem 2.1. *For $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{C}_p$ and positive integers n, m,*

$$H_n^{(m)}(u, \alpha_1+\alpha_2+\cdots+\alpha_m) = \sum_{\substack{i_1,\ldots,i_m \\ n=i_1+\cdots+i_m}} \binom{n}{i_1,\ldots,i_m} H_{i_1}(u,\alpha_1) H_{i_2}(u,\alpha_2) \cdots H_{i_m}(u,\alpha_m), \tag{2.8}$$

where $\binom{n}{i_1,\ldots,i_m}$ is the multinomial coefficient.

Remark 2.2. The above theorem is an answer to the problem which was introduced in [2, page 179].

Remark 2.3. Note that $H_n(-1) = \sum_{k=0}^{n} \binom{n+1}{k} 2^k B_k$, where B_k are the kth ordinary Bernoulli numbers.

Remark 2.4. By using Volkenborn integral, it was well known that

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} x^n d\mu_1(x) \frac{t^n}{n!} \quad \text{(cf. [3, 7, 10])}. \tag{2.9}$$

In [1, 9], note that

$$\left(\frac{t}{e^t-1}\right)^k=\sum_{n=0}^{\infty}\underbrace{\iint_X\cdots\int_X}_{k\text{ times}}(x+x_1+\cdots+x_k)^n d\mu_1(x_1)d\mu_1(x_2)\cdots d\mu_1(x_k)\frac{t^n}{n!}. \tag{2.10}$$

The Bernoulli polynomials with order k, $B_n^{(k)}(x)$, were defined by

$$B_n^{(k)}(x)=\underbrace{\iint_X\cdots\int_X}_{k\text{ times}}(x+x_1+\cdots+x_k)^n d\mu_1(x_1)d\mu_1(x_2)\cdots d\mu_1(x_k) \quad \text{(cf. [7, 9, 10])}. \tag{2.11}$$

In the case $x=0$, we write $B_n^{(k)}(0)=B_n^{(k)}$ (cf. [9]).

In [2], the authors proved the formulae of sums of products of Bernoulli numbers of higher order by using theory of residues. By using the properties of invariant p-adic integrals in this paper, we can also give the same formulae on the sums of products for $B_n^{(k)}$ in [2]. Let χ be a Dirichlet character with conductor f. We set $p^*=p$ for $p\geq 2$, and $p^*=4$ for $p=2$. Let $\bar{f}=(f,p^*)$ be denoted by the least common multiple of the conductor f of χ and p^*.

Now, we define the generalized Bernoulli numbers of higher order with χ as

$$B_{n,\chi}^{(m)}=\int_X\cdots\int_X\chi(x_1+\cdots+x_m)(x_1+\cdots+x_m)^n d\mu_1(x_1)\cdots d\mu_1(x_m). \tag{2.12}$$

We easily get in (2.12)

$$B_{n,\chi}^{(m)}=l^{n-m}\sum_{x_1,\ldots,x_m=0}^{l-1}B_n^{(m)}\left(\frac{x_1+\cdots+x_m}{l}\right)\chi(x_1+\cdots+x_m), \tag{2.13}$$

where $B_{n,\chi}$ is the generalized ordinary Bernoulli number with χ.

By (2.12), we have

$$B_{n,\chi}^{(m)}=\lim_{\rho\to\infty}\frac{1}{(\bar{f}p^{\rho})^m}\sum_{1\leq x_1\leq\bar{f}p^{\rho}}\cdots\sum_{1\leq x_m\leq\bar{f}p^{\rho}}\chi(x_1+\cdots+x_m)(x_1+\cdots+x_m)^n. \tag{2.14}$$

The investigation of these numbers is left to the interested reader.

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