

SOME ELEMENTARY INEQUALITIES IN GAS DYNAMICS EQUATION

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We describe the sets on which difference of solutions of the gas dynamics equation satisfy some special conditions. By virtue of nonlinearity of the equation the sets depend on the solution gradient quantity. We show double-ended estimates of the given sets and some properties of these estimates.

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1. Main results

Consider the gas dynamics equation

$$\sum_{c=1}^n \frac{\partial}{\partial x_i} (\sigma(|\nabla f|) f_{x_i}) = 0, \quad (1.1)$$

where

$$\sigma(t) = \left(1 - \frac{\gamma - 1}{2} t^2\right)^{1/(\gamma - 1)}. \quad (1.2)$$

Here γ is a constant, $-\infty < \gamma < +\infty$. This equation describes the velocity potential of a steady-state flow of ideal gas in the adiabatic process. In the case $n = 2$ the parameter γ characterizes the flow of substance. For different values γ it can be a flow of gas, fluid, plastic, electric or chemical field in different mediums, and so forth (see, e.g., [1, Section 2], [2, Section 15, Chapter IV]). For $\gamma = 1 \pm 0$ we assume

$$\sigma(t) = \exp\left\{-\frac{1}{2}t^2\right\}. \quad (1.3)$$

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The case of $\gamma = -1$ is known as the minimal surface equation (Chaplygin's gas):

$$\operatorname{div} \left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 0. \quad (1.4)$$

For $\gamma = -\infty$, (1.1) becomes the Laplace equation.

In general, a solution of (1.1) with a function σ of variables (x_1, \dots, x_n) is called σ -harmonic function. Such functions were studied in many works (see., e.g., [3, 4] and literature quoted therein).

We set $\Omega_\gamma = \mathbb{R}^n$ for $\gamma \leq 1$,

$$\Omega_\gamma = \left\{ \xi \in \mathbb{R}^n : |\xi| < \sqrt{\frac{2}{\gamma - 1}} \right\} \quad \text{for } \gamma > 1. \quad (1.5)$$

The following inequalities were crucial in previous analysis of solutions to (1.1) for $\gamma = -1$ (see [5–9]):

$$c_1 \sum_{i=1}^n (\xi_i - \eta_i)^2 \leq \sum_{i=1}^n (\sigma(|\xi|)\xi_i - \sigma(|\eta|)\eta_i)(\xi_i - \eta_i), \quad \xi, \eta \in \Omega_\gamma, \quad (1.6)$$

$$\sum_{i=1}^n (\sigma(|\xi|)\xi_i - \sigma(|\eta|)\eta_i)^2 \leq c_2 \sum_{i=1}^n (\sigma(|\xi|)\xi_i - \sigma(|\eta|)\eta_i)(\xi_i - \eta_i), \quad \xi, \eta \in \Omega_\gamma. \quad (1.7)$$

Here $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ and $c_1 > 0$, $c_2 > 0$ are constants not depending on ξ and η .

In general, the latter inequalities are valid only on subsets of $\Omega_\gamma \times \Omega_\gamma$ with c_1 and c_2 depending on these subsets. The purpose of the present paper is to describe that dependence.

Introduce the sets

$$\mathcal{A}_\gamma(c_1) = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : \xi, \eta \text{ satisfy (1.6)}\}, \quad (1.8)$$

$$\mathcal{B}_\gamma(c_2) = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : \xi, \eta \text{ satisfy (1.7)}\}. \quad (1.9)$$

Generally, the sets $\mathcal{A}_\gamma(c_1)$ and $\mathcal{B}_\gamma(c_2)$ have a complicated structure. We will describe them by comparing with canonical sets of the “simplest form.”

We set $\Sigma_\gamma = \{x \in \mathbb{R} : x \geq 0\}$ for $\gamma \leq 1$ and

$$\Sigma_\gamma = \left\{ x \in \mathbb{R} : 0 \leq x < \sqrt{\frac{2}{\gamma - 1}} \right\} \quad \text{for } \gamma > 1. \quad (1.10)$$

For every $\gamma \in \mathbb{R}$, define the functions I_γ^- and I_γ^+ on $\Sigma_\gamma \times \Sigma_\gamma$ by

$$I_\gamma^-(x, y) = \begin{cases} \frac{x\sigma(x) - y\sigma(y)}{x - y} & \text{if } x \neq y, \\ \sigma(x) + \sigma'(x)x & \text{if } x = y, \end{cases} \quad (1.11)$$

$$I_\gamma^+(x, y) = \begin{cases} \frac{x\sigma(x) + y\sigma(y)}{x + y} & \text{if } x^2 + y^2 > 0, \\ 1 & \text{if } x = y = 0. \end{cases}$$

Note that the functions I_γ^- and I_γ^+ are continuous on the closing of $\Sigma_\gamma \times \Sigma_\gamma$ and they are infinitely differentiable at each inner point of $\Sigma_\gamma \times \Sigma_\gamma$.

For arbitrary $\varepsilon \geq 0$ we put $W_\gamma^-(\varepsilon) = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : I_\gamma^-(|\xi|, |\eta|) \geq \varepsilon\}$, $W_\gamma^+(\varepsilon) = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : I_\gamma^+(|\xi|, |\eta|) \geq \varepsilon\}$, $V_\gamma^-(\varepsilon) = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : I_\gamma^-(|\xi|, |\eta|) \leq \varepsilon\}$, $V_\gamma^+(\varepsilon) = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : I_\gamma^+(|\xi|, |\eta|) \leq \varepsilon\}$.

Also we will need the sets $D_\gamma = \{(\xi, \xi) \in \Omega_\gamma \times \Omega_\gamma\}$, $Q_\gamma = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : \xi\sigma(|\xi|) = \eta\sigma(|\eta|)\}$.

The main result of our paper are the following theorems.

THEOREM 1.1. *For every $\gamma \in \mathbb{R}$,*

$$(W_\gamma^-(\varepsilon) \cup D_\gamma) \subset \mathcal{A}_\gamma(\varepsilon) \subset (W_\gamma^+(\varepsilon) \cup D_\gamma) \quad \forall \varepsilon \in (0, 1), \quad (1.12)$$

$$\mathcal{A}_\gamma(\varepsilon) = D_\gamma \quad \forall \varepsilon \in [1, +\infty).$$

THEOREM 1.2. (a) *If $\gamma \in (-\infty, -1]$, then*

$$(V_\gamma^+(\varepsilon) \cup D_\gamma) \subset \mathcal{B}_\gamma(\varepsilon) \subset (V_\gamma^-(\varepsilon) \cup D_\gamma) \quad \forall \varepsilon \in (0, 1), \quad (1.13)$$

$$\mathcal{B}_\gamma(\varepsilon) = \mathbb{R}^{2n} \quad \forall \varepsilon \in [1, +\infty). \quad (1.14)$$

(b) *If $\gamma \in (-1, +\infty)$, then*

$$(V_\gamma^+(\varepsilon) \cap W_\gamma^-(0)) \subset \mathcal{B}_\gamma(\varepsilon) \subset (V_\gamma^-(\varepsilon) \cup Q_\gamma) \quad \forall \varepsilon \in (0, 1), \quad (1.15)$$

$$W_\gamma^-(0) \subset \mathcal{B}_\gamma(\varepsilon) \quad \forall \varepsilon \in [1, +\infty).$$

Relation (1.14) was first proved for $\gamma = -1$ and $\varepsilon = 1$ in [5] and later repeatedly in [6–9].

2. Properties of σ

Consider the equation

$$\theta'(t) = \varepsilon, \quad (2.1)$$

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where $\theta(t) = t\sigma(t)$ and ε is an arbitrary parameter. It is easy to verify that for $\gamma \neq 1$, (2.1) can be rewritten in the following form:

$$\frac{2}{\gamma-1}\sigma^{2-\gamma}(t) - \frac{\gamma+1}{\gamma-1}\sigma(t) + \varepsilon = 0. \quad (2.2)$$

For arbitrary $\varepsilon \in (0, 1)$ we set

$$r_\gamma(\varepsilon) = \sqrt{\frac{2(1-\varepsilon^{\gamma-1})}{\gamma-1}} \quad \text{if } \gamma \neq 1, \quad (2.3)$$

$$r_1(\varepsilon) = \sqrt{-2\ln\varepsilon}.$$

Observe that $r_\gamma(\varepsilon) \in \Sigma_\gamma$ for every $\gamma \in \mathbb{R}$ and every $\varepsilon \in (0, 1)$.

The following assertions hold.

(1) Let $\gamma \in \mathbb{R}$. Then the domain of σ is the set Σ_γ . Moreover, $\sigma(0) = 1$, $\sigma(+\infty) = 0$ for $\gamma \leq 1$ and $\sigma(\sqrt{2/(\gamma-1)}) = 0$ for $\gamma > 1$.

(2) For each $\gamma \in \mathbb{R}$ we have

$$0 < \sigma(t) \leq 1 \quad \forall t \in \Sigma_\gamma. \quad (2.4)$$

(3) Let $\gamma \in \mathbb{R}$. Then $\sigma'(0) = 0$ and

$$\sigma'(t) < 0 \quad \forall t > 0, t \in \Sigma_\gamma. \quad (2.5)$$

(4) If $\gamma \in (-\infty, -1]$, then

$$\begin{aligned} \theta'(0) &= 1, & \theta'(+\infty) &= 0, \\ \theta'(t) &> 0 & \forall t &\in [0, +\infty). \end{aligned} \quad (2.6)$$

(5) If $\gamma \in (-1, +\infty)$, then

$$\begin{aligned} \theta'(0) &= 1, & \theta' \left(\sqrt{\frac{2}{\gamma+1}} \right) &= 0, \\ \theta'(t) &> 0 & \forall t &\in \left(0, \sqrt{\frac{2}{\gamma+1}} \right), \\ \theta'(t) &< 0 & \forall t &> \sqrt{\frac{2}{\gamma+1}}, t \in \Sigma_\gamma. \end{aligned} \quad (2.7)$$

Moreover,

$$\begin{aligned}
 \theta'(+\infty) &= 0 \quad \text{if } \gamma \in (-1, 1], \\
 \theta' \left(\sqrt{\frac{2}{\gamma-1}} \right) &= 0 \quad \text{if } \gamma \in (1, 2), \\
 \theta' \left(\sqrt{\frac{2}{\gamma-1}} \right) &= -2 \quad \text{if } \gamma = 2, \\
 \theta' \left(\sqrt{\frac{2}{\gamma-1}} - 0 \right) &= -\infty \quad \text{if } \gamma \in (2, +\infty).
 \end{aligned} \tag{2.8}$$

(6) If $\gamma \in (-\infty, -1] \cup [2, +\infty)$, then $\theta''(0) = 0$ and

$$\theta''(t) < 0 \quad \forall t > 0, t \in \Sigma_\gamma. \tag{2.9}$$

(7) If $\gamma \in (-1, 2)$, then

$$\begin{aligned}
 \theta''(0) &= 0, \quad \theta'' \left(\sqrt{\frac{6}{\gamma+1}} \right) = 0, \\
 \theta''(t) &< 0 \quad \forall t \in \left(0, \sqrt{\frac{6}{\gamma+1}} \right), \\
 \theta''(t) &> 0 \quad \forall t > \sqrt{\frac{6}{\gamma+1}}, t \in \Sigma_\gamma.
 \end{aligned} \tag{2.10}$$

(8) For every $\gamma \in \mathbb{R}$ and every $\epsilon \in (0, 1)$, (2.1) has a unique positive solution $s_\gamma(\epsilon) \in (0, r_\gamma(\epsilon))$ and

$$\theta'(t) > \epsilon \quad \forall t \in [0, s_\gamma(\epsilon)), \quad \theta'(t) < \epsilon \quad \forall t > s_\gamma(\epsilon), t \in \Sigma_\gamma. \tag{2.11}$$

Moreover, for every $\gamma > -1$ and $\epsilon \in (0, 1)$,

$$s_\gamma(\epsilon) < \sqrt{\frac{2}{\gamma+1}}. \tag{2.12}$$

(9) Let $\gamma \in \mathbb{R}$. Then for all $x, y \in \Sigma_\gamma$, $x^2 + y^2 > 0$,

$$I_\gamma^-(x, y) \leq I_\gamma^+(x, y) < 1. \tag{2.13}$$

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Proof. The proof of assertions (1)–(7) follows from the equalities

$$\begin{aligned}
 \sigma'(t) &= \begin{cases} -t \left(1 - \frac{\gamma-1}{2} t^2\right)^{(2-\gamma)/(\gamma-1)} & \text{if } \gamma \neq 1, \\ -t \exp\left\{-\frac{1}{2} t^2\right\} & \text{if } \gamma = 1, \end{cases} \\
 \theta'(t) &= \begin{cases} \left(1 - \frac{\gamma+1}{2} t^2\right) \left(1 - \frac{\gamma-1}{2} t^2\right)^{(2-\gamma)/(\gamma-1)} & \text{if } \gamma \neq 1, \\ (1 - t^2) \exp\left\{-\frac{1}{2} t^2\right\} & \text{if } \gamma = 1, \end{cases} \\
 \theta''(t) &= \begin{cases} -t \left(3 - \frac{\gamma+1}{2} t^2\right) \left(1 - \frac{\gamma-1}{2} t^2\right)^{(3-2\gamma)/(\gamma-1)} & \text{if } \gamma \neq 1, \\ t(t^2 - 3) \exp\left\{-\frac{1}{2} t^2\right\} & \text{if } \gamma = 1. \end{cases}
 \end{aligned} \tag{2.14}$$

Let $\gamma \in \mathbb{R}$ and $\epsilon \in (0, 1)$. Suppose that $s_\gamma(\epsilon) \in \Sigma_\gamma$ satisfies (2.1). We have

$$\sigma(r_\gamma(\epsilon)) = \epsilon = \theta'(s_\gamma(\epsilon)) = \sigma(s_\gamma(\epsilon)) + s_\gamma(\epsilon) \sigma'(s_\gamma(\epsilon)) < \sigma(s_\gamma(\epsilon)). \tag{2.15}$$

From this $s_\gamma(\epsilon) < r_\gamma(\epsilon)$. Next, using assertions (4)–(7), we obtain assertion (8).

We prove assertion (9). Let $x, y \in \Sigma_\gamma$, $x^2 + y^2 > 0$. If $x = y$, then

$$I_\gamma^-(x, y) = \sigma(x) + x \sigma'(x) < \sigma(x) = I_\gamma^+(x, y) < 1. \tag{2.16}$$

Suppose that $x > y$. Since

$$\sigma(x) < \sigma(y), \tag{2.17}$$

we obtain

$$\begin{aligned}
 I_\gamma^-(x, y) &= \frac{x\sigma(x) - y\sigma(y)}{x - y} \leq \frac{x\sigma(x) - y\sigma(x)}{x - y} = \sigma(x) \\
 &= \frac{x\sigma(x) + y\sigma(x)}{x + y} \leq \frac{x\sigma(x) + y\sigma(y)}{x + y} = I_\gamma^+(x, y) \\
 &< \frac{x\sigma(y) + y\sigma(y)}{x + y} = \sigma(y) \leq 1.
 \end{aligned} \tag{2.18}$$

The case $x < y$ is analogous. □

3. Properties of $W_\gamma^-(\epsilon)$, $W_\gamma^+(\epsilon)$, $V_\gamma^-(\epsilon)$, and $V_\gamma^+(\epsilon)$

Here we study the sets $W_\gamma^-(\epsilon)$, $W_\gamma^+(\epsilon)$, $V_\gamma^-(\epsilon)$ and $V_\gamma^+(\epsilon)$.

We say that a set $G \subset \mathbf{R}^n$ is *linearly connected* if any pair of points $x, y \in G$ can be joined on D by an arc.

The following assertions hold.

- (1) $W_\gamma^-(\varepsilon) = W_\gamma^+(\varepsilon) = \emptyset$ for every $\gamma \in \mathbb{R}$ and $\varepsilon > 1$.
- (2) $W_\gamma^-(1) = W_\gamma^+(1) = \{0\}$ for every $\gamma \in \mathbb{R}$.
- (3) $W_\gamma^-(0) = \mathbb{R}^4$ for every $\gamma \leq -1$.
- (4) $W_\gamma^+(0) = \Omega_\gamma \times \Omega_\gamma$ for every $\gamma \in \mathbb{R}$.
- (5) $W_\gamma^-(\varepsilon) \subset W_\gamma^+(\varepsilon)$ for every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$.
- (6) $V_\gamma^-(\varepsilon) = V_\gamma^+(\varepsilon) = \Omega_\gamma \times \Omega_\gamma$ for every $\gamma \in \mathbb{R}$ and $\varepsilon \geq 1$.
- (7) $V_\gamma^-(0) = \emptyset$ for every $\gamma \leq -1$.
- (8) $V_\gamma^+(0) = \emptyset$ for every $\gamma \in \mathbb{R}$.
- (9) $V_\gamma^+(\varepsilon) \subset V_\gamma^-(\varepsilon)$ for every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$.
- (10) The set $W_\gamma^-(\varepsilon)$ is linearly connected for every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$.
- (11) The set $W_\gamma^-(0)$ is linearly connected for every $\gamma > -1$.
- (12) The set $W_\gamma^+(\varepsilon)$ is linearly connected for every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$.
- (13) For every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$, we have

$$\{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \leq s_\gamma(\varepsilon), |\eta| \leq s_\gamma(\varepsilon)\} \subset W_\gamma^-(\varepsilon). \quad (3.1)$$

Here $s_\gamma(\varepsilon)$ is a unique positive solution of (2.1).

- (14) For every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$, we have

$$W_\gamma^-(\varepsilon) \subset \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \leq r_\gamma(\varepsilon), |\eta| \leq r_\gamma(\varepsilon)\}. \quad (3.2)$$

- (15) If $\gamma > -1$, then

$$\left\{ (\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \leq \sqrt{\frac{2}{\gamma+1}}, |\eta| \leq \sqrt{\frac{2}{\gamma+1}} \right\} \subset W_\gamma^-(0). \quad (3.3)$$

- (16) For every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$, we have

$$\{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \leq r_\gamma(\varepsilon), |\eta| \leq r_\gamma(\varepsilon)\} \subset W_\gamma^+(\varepsilon). \quad (3.4)$$

- (17) For every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$, we have

$$V_\gamma^-(\varepsilon) \subset \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \geq s_\gamma(\varepsilon) \text{ or } |\eta| \geq s_\gamma(\varepsilon)\}. \quad (3.5)$$

- (18) For every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$, we have

$$\{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \geq r_\gamma(\varepsilon) \text{ or } |\eta| \geq r_\gamma(\varepsilon)\} \subset V_\gamma^-(\varepsilon). \quad (3.6)$$

- (19) If $\gamma > -1$, then

$$V_\gamma^-(0) \subset \left\{ (\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \geq \sqrt{\frac{2}{\gamma+1}} \text{ or } |\eta| \geq \sqrt{\frac{2}{\gamma+1}} \right\}. \quad (3.7)$$

- (20) For every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$, we have

$$V_\gamma^+(\varepsilon) \subset \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \geq r_\gamma(\varepsilon) \text{ or } |\eta| \geq r_\gamma(\varepsilon)\}. \quad (3.8)$$

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Proof of assertions (1)–(9). The proof follows from assertions (4) and (9) of Section 2. \square

Proof of assertions (10)–(12). We prove assertion (10). Fix $\gamma \in \mathbb{R}$, $\varepsilon \in (0, 1)$, and a nonzero point $\zeta = (\xi, \eta) \in W_\gamma^-(\varepsilon)$. To prove the statement, it is sufficient to show that $W_\gamma^-(\varepsilon)$ contains the segment $\mathcal{L} = \{(\xi t, \eta t) : 0 \leq t \leq 1\}$ with the endpoints 0 and ζ .

Indeed, let $\zeta', \zeta'' \in W_\gamma^-(\varepsilon)$ be arbitrary. Let $\mathcal{L}', \mathcal{L}''$ be the segments with the endpoints 0, ζ' and 0, ζ'' , respectively. Denote by $\mathcal{L}' \cup \mathcal{L}''$ the double curve which consists of two segments \mathcal{L}' and \mathcal{L}'' . Then this double curve will join the points ζ', ζ'' and it will lie on $W_\gamma^-(\varepsilon)$.

Assume that $I_\gamma(x, y) \geq \varepsilon$. As above, for the case $x > y$ we obtain

$$\varepsilon \leq I_\gamma^-(x, y) \leq \sigma(x) < \sigma(y). \quad (3.9)$$

From this $x, y \in [0, r_\gamma(\varepsilon)]$. The case $x < y$ is analogous. Suppose that $x = y$. Then

$$\varepsilon \leq I_\gamma^-(x, y) = \theta'(x) = \sigma(x) + x\sigma'(x) \leq \sigma(x) = \sigma(y), \quad (3.10)$$

and consequently $x, y \in [0, r_\gamma(\varepsilon)]$. Thus if $I_\gamma(x, y) \geq \varepsilon$, then $x, y \in [0, r_\gamma(\varepsilon)]$.

Further we will need the function

$$\mu(x) = x(\sigma(x) - \varepsilon). \quad (3.11)$$

It is easy to see that for all $x, y \in [0, r_\gamma(\varepsilon)]$, $x \neq y$,

$$I_\gamma^-(x, y) = \varepsilon \iff \mu(x) = \mu(y). \quad (3.12)$$

Define the monotonicity intervals of μ . Since

$$\mu'(x) = \theta'(x) - \varepsilon, \quad (3.13)$$

from assertion (8) of Section 2 it follows that the function μ is strictly increasing on $[0, s_\gamma(\varepsilon)]$ and strictly decreasing on $[s_\gamma(\varepsilon), r_\gamma(\varepsilon)]$. Moreover,

$$\mu(0) = \mu(r_\gamma(\varepsilon)) = 0. \quad (3.14)$$

Note that if $I_\gamma^-(x, y) = \varepsilon$ and $x = y$, then $x = y = s_\gamma(\varepsilon)$. Consequently for each $x \in [0, r_\gamma(\varepsilon)]$ there is a unique number $y \in [0, r_\gamma(\varepsilon)]$, satisfying (3.12). Therefore there exists the function $g : [0, r_\gamma(\varepsilon)] \rightarrow [0, r_\gamma(\varepsilon)]$ such that for all $x, y \in [0, r_\gamma(\varepsilon)]$,

$$I_\gamma^-(x, y) = \varepsilon \iff y = g(x). \quad (3.15)$$

In addition

$$\begin{aligned} s_\gamma(\varepsilon) < g(x) \leq r_\gamma(\varepsilon) & \quad \text{if } x \in [0, s_\gamma(\varepsilon)], \\ 0 \leq g(x) < s_\gamma(\varepsilon) & \quad \text{if } x \in (s_\gamma(\varepsilon), r_\gamma(\varepsilon)), \end{aligned} \quad (3.16)$$

as well as

$$g(0) = r_\gamma(\varepsilon), \quad g(s_\gamma(\varepsilon)) = s_\gamma(\varepsilon), \quad g(r_\gamma(\varepsilon)) = 0. \quad (3.17)$$

Note that the function $I_\gamma^-(x, y)$ is infinitely differentiable at each point of $[0, r_\gamma(\varepsilon)] \times [0, r_\gamma(\varepsilon)]$. Fix arbitrary $x_0, y_0 \in [0, r_\gamma(\varepsilon)]$, $x_0 \neq y_0$, satisfying (3.15). We have

$$\begin{aligned} \frac{\partial}{\partial x} I_\gamma^-(x_0, y_0) &= \frac{\theta'(x_0)(x_0 - y_0) - (\theta(x_0) - \theta(y_0))}{(x_0 - y_0)^2} = \frac{\theta'(x_0) - \varepsilon}{x_0 - y_0} \neq 0, \\ \frac{\partial}{\partial y} I_\gamma^-(x_0, y_0) &= \frac{\theta'(y_0)(y_0 - x_0) - (\theta(y_0) - \theta(x_0))}{(y_0 - x_0)^2} = \frac{\theta'(y_0) - \varepsilon}{y_0 - x_0} \neq 0. \end{aligned} \quad (3.18)$$

Using the implicit function theorem, we obtain

$$g'(x_0) = - \left[\frac{\partial}{\partial y} I_\gamma^-(x_0, g(x_0)) \right]^{-1} \left[\frac{\partial}{\partial x} I_\gamma^-(x_0, g(x_0)) \right] = \frac{\theta'(x_0) - \varepsilon}{\theta'(g(x_0)) - \varepsilon}. \quad (3.19)$$

By assertion (8) of Section 2, (3.16), it follows that

$$g'(x_0) < 0. \quad (3.20)$$

Thus the function $y = g(x)$ is strictly decreasing on $[0, r_\gamma(\varepsilon)]$.

We prove that the segment \mathcal{L} lies in $W_\gamma^-(\varepsilon)$.

Indeed, assume that $|\xi| \leq |\eta|$ and for some $t \in (0, 1)$,

$$I_\gamma^-(|\xi t|, |\eta t|) < \varepsilon. \quad (3.21)$$

Then there is a number $t_0 \in (0, 1)$ such that

$$I_\gamma^-(|\xi t_0|, |\eta t_0|) = \varepsilon \quad (3.22)$$

and hence $|\eta t_0| = g(|\xi t_0|)$.

Since $0 < r_\gamma(\varepsilon) = g(0)$ and $I_\gamma^-(0, 0) = 1 > \varepsilon$, we have $|\eta| \leq g(|\xi|)$. We deduce

$$\frac{g(|\xi|)}{t_0} \leq \frac{g(|\xi t_0|)}{t_0} = |\eta| \leq g(|\xi|). \quad (3.23)$$

From this $t_0 \geq 1$ and we arrive at a contradiction. The case $|\xi| > |\eta|$ is analogous. Thus $W_\gamma^-(\varepsilon)$ contains \mathcal{L} .

The proof of assertion (11) is analogous.

Now we prove assertion (12). We fix $\gamma \in \mathbb{R}$, $\varepsilon \in (0, 1)$, and a nonzero point $\zeta = (\xi, \eta) \in W_\gamma^+(\varepsilon)$. As above, to prove this statement, it is sufficient to show that $W_\gamma^+(\varepsilon)$ contains the segment \mathcal{L} . We have

$$I_\gamma^+(|\xi t|, |\eta t|) = \frac{|\xi| \sigma(|\xi t|) + |\eta| \sigma(|\eta t|)}{|\xi| + |\eta|} > \frac{|\xi| \sigma(|\xi|) + |\eta| \sigma(|\eta|)}{|\xi| + |\eta|} \geq \varepsilon \quad (3.24)$$

for all $t \in (0, 1)$. Thus $W_\gamma^+(\varepsilon)$ contains \mathcal{L} . \square

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Proof of assertions (13), (15), (17), and (19). Let

$$(\xi, \eta) \in \{(\xi, \eta) \in \Omega_y \times \Omega_y : |\xi| \leq s_y(\varepsilon), |\eta| \leq s_y(\varepsilon)\}. \quad (3.25)$$

By assertion (8) of Section 2 it follows that

$$\theta'(|\xi|) \geq \varepsilon, \quad \theta'(|\eta|) \geq \varepsilon. \quad (3.26)$$

Suppose that $|\xi| = |\eta|$. We have

$$I_y^-(|\xi|, |\eta|) = \theta'(|\xi|) = \theta'(|\eta|) \geq \varepsilon. \quad (3.27)$$

From this $(\xi, \eta) \in W_y^-(\varepsilon)$.

Assume that $|\xi| < |\eta|$. Using the well-known Lagrange mean value theorem, we obtain

$$I_y^-(|\xi|, |\eta|) = \theta'(c), \quad |\xi| < c < |\eta|. \quad (3.28)$$

By assertion (8) of Section 2,

$$\theta'(c) > \varepsilon. \quad (3.29)$$

Therefore $(\xi, \eta) \in W_y^-(\varepsilon)$. The case $|\xi| > |\eta|$ is analogous.

The proof of assertion (15) is analogous. Assertion (17) follows from assertion (13), and assertion (19) follows from assertion (15). \square

Proof of assertions (14) and (18). Let $(\xi, \eta) \in W_y^-(\varepsilon)$. Assume that $|\xi| = |\eta|$. We have

$$\varepsilon \leq I_y^-(|\xi|, |\eta|) = \theta'(|\xi|) = \sigma(|\xi|) + |\xi|\sigma'(|\xi|) \leq \sigma(|\xi|) = \sigma(|\eta|). \quad (3.30)$$

Then the inequalities

$$\sigma(|\xi|) = \sigma(|\eta|) \geq \varepsilon \quad (3.31)$$

imply

$$|\xi| = |\eta| \leq r_y(\varepsilon). \quad (3.32)$$

Hence

$$(\xi, \eta) \in \{(\xi, \eta) \in \Omega_y \times \Omega_y : |\xi| \leq r_y(\varepsilon), |\eta| \leq r_y(\varepsilon)\}. \quad (3.33)$$

Now we assume that $|\xi| > |\eta|$. We have

$$\varepsilon \leq I_y^-(|\xi|, |\eta|) = \frac{|\xi|\sigma(|\xi|) - |\eta|\sigma(|\eta|)}{|\xi| - |\eta|} \leq \frac{|\xi|\sigma(|\xi|) - |\eta|\sigma(|\xi|)}{|\xi| - |\eta|} = \sigma(|\xi|) < \sigma(|\eta|). \quad (3.34)$$

From this

$$(\xi, \eta) \in \{(\xi, \eta) \in \Omega_y \times \Omega_y : |\xi| \leq r_y(\varepsilon), |\eta| \leq r_y(\varepsilon)\}. \quad (3.35)$$

The case $|\xi| < |\eta|$ is analogous.

Assertion (18) follows from assertion (14). \square

Proof of assertions (16) and (20). Let

$$(\xi, \eta) \in \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \leq r_\gamma(\varepsilon), |\eta| \leq r_\gamma(\varepsilon)\}. \quad (3.36)$$

Then

$$\sigma(|\xi|) \geq \varepsilon, \quad \sigma(|\eta|) \geq \varepsilon. \quad (3.37)$$

Suppose $|\xi| = |\eta|$. Then

$$I_\gamma^+(|\xi|, |\eta|) = \sigma(|\xi|) \geq \varepsilon. \quad (3.38)$$

Hence $(\xi, \eta) \in W_\gamma^+(\varepsilon)$.

Assume that $|\xi| > |\eta|$. We have

$$I_\gamma^+(|\xi|, |\eta|) = \frac{|\xi|\sigma(|\xi|) + |\eta|\sigma(|\eta|)}{|\xi| + |\eta|} \geq \frac{|\xi|\sigma(|\xi|) + |\eta|\sigma(|\xi|)}{|\xi| + |\eta|} = \sigma(|\xi|) \geq \varepsilon. \quad (3.39)$$

From this $(\xi, \eta) \in W_\gamma^+(\varepsilon)$. The case $|\xi| < |\eta|$ is analogous.

Assertion (20) follows from assertion (16). \square

4. Proofs of main theorems

Introduce the sets $H_\gamma = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| = |\eta|, \xi \neq \eta\}$, $G_\gamma = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \neq |\eta|\}$, $U_\gamma^- = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : I_\gamma^-(|\xi|, |\eta|) < 0\}$, $U_\gamma^+ = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : I_\gamma^-(|\xi|, |\eta|) > 0\}$, $P_\gamma = \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi|\sigma(|\xi|) = |\eta|\sigma(|\eta|), \xi\sigma(|\xi|) \neq \eta\sigma(|\eta|)\}$, $F_\gamma^+(\varepsilon) = (V_\gamma^+(\varepsilon) \cap U_\gamma^+) \cup Q_\gamma \cup (V_\gamma^+(\varepsilon) \cap P_\gamma)$, $F_\gamma^-(\varepsilon) = (V_\gamma^-(\varepsilon) \cap U_\gamma^-) \cup Q_\gamma \cup (V_\gamma^-(\varepsilon) \cap P_\gamma) \cup (V_\gamma^+(\varepsilon) \cap U_\gamma^-)$.

For any $\xi, \eta \in \mathbb{R}^n$, their inner product is denoted by $\langle \xi, \eta \rangle$. Obviously inequalities (1.6) and (1.7) with some constant $\varepsilon > 0$ can be written as

$$\varepsilon|\xi - \eta|^2 \leq \langle \sigma(|\xi|)\xi - \sigma(|\eta|)\eta, \xi - \eta \rangle, \quad (4.1)$$

$$|\sigma(|\xi|)\xi - \sigma(|\eta|)\eta|^2 \leq \varepsilon \langle \sigma(|\xi|)\xi - \sigma(|\eta|)\eta, \xi - \eta \rangle, \quad (4.2)$$

respectively. Let φ be the angle between the vectors ξ and η . Then

$$|\xi - \eta|^2 = |\xi|^2 + |\eta|^2 - 2|\xi||\eta|\cos\varphi,$$

$$\langle \sigma(|\xi|)\xi - \sigma(|\eta|)\eta, \xi - \eta \rangle = \sigma(|\xi|)|\xi|^2 + \sigma(|\eta|)|\eta|^2 - (\sigma(|\xi|) + \sigma(|\eta|))|\xi||\eta|\cos\varphi,$$

$$|\sigma(|\xi|)\xi - \sigma(|\eta|)\eta|^2 = \sigma^2(|\xi|)|\xi|^2 + \sigma^2(|\eta|)|\eta|^2 - 2\sigma(|\xi|)\sigma(|\eta|)|\xi||\eta|\cos\varphi. \quad (4.3)$$

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We set

$$\begin{aligned}\Upsilon(\varphi) &= |\xi|^2 + |\eta|^2 - 2|\xi||\eta|\cos\varphi, \\ \Phi(\varphi) &= \sigma(|\xi|)|\xi|^2 + \sigma(|\eta|)|\eta|^2 - (\sigma(|\xi|) + \sigma(|\eta|))|\xi||\eta|\cos\varphi, \\ \Psi(\varphi) &= \sigma^2(|\xi|)|\xi|^2 + \sigma^2(|\eta|)|\eta|^2 - 2\sigma(|\xi|)\sigma(|\eta|)|\xi||\eta|\cos\varphi.\end{aligned}\quad (4.4)$$

Proof of Theorem 1.1. Fix $\gamma \in \mathbb{R}$ and $\varepsilon > 0$. It is clear that inequality (4.1) holds for all $(\xi, \eta) \in D_\gamma$.

Let $(\xi, \eta) \in \mathcal{A}_\gamma(\varepsilon) \cap H_\gamma$. In this case inequality (4.1) is rewritten in the form

$$\varepsilon \leq \sigma(|\xi|) = \sigma(|\eta|). \quad (4.5)$$

Obviously

$$\mathcal{A}_\gamma(\varepsilon) \cap H_\gamma = W_\gamma^+(\varepsilon) \cap H_\gamma. \quad (4.6)$$

Using assertion (5) of Section 3, we see that

$$(W_\gamma^-(\varepsilon) \cap H_\gamma) \subset (\mathcal{A}_\gamma(\varepsilon) \cap H_\gamma) \subset (W_\gamma^+(\varepsilon) \cap H_\gamma). \quad (4.7)$$

Let $(\xi, \eta) \in G_\gamma$. Then $\Upsilon(\varphi) > 0$ and after simple calculations we find

$$\frac{\partial}{\partial \varphi} \left(\frac{\Phi(\varphi)}{\Upsilon(\varphi)} \right) = \frac{(\sigma(|\eta|) - \sigma(|\xi|))(|\xi|^2 - |\eta|^2)|\xi||\eta|\sin\varphi}{\Upsilon^2(\varphi)}. \quad (4.8)$$

It is clear that

$$(\sigma(|\eta|) - \sigma(|\xi|))(|\xi|^2 - |\eta|^2) > 0. \quad (4.9)$$

Therefore

$$\begin{aligned}\min_{\varphi \in [0, \pi]} \left(\frac{\Phi(\varphi)}{\Upsilon(\varphi)} \right) &= \frac{\Phi(0)}{\Upsilon(0)} \\ &= \frac{\sigma(|\xi|)|\xi|^2 + \sigma(|\eta|)|\eta|^2 - (\sigma(|\xi|) + \sigma(|\eta|))|\xi||\eta|}{(|\xi| - |\eta|)^2} = I_\gamma^-(|\xi|, |\eta|), \\ \max_{\varphi \in [0, \pi]} \left(\frac{\Phi(\varphi)}{\Upsilon(\varphi)} \right) &= \frac{\Phi(\pi)}{\Upsilon(\pi)} \\ &= \frac{\sigma(|\xi|)|\xi|^2 + \sigma(|\eta|)|\eta|^2 + (\sigma(|\xi|) + \sigma(|\eta|))|\xi||\eta|}{(|\xi| + |\eta|)^2} = I_\gamma^+(|\xi|, |\eta|).\end{aligned}\quad (4.10)$$

Thus for all $(\xi, \eta) \in G_\gamma$,

$$I_\gamma^-(|\xi|, |\eta|) \leq \frac{\langle \sigma(|\xi|)\xi - \sigma(|\eta|)\eta, \xi - \eta \rangle}{|\xi - \eta|^2} \leq I_\gamma^+(|\xi|, |\eta|). \quad (4.11)$$

This implies

$$(W_{\gamma}^{-}(\varepsilon) \cap G_{\gamma}) \subset (\mathcal{A}_{\gamma}(\varepsilon) \cap G_{\gamma}) \subset (W_{\gamma}^{+}(\varepsilon) \cap G_{\gamma}). \quad (4.12)$$

From this, by (4.7), and assertions (1), (2) of Section 3 we obtain (1.12). \square

Proof of Theorem 1.2. (a) We fix $\gamma \leq -1$ and $\varepsilon > 0$. It is clear that inequality (4.2) holds for all $(\xi, \eta) \in D_{\gamma}$.

Let $(\xi, \eta) \in \mathcal{B}_{\gamma}(\varepsilon) \cap H_{\gamma}$. In this case inequality (4.2) becomes

$$\sigma(|\xi|) = \sigma(|\eta|) \leq \varepsilon. \quad (4.13)$$

Then

$$\mathcal{B}_{\gamma}(\varepsilon) \cap H_{\gamma} = V_{\gamma}^{+}(\varepsilon) \cap H_{\gamma}. \quad (4.14)$$

Using assertion (9) of Section 3, we see that

$$(V_{\gamma}^{+}(\varepsilon) \cap H_{\gamma}) \subset (\mathcal{B}_{\gamma}(\varepsilon) \cap H_{\gamma}) \subset (V_{\gamma}^{-}(\varepsilon) \cap H_{\gamma}). \quad (4.15)$$

Let $(\xi, \eta) \in G_{\gamma}$. Then by the inequality

$$\Psi(\varphi) \geq (\sigma(|\xi|)|\xi| - \sigma(|\eta|)|\eta|)^2 \quad (4.16)$$

and by assertion (4) of Section 2, we conclude that $\Psi(\varphi) > 0$ for all $\varphi \in [0, \pi]$. After simple calculations, we obtain

$$\frac{\partial}{\partial \varphi} \left(\frac{\Phi(\varphi)}{\Psi(\varphi)} \right) = \frac{(\sigma(|\xi|) - \sigma(|\eta|))(|\xi|^2 \sigma^2(|\xi|) - |\eta|^2 \sigma^2(|\eta|))|\xi||\eta| \sin \varphi}{\Psi^2(\varphi)}. \quad (4.17)$$

By assertions (3) and (4) of Section 2, it follows that

$$(\sigma(|\xi|) - \sigma(|\eta|))(|\xi|^2 \sigma^2(|\xi|) - |\eta|^2 \sigma^2(|\eta|)) < 0. \quad (4.18)$$

Therefore

$$\begin{aligned} \min_{\varphi \in [0, \pi]} \left(\frac{\Phi(\varphi)}{\Psi(\varphi)} \right) &= \frac{\Phi(\pi)}{\Psi(\pi)} \\ &= \frac{\sigma(|\xi|)|\xi|^2 + \sigma(|\eta|)|\eta|^2 + (\sigma(|\xi|) + \sigma(|\eta|))|\xi||\eta|}{\sigma^2(|\xi|)|\xi|^2 + \sigma^2(|\eta|)|\eta|^2 + 2\sigma(|\xi|)\sigma(|\eta|)|\xi||\eta|} = \frac{1}{I_{\gamma}^{+}(|\xi|, |\eta|)}, \\ \max_{\varphi \in [0, \pi]} \left(\frac{\Phi(\varphi)}{\Psi(\varphi)} \right) &= \frac{\Phi(0)}{\Psi(0)} \\ &= \frac{\sigma(|\xi|)|\xi|^2 + \sigma(|\eta|)|\eta|^2 - (\sigma(|\xi|) + \sigma(|\eta|))|\xi||\eta|}{\sigma^2(|\xi|)|\xi|^2 + \sigma^2(|\eta|)|\eta|^2 - 2\sigma(|\xi|)\sigma(|\eta|)|\xi||\eta|} = \frac{1}{I_{\gamma}^{-}(|\xi|, |\eta|)}. \end{aligned} \quad (4.19)$$

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Thus for all $(\xi, \eta) \in G_\gamma$,

$$\frac{1}{I_\gamma^+(\lvert\xi\rvert, \lvert\eta\rvert)} \leq \frac{\langle \sigma(\lvert\xi\rvert)\xi - \sigma(\lvert\eta\rvert)\eta, \xi - \eta \rangle}{\lvert \sigma(\lvert\xi\rvert)\xi - \sigma(\lvert\eta\rvert)\eta \rvert^2} \leq \frac{1}{I_\gamma^-(\lvert\xi\rvert, \lvert\eta\rvert)}. \quad (4.20)$$

This implies that

$$(V_\gamma^+(\varepsilon) \cap G_\gamma) \subset (\mathcal{B}_\gamma(\varepsilon) \cap G_\gamma) \subset (V_\gamma^-(\varepsilon) \cap G_\gamma). \quad (4.21)$$

From this, by (4.15) and assertion (6) of Section 3, we obtain (1.13) and (1.14).

(b) We fix $\gamma > -1$ and $\varepsilon > 0$. It is clear that the inequality (4.2) holds for all $(\xi, \eta) \in Q_\gamma$. By assertion (5) of Section 2, $Q_\gamma \neq D_\gamma$.

Let $(\xi, \eta) \in \mathcal{B}_\gamma(\varepsilon) \cap P_\gamma$. Similarly we establish that $P_\gamma \neq H_\gamma$. We have

$$\begin{aligned} \Psi(\varphi) &= \sigma^2(\lvert\xi\rvert)\lvert\xi\rvert^2 + \sigma^2(\lvert\eta\rvert)\lvert\eta\rvert^2 - 2\sigma(\lvert\xi\rvert)\sigma(\lvert\eta\rvert)\lvert\xi\rvert\lvert\eta\rvert \cos\varphi = 2\sigma^2(\lvert\xi\rvert)\lvert\xi\rvert^2(1 - \cos\varphi), \\ \Phi(\varphi) &= \sigma(\lvert\xi\rvert)\lvert\xi\rvert^2 + \sigma(\lvert\eta\rvert)\lvert\eta\rvert^2 - (\sigma(\lvert\xi\rvert) + \sigma(\lvert\eta\rvert))\lvert\xi\rvert\lvert\eta\rvert \cos\varphi \\ &= \sigma(\lvert\xi\rvert)\lvert\xi\rvert^2 + \sigma(\lvert\xi\rvert)\lvert\xi\rvert\lvert\eta\rvert - \sigma(\lvert\xi\rvert)\lvert\xi\rvert\lvert\eta\rvert \cos\varphi - \sigma(\lvert\xi\rvert)\lvert\xi\rvert^2 \cos\varphi \\ &= \sigma(\lvert\xi\rvert)\lvert\xi\rvert(\lvert\xi\rvert + \lvert\eta\rvert)(1 - \cos\varphi). \end{aligned} \quad (4.22)$$

It is easy to see that $\cos\varphi \neq 1$. Indeed, if $\cos\varphi = 1$, then $\xi\sigma(\lvert\xi\rvert) = \eta\sigma(\lvert\eta\rvert)$. Next, we find

$$\frac{\Psi(\varphi)}{\Phi(\varphi)} = \frac{2\lvert\xi\rvert\sigma(\lvert\xi\rvert)}{\lvert\xi\rvert + \lvert\eta\rvert} = I_\gamma^+(\lvert\xi\rvert, \lvert\eta\rvert). \quad (4.23)$$

Thus inequality (4.2) assumes the form

$$I_\gamma^+(\lvert\xi\rvert, \lvert\eta\rvert) \leq \varepsilon. \quad (4.24)$$

Then

$$\mathcal{B}_\gamma(\varepsilon) \cap P_\gamma = V_\gamma^+(\varepsilon) \cap P_\gamma. \quad (4.25)$$

Let $(\xi, \eta) \in U_\gamma^+$. By assertion (3) of Section 2 we find that inequality (4.18) is valid. Therefore inequalities (4.20) are true. Hence

$$(V_\gamma^+(\varepsilon) \cap U_\gamma^+) \subset (\mathcal{B}_\gamma(\varepsilon) \cap U_\gamma^+) \subset (V_\gamma^-(\varepsilon) \cap U_\gamma^+). \quad (4.26)$$

Consider the remaining case $(\xi, \eta) \in U_\gamma^-$. Observe that the set U_γ^- is not empty. It is easy to see that

$$(\sigma(\lvert\xi\rvert) - \sigma(\lvert\eta\rvert))(\lvert\xi\rvert^2\sigma^2(\lvert\xi\rvert) - \lvert\eta\rvert^2\sigma^2(\lvert\eta\rvert)) > 0. \quad (4.27)$$

Hence for all $(\xi, \eta) \in U_\gamma^-$,

$$\frac{1}{I_\gamma^-(\lvert\xi\rvert, \lvert\eta\rvert)} \leq \frac{\langle \sigma(\lvert\xi\rvert)\xi - \sigma(\lvert\eta\rvert)\eta, \xi - \eta \rangle}{\lvert \sigma(\lvert\xi\rvert)\xi - \sigma(\lvert\eta\rvert)\eta \rvert^2} \leq \frac{1}{I_\gamma^+(\lvert\xi\rvert, \lvert\eta\rvert)}, \quad (4.28)$$

which implies that

$$(\mathcal{B}_\gamma(\varepsilon) \cap U_\gamma^-) \subset (V_\gamma^+(\varepsilon) \cap U_\gamma^-). \quad (4.29)$$

From this, by (4.25) and (4.26),

$$F_\gamma^+(\varepsilon) \subset \mathcal{B}_\gamma(\varepsilon) \subset F_\gamma^-(\varepsilon). \quad (4.30)$$

It is not hard to establish that

$$W_\gamma^-(0) \subset (P_\gamma \cup Q_\gamma \cup U_\gamma^+), \quad (P_\gamma \cup Q_\gamma \cup U_\gamma^+ \cup U_\gamma^-) = \Omega_\gamma \times \Omega_\gamma. \quad (4.31)$$

Then, using assertion (9) of Section 3, we find

$$(V_\gamma^+(\varepsilon) \cap W_\gamma^-(0)) \subset F_\gamma^+(\varepsilon), \quad F_\gamma^-(\varepsilon) \subset (V_\gamma^-(\varepsilon) \cup Q_\gamma). \quad (4.32)$$

From this, by assertion (6) of Section 3 we obtain (1.15). \square

5. Properties of $x_\gamma(\varepsilon)$

For every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$ we set

$$\begin{aligned} X_\gamma(\varepsilon) &= \{x \in \Sigma_\gamma : \exists y \in \Sigma_\gamma, I_\gamma^+(x, y) \geq \varepsilon\}, \\ x_\gamma(\varepsilon) &= \sup_x X_\gamma(\varepsilon). \end{aligned} \quad (5.1)$$

If $x_\gamma(\varepsilon) \in \Sigma_\gamma$, then the following relations are true:

$$\begin{aligned} W_\gamma^+(\varepsilon) &\subset \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \leq x_\gamma(\varepsilon), |\eta| \leq x_\gamma(\varepsilon)\}, \\ \{(\xi, \eta) \in \Omega_\gamma \times \Omega_\gamma : |\xi| \geq x_\gamma(\varepsilon) \text{ or } |\eta| \geq x_\gamma(\varepsilon)\} &\subset V_\gamma^+(\varepsilon). \end{aligned} \quad (5.2)$$

For every $\gamma \in \mathbb{R}$ we will study the function $x_\gamma(\varepsilon)$ of variable ε . Let $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$. Then

$$I_\gamma^+(0, r_\gamma(\varepsilon)) = \sigma(r_\gamma(\varepsilon)) = \varepsilon, \quad (5.3)$$

where $r_\gamma(\varepsilon)$ is defined in Section 2. From this, $r_\gamma(\varepsilon) \in X_\gamma(\varepsilon)$. Therefore for every $\gamma \in \mathbb{R}$ the function $x_\gamma(\varepsilon)$ is defined everywhere on $(0, 1)$. Moreover, for every $\gamma \in \mathbb{R}$,

$$r_\gamma(\varepsilon) \leq x_\gamma(\varepsilon) \quad \forall \varepsilon \in (0, 1). \quad (5.4)$$

As above, let $s_\gamma(\varepsilon)$ be a unique positive solution of (2.1) for every $\gamma \in \mathbb{R}$ and $\varepsilon \in (0, 1)$. For $\gamma > 1$ we put

$$\hat{\varepsilon}_\gamma = \max_{y \in [0, \sqrt{2/(\gamma-1)}]} I_\gamma^+\left(\sqrt{\frac{2}{\gamma-1}}, y\right). \quad (5.5)$$

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The function $x_\gamma(\varepsilon)$ has the following properties.

(1) Let $\gamma > 1$. Then

$$x_\gamma(\varepsilon) = \sqrt{\frac{2}{\gamma-1}} \quad \forall \varepsilon \in (0, \hat{\varepsilon}_\gamma], \quad (5.6)$$

$$x_\gamma(\varepsilon) < \sqrt{\frac{2}{\gamma-1}} \quad \forall \varepsilon \in (\hat{\varepsilon}_\gamma, 1). \quad (5.7)$$

(2) Let

$$\gamma \in (-\infty, 1], \quad \varepsilon \in (0, 1), \quad (5.8)$$

or

$$\gamma \in (1, +\infty), \quad \varepsilon \in (\hat{\varepsilon}_\gamma, 1). \quad (5.9)$$

Then $x_\gamma(\varepsilon) \in \Sigma_\gamma$ and

$$I_\gamma^+(x_\gamma(\varepsilon), s_\gamma(\varepsilon)) = \varepsilon. \quad (5.10)$$

(3) For every $\gamma > 1$ we have

$$\lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} x_\gamma(\varepsilon) = \sqrt{\frac{2}{\gamma-1}}. \quad (5.11)$$

(4) The function $x_\gamma(\varepsilon)$ is strictly decreasing on $(0, 1)$ for $\gamma \leq 1$ and strictly decreasing on $(\hat{\varepsilon}_\gamma, 1)$ for $\gamma > 1$. Moreover,

$$x'_\gamma(\varepsilon) = \frac{x_\gamma(\varepsilon) + s_\gamma(\varepsilon)}{\theta'(x_\gamma(\varepsilon)) - \varepsilon} < 0 \quad (5.12)$$

for every γ and ε , satisfying (5.8) or (5.9).

(5) (a) If $\gamma \in (-\infty, 1]$, then the function $x_\gamma(\varepsilon) \in C^\infty(0, 1)$.

(b) If $\gamma \in (1, 2]$, then the function $x_\gamma(\varepsilon) \in C^\infty((0, \hat{\varepsilon}_\gamma) \cup (\hat{\varepsilon}_\gamma, 1))$ and it is continuous at the point $\hat{\varepsilon}_\gamma$;

(c) If $\gamma \in (2, 3]$, then the function $x_\gamma(\varepsilon) \in C^\infty((0, \hat{\varepsilon}_\gamma) \cup (\hat{\varepsilon}_\gamma, 1))$ and it has the continuous derivative at the point $\hat{\varepsilon}_\gamma$;

(d) If $\gamma \in (3, +\infty]$, then the function $x_\gamma(\varepsilon) \in C^\infty((0, \hat{\varepsilon}_\gamma) \cup (\hat{\varepsilon}_\gamma, 1))$ and it has the second continuous derivative at the point $\hat{\varepsilon}_\gamma$.

(6) For every $\gamma \in \mathbb{R}$ we have

$$\lim_{\varepsilon \rightarrow 1-0} x_\gamma(\varepsilon) = 0. \quad (5.13)$$

(7) For every $\gamma \leq 1$ we have

$$\lim_{\varepsilon \rightarrow 0+} x_\gamma(\varepsilon) = +\infty. \quad (5.14)$$

(8) (a) If $\gamma \in (-\infty, -1)$, then

$$\lim_{\varepsilon \rightarrow 0^+} x_\gamma(\varepsilon) \varepsilon^{-\alpha} = 0 \quad \text{for every } \alpha < \frac{\gamma - 1}{2}. \quad (5.15)$$

(b) If $\gamma = -1$, then

$$\lim_{\varepsilon \rightarrow 0^+} x_\gamma(\varepsilon) \varepsilon = 2. \quad (5.16)$$

(c) If $\gamma \in (-1, 1)$, then

$$\lim_{\varepsilon \rightarrow 0^+} x_\gamma(\varepsilon) \varepsilon = \left(\frac{\gamma + 1}{2} \right)^{(y+1)/(2y-2)}. \quad (5.17)$$

(d) If $\gamma = 1$, then

$$\lim_{\varepsilon \rightarrow 0^+} x_\gamma(\varepsilon) \varepsilon = \exp \left\{ -\frac{1}{2} \right\}. \quad (5.18)$$

(9) For every $\gamma \in \mathbb{R}$ we have

$$\lim_{\varepsilon \rightarrow 1-0} \frac{x_\gamma(\varepsilon)}{(1-\varepsilon)^\alpha} = +\infty \quad \text{for every } \alpha > \frac{1}{2}. \quad (5.19)$$

Proof of property (1). Let $\gamma > 1$. We set

$$\alpha(y) = I_\gamma^+ \left(\sqrt{\frac{2}{\gamma-1}}, y \right) = \frac{\theta(y)}{y + \sqrt{2/(\gamma-1)}}. \quad (5.20)$$

It is easy to see that the function $\alpha(y)$ is positive on $(0, \sqrt{2/(\gamma-1)})$ and it is continuous on $[0, \sqrt{2/(\gamma-1)}]$. Therefore there exists

$$\hat{\varepsilon}_\gamma = \max_{y \in [0, \sqrt{2/(\gamma-1)}]} \alpha(y) > 0. \quad (5.21)$$

Next,

$$\alpha(y) \leq \frac{y}{y + \sqrt{2/(\gamma-1)}} < 1 \quad \forall y \in \left[0, \sqrt{\frac{2}{\gamma-1}} \right]. \quad (5.22)$$

Hence $\hat{\varepsilon}_\gamma < 1$. Therefore for every $\varepsilon \in (0, \hat{\varepsilon}_\gamma]$ the equation

$$\alpha(y) = \varepsilon \quad (5.23)$$

has at the least one solution $y_0 \in (0, \sqrt{2/(\gamma-1)})$. Otherwise the equation does not have any solution.

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Fix $\varepsilon \in (0, \hat{\varepsilon}_\gamma]$ and $x \in \Sigma_\gamma$. Let $y_0 \in \Sigma_\gamma$ be a solution of (5.23). We have

$$\varepsilon = \alpha(y_0) = \frac{\theta(y_0)}{y_0 + \sqrt{2/(\gamma-1)}} \leq \frac{\theta(x) + \theta(y_0)}{x + y_0} = I_\gamma^+(x, y_0). \quad (5.24)$$

From this $x \in X_\gamma(\varepsilon)$. Hence $X_\gamma(\varepsilon) = \Sigma_\gamma$ for all $\varepsilon \in (0, \hat{\varepsilon}_\gamma]$. This proves (5.6).

Now we prove (5.7). Fix $\varepsilon \in (\hat{\varepsilon}_\gamma, 1)$. Suppose that

$$x_\gamma(\varepsilon) = \sqrt{\frac{2}{\gamma-1}}. \quad (5.25)$$

Then for arbitrary $n \in \mathbb{N}$ there exists a number $x_n \in X_\gamma(\varepsilon)$ such that

$$\sqrt{\frac{2}{\gamma-1}} - \frac{1}{n} < x_n. \quad (5.26)$$

Moreover,

$$\lim_{n \rightarrow \infty} x_n = \sqrt{\frac{2}{\gamma-1}} \quad (5.27)$$

and for arbitrary $n \in \mathbb{N}$ there exists $y_n \in \Sigma_\gamma$ satisfying the inequality

$$I_\gamma^+(x_n, y_n) \geq \varepsilon, \quad (5.28)$$

which implies

$$\theta(x_n) - \varepsilon x_n \geq \varepsilon y_n - \theta(y_n). \quad (5.29)$$

Further, we have

$$\alpha(y_n) = \frac{\theta(y_n)}{y_n + \sqrt{2/(\gamma-1)}} \leq \hat{\varepsilon}_\gamma \quad \forall n \in \mathbb{N}. \quad (5.30)$$

Then

$$\theta(y_n) \leq \hat{\varepsilon}_\gamma \left(y_n + \sqrt{\frac{2}{\gamma-1}} \right) \quad \forall n \in \mathbb{N}. \quad (5.31)$$

Using (5.29), for all $n \in \mathbb{N}$ we deduce

$$\begin{aligned} \theta(x_n) - \varepsilon x_n &\geq \varepsilon y_n - \theta(y_n) \\ &\geq \varepsilon y_n - \hat{\varepsilon}_\gamma \left(y_n + \sqrt{\frac{2}{\gamma-1}} \right) \geq -\hat{\varepsilon}_\gamma \sqrt{\frac{2}{\gamma-1}}. \end{aligned} \quad (5.32)$$

Letting $n \rightarrow \infty$ in the inequality

$$\theta(x_n) - \varepsilon x_n \geq -\widehat{\varepsilon}_\gamma \sqrt{\frac{2}{\gamma-1}}, \quad (5.33)$$

we see that $\varepsilon \leq \widehat{\varepsilon}_\gamma$ and we arrive at a contradiction. \square

Further we will need the following lemma.

LEMMA 5.1. *If (5.8) or (5.9) holds, then $x_\gamma(\varepsilon) \in \Sigma_\gamma$ and there exists a number $y_\gamma(\varepsilon) \in \Sigma_\gamma$ such that*

$$I_\gamma^+(x_\gamma(\varepsilon), y_\gamma(\varepsilon)) = \varepsilon. \quad (5.34)$$

Proof. Show that the set $X_\gamma(\varepsilon)$ is compact for every γ and every ε satisfying (5.8) or (5.9). Introduce the set

$$Z_\gamma(\varepsilon) = \{(x, y) \in \Sigma_\gamma \times \Sigma_\gamma : I_\gamma^+(x, y) \geq \varepsilon\}. \quad (5.35)$$

Let $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}, \pi(x, y) = x$ be natural projection. It is clear that $\pi(Z_\gamma(\varepsilon)) = X_\gamma(\varepsilon)$.

Assume that (5.8) holds. The set $Z_\gamma(\varepsilon)$ is closed since the function $I_\gamma^+(x, y)$ is continuous. The set $Z_\gamma(\varepsilon)$ is bounded. Indeed, we can find a sequence $Z_\gamma(\varepsilon) \ni (x_n, y_n) \rightarrow \infty$. Assume that $x_n \rightarrow \infty$. Then for the bounded subsequence of $\{y_n\}$ we have

$$\varepsilon \leq I_\gamma^+(x_n, y_n) = \frac{x_n \sigma(x_n) + y_n \sigma(y_n)}{x_n + y_n} \leq \frac{x_n \sigma(x_n) + y_n}{x_n}. \quad (5.36)$$

The right part of this inequality tends to zero as $n \rightarrow \infty$. Thus we obtain a contradiction to (5.8). For an unbounded subsequence of $\{y_n\}$ we have

$$\varepsilon \leq I_\gamma^+(x_n, y_n) \leq \sigma(x_n) + \sigma(y_n). \quad (5.37)$$

The right part of this inequality tends to zero as $n \rightarrow \infty$. Again we obtain a contradiction to (5.8). Hence $Z_\gamma(\varepsilon)$ is bounded. Therefore $Z_\gamma(\varepsilon)$ is compact. Because the mapping π is continuous, the set $X_\gamma(\varepsilon) = \pi(Z_\gamma(\varepsilon))$ is compact too.

Assume that (5.9) holds. By (5.7) it follows that $\overline{Z_\gamma(\varepsilon)} \subset \Sigma_\gamma \times \Sigma_\gamma$. Here $\overline{Z_\gamma(\varepsilon)}$ denotes the closure of $Z_\gamma(\varepsilon)$. Since the function $I_\gamma^+(x, y)$ is continuous, $Z_\gamma(\varepsilon)$ is compact. Therefore $X_\gamma(\varepsilon)$ is compact too.

Similarly we establish that the set

$$\overline{X_\gamma(\varepsilon)} = \{x \in \Sigma_\gamma : \exists y \in \Sigma_\gamma, I_\gamma^+(x, y) = \varepsilon\} \quad (5.38)$$

is compact for every γ and ε satisfying (5.8) or (5.9).

We fix γ and ε satisfying (5.8) or (5.9). Prove that

$$\max_x X_\gamma(\varepsilon) = \max_x \overline{X_\gamma(\varepsilon)}. \quad (5.39)$$

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We set

$$a = \max_x X_\gamma(\varepsilon), \quad b = \max_x \bar{X}_\gamma(\varepsilon). \quad (5.40)$$

Obviously, $a \geq b$. Show that $a \leq b$. Since $a \in X_\gamma(\varepsilon)$, there exists a number $y_0 \in \Sigma_\gamma$ such that

$$I_\gamma^+(a, y_0) \geq \varepsilon. \quad (5.41)$$

Assume that

$$I_\gamma^+(a, y_0) = \varepsilon. \quad (5.42)$$

Then $a \in \bar{X}_\gamma(\varepsilon)$ and hence $a \leq b$.

Now we assume

$$I_\gamma^+(a, y_0) > \varepsilon. \quad (5.43)$$

For $\gamma \leq 1$ we have

$$\lim_{x \rightarrow +\infty} I_\gamma^+(x, y_0) = 0. \quad (5.44)$$

Since the function $I_\gamma^+(x, y)$ is continuous, there exists a number $x' > a$ such that

$$I_\gamma^+(x', y_0) = \varepsilon. \quad (5.45)$$

Then $x' \in \bar{X}_\gamma(\varepsilon)$. Hence $a < x' \leq b$ and we arrive at a contradiction. For $\gamma > 1$ we have

$$I_\gamma^+\left(\sqrt{\frac{2}{\gamma-1}}, y_0\right) \leq \hat{\varepsilon}_\gamma < \varepsilon. \quad (5.46)$$

Then there exists a number $x' \in (a, \sqrt{2/(\gamma-1)})$ satisfying (5.45). Hence $x' \in \bar{X}_\gamma(\varepsilon)$. Therefore $a < x' \leq b$ and we arrive at a contradiction.

Thus we establish that

$$x_\gamma(\varepsilon) = \max_x \bar{X}_\gamma(\varepsilon) \quad (5.47)$$

and arrive at the desired result. \square

Proof of properties (2)–(5). Fix γ and ε_0 satisfying (5.8) or (5.9). By Lemma 5.1 the number $x_\gamma(\varepsilon_0) \in \Sigma_\gamma$ and there exists a number $y_\gamma(\varepsilon_0) \in \Sigma_\gamma$ such that

$$I_\gamma^+(x_\gamma(\varepsilon_0), y_\gamma(\varepsilon_0)) = \varepsilon_0. \quad (5.48)$$

We set

$$F(x, y, \varepsilon) = I_\gamma^+(x, y) - \varepsilon. \quad (5.49)$$

Observe that the function $F(x, y, \varepsilon)$ is C^∞ -differentiable in some neighborhood $U \subset \mathbb{R}^3$ of the point $p_0 = (x_\gamma(\varepsilon_0), y_\gamma(\varepsilon_0), \varepsilon_0)$ and $F(p_0) = 0$. We have

$$\frac{\partial F}{\partial x}(p_0) = \frac{\theta'(x_\gamma(\varepsilon_0)) - I_\gamma^+(x_\gamma(\varepsilon_0), y_\gamma(\varepsilon_0))}{x_\gamma(\varepsilon_0) + y_\gamma(\varepsilon_0)} = \frac{\theta'(x_\gamma(\varepsilon_0)) - \varepsilon_0}{x_\gamma(\varepsilon_0) + y_\gamma(\varepsilon_0)}. \quad (5.50)$$

By assertion (8) of Section 2, $0 < s_\gamma(\varepsilon_0) < r_\gamma(\varepsilon_0)$. Therefore the inequality $r_\gamma(\varepsilon_0) \leq x_\gamma(\varepsilon_0)$ yields

$$\frac{\partial F}{\partial x}(p_0) < 0. \quad (5.51)$$

By the well-known implicit function theorem, there exist a 3-dimensional interval $I = I_x \times I_y \times I_\varepsilon \subset U$ and a function $f \in C^\infty(I_y \times I_\varepsilon)$ such that for all $(x, y, \varepsilon) \in I_x \times I_y \times I_\varepsilon$,

$$F(x, y, \varepsilon) = 0 \iff x = f(y, \varepsilon). \quad (5.52)$$

Here

$$\begin{aligned} I_x &= \{x \in \mathbb{R} : |x - x_\gamma(\varepsilon_0)| < a\}, & I_y &= \{y \in \mathbb{R} : |y - y_\gamma(\varepsilon_0)| < b\}, \\ I_\varepsilon &= \{\varepsilon \in \mathbb{R} : |\varepsilon - \varepsilon_0| < c\}. \end{aligned} \quad (5.53)$$

Moreover,

$$\begin{aligned} \frac{\partial f}{\partial y}(y_\gamma(\varepsilon_0), \varepsilon_0) &= -[F'_x(p_0)]^{-1}[F'_y(p_0)] = -\frac{\theta'(y_\gamma(\varepsilon_0)) - \varepsilon_0}{\theta'(x_\gamma(\varepsilon_0)) - \varepsilon_0}, \\ \frac{\partial f}{\partial \varepsilon}(y_\gamma(\varepsilon_0), \varepsilon_0) &= -[F'_x(p_0)]^{-1}[F'_\varepsilon(p_0)] = \frac{x_\gamma(\varepsilon_0) + y_\gamma(\varepsilon_0)}{\theta'(x_\gamma(\varepsilon_0)) - \varepsilon_0}. \end{aligned} \quad (5.54)$$

It is easy to see that at the point $y_\gamma(\varepsilon_0)$ the function $x = f(y, \varepsilon_0)$ reaches a maximum on I_y . Therefore

$$\frac{\partial f}{\partial y}(y_\gamma(\varepsilon_0), \varepsilon_0) = 0. \quad (5.55)$$

From this

$$\theta'(y_\gamma(\varepsilon_0)) = \varepsilon_0. \quad (5.56)$$

Hence $y_\gamma(\varepsilon_0) = s_\gamma(\varepsilon_0)$ and property (2) is proved.

Further, we set

$$G(y, \varepsilon) = \theta'(y) - \varepsilon. \quad (5.57)$$

Observe that the function $G(y, \varepsilon)$ is C^∞ -differentiable in some neighborhood $V \subset \mathbb{R}^2$ of the point $q_0 = (s_\gamma(\varepsilon_0), \varepsilon_0)$ and $G(q_0) = 0$. By assertions (6)–(8) of Section 2 we have

$$\frac{\partial G}{\partial y}(q_0) = \theta''(s_\gamma(\varepsilon_0)) < 0. \quad (5.58)$$

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By the implicit function theorem, the function $s_\gamma(\varepsilon)$ is C^∞ -differentiable at the point ε_0 . Therefore there is an interval

$$I'_\varepsilon = \{\varepsilon \in \mathbb{R} : |\varepsilon - \varepsilon_0| < c'\} \subset I_\varepsilon \quad (5.59)$$

such that

$$s_\gamma(\varepsilon) \in I_y \quad \forall \varepsilon \in I'_\varepsilon. \quad (5.60)$$

Hence for all $(x, \varepsilon) \in I_x \times I'_\varepsilon$,

$$F(x, s_\gamma(\varepsilon), \varepsilon) = 0 \iff x = f(s_\gamma(\varepsilon), \varepsilon). \quad (5.61)$$

We fix $\varepsilon \in I'_\varepsilon$. Next,

$$x = f(s_\gamma(\varepsilon), \varepsilon) \quad (5.62)$$

and hence

$$F(x, s_\gamma(\varepsilon), \varepsilon) = 0. \quad (5.63)$$

Rewrite the latter equality in the form

$$\mu(x) = -\mu(s_\gamma(\varepsilon)), \quad (5.64)$$

where

$$\mu(t) = \mu(t, \varepsilon) = \theta(t) - t\varepsilon. \quad (5.65)$$

We have

$$\mu'(t) = \theta'(t) - \varepsilon. \quad (5.66)$$

By assertion (8) of Section 2 we conclude that the function $\mu(t)$ is strictly increasing on $(0, s_\gamma(\varepsilon))$ and strictly decreasing on $(s_\gamma(\varepsilon), +\infty) \cap \Sigma_\gamma$. Moreover, $\mu(0) = \mu(r_\gamma(\varepsilon)) = 0$ and by property (2), $\mu(x_\gamma(\varepsilon)) = -\mu(s_\gamma(\varepsilon))$. Then it is not hard to check that $x = x_\gamma(\varepsilon)$. Thus

$$x_\gamma(\varepsilon) = f(s_\gamma(\varepsilon), \varepsilon) \quad \forall \varepsilon \in I'_\varepsilon. \quad (5.67)$$

Hence the function $x_\gamma(\varepsilon)$ is C^∞ -differentiable at the point ε_0 and

$$x'_\gamma(\varepsilon_0) = \frac{\partial f}{\partial y}(s_\gamma(\varepsilon_0), \varepsilon_0) s'_\gamma(\varepsilon_0) + \frac{\partial f}{\partial \varepsilon}(s_\gamma(\varepsilon_0), \varepsilon_0) = \frac{\partial f}{\partial \varepsilon}(s_\gamma(\varepsilon_0), \varepsilon_0) = \frac{x_\gamma(\varepsilon_0) + s_\gamma(\varepsilon_0)}{\theta'(x_\gamma(\varepsilon_0)) - \varepsilon_0} < 0. \quad (5.68)$$

This proves property (4).

Let $\gamma > 1$. We show that

$$\lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} x_\gamma(\varepsilon) = \sqrt{\frac{2}{\gamma - 1}}. \quad (5.69)$$

Let $y_0 \in \Sigma_\gamma$ be a solution of the equation

$$\alpha(y) = \hat{\varepsilon}_\gamma. \quad (5.70)$$

Here, as above,

$$\alpha(y) = I_\gamma^+ \left(\sqrt{\frac{2}{\gamma-1}}, y \right). \quad (5.71)$$

Then

$$\begin{aligned} \frac{\theta(y_0)}{y_0 + \sqrt{2/(\gamma-1)}} &= \hat{\varepsilon}_\gamma, \\ \alpha'(y_0) &= \frac{\theta'(y_0) \left(y_0 + \sqrt{2/(\gamma-1)} \right) - \theta(y_0)}{\left(y_0 + \sqrt{2/(\gamma-1)} \right)^2} = 0. \end{aligned} \quad (5.72)$$

From this

$$\theta(y_0) = \theta'(y_0) \left(y_0 + \sqrt{\frac{2}{\gamma-1}} \right). \quad (5.73)$$

Using (5.72), we conclude that

$$\theta'(y_0) = \hat{\varepsilon}_\gamma, \quad (5.74)$$

that is, $y_0 = s_\gamma(\hat{\varepsilon}_\gamma)$.

We rewrite the equality

$$I_\gamma^+(x_\gamma(\varepsilon), s_\gamma(\varepsilon)) = \varepsilon \quad (5.75)$$

in the form

$$\theta(x_\gamma(\varepsilon)) - x_\gamma(\varepsilon)\varepsilon = -(\theta(s_\gamma(\varepsilon)) - s_\gamma(\varepsilon)\varepsilon). \quad (5.76)$$

Using (5.72), we obtain

$$\lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} (\theta(x_\gamma(\varepsilon)) - x_\gamma(\varepsilon)\varepsilon) = -(\theta(s_\gamma(\hat{\varepsilon}_\gamma)) - s_\gamma(\hat{\varepsilon}_\gamma)\hat{\varepsilon}_\gamma) = -\hat{\varepsilon}_\gamma \sqrt{\frac{2}{\gamma-1}}. \quad (5.77)$$

Thus

$$\lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \mu(x_\gamma(\varepsilon), \varepsilon) = -\hat{\varepsilon}_\gamma \sqrt{\frac{2}{\gamma - 1}}. \quad (5.78)$$

Suppose that (5.69) is not true. That is, for some sequence $\varepsilon_i \rightarrow \hat{\varepsilon}_\gamma + 0$ of numbers the inequality

$$x_\gamma(\varepsilon_i) \leq \sqrt{\frac{2}{\gamma - 1}} - m \quad (5.79)$$

holds with some constant $m > 0$. Note that $x_\gamma(\varepsilon) \in [r_\gamma(\varepsilon), \sqrt{(2/\gamma - 1)}]$ for every $\varepsilon \in (\hat{\varepsilon}_\gamma, 1)$. By assertion (8) of Section 2 it follows that the function $\mu(t)$ is strictly decreasing on $[r_\gamma(\varepsilon), \sqrt{(2/\gamma - 1)}]$. We have

$$\mu(x_\gamma(\varepsilon_i), \varepsilon_i) \geq \mu\left(\sqrt{\frac{2}{\gamma - 1}} - m, \varepsilon_i\right) > -\left(\sqrt{\frac{2}{\gamma - 1}} - m\right)\varepsilon_i. \quad (5.80)$$

Letting $\varepsilon_i \rightarrow \hat{\varepsilon}_\gamma + 0$, we obtain a contradiction to (5.78). Thus property (3) is proved. Hence the function $x_\gamma(\varepsilon)$ is continuous at the point $\hat{\varepsilon}_\gamma$ for every $\gamma > 1$.

For $\gamma > 1$ we have

$$A_\gamma = \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \frac{x_\gamma(\varepsilon) - x_\gamma(\hat{\varepsilon}_\gamma)}{\varepsilon - \hat{\varepsilon}_\gamma} = \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} x'_\gamma(\varepsilon) = \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \frac{x_\gamma(\varepsilon) + s_\gamma(\varepsilon)}{\theta'(x_\gamma(\varepsilon)) - \varepsilon}. \quad (5.81)$$

By assertion (5) of Section 2 and by property (3) we obtain

$$\begin{aligned} A_\gamma &= -\frac{\sqrt{2/(\gamma - 1)} + s_\gamma(\hat{\varepsilon}_\gamma)}{\hat{\varepsilon}_\gamma} < 0 \quad \text{for } \gamma \in (1, 2), \\ A_\gamma &= -\frac{\sqrt{2/(\gamma - 1)} + s_\gamma(\hat{\varepsilon}_\gamma)}{2 + \hat{\varepsilon}_\gamma} < 0 \quad \text{for } \gamma = 2, \\ A_\gamma &= 0 \quad \text{for } \gamma \in (2, +\infty). \end{aligned} \quad (5.82)$$

Hence $x_\gamma(\varepsilon)$ is not differentiable at the point $\hat{\varepsilon}_\gamma$ for $\gamma \in (1, 2]$ and it has the continuous derivative at the point $\hat{\varepsilon}_\gamma$ for $\gamma \in (2, +\infty)$.

For $\gamma > 2$ we have

$$B_\gamma = \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \frac{x'_\gamma(\varepsilon) - x'_\gamma(\hat{\varepsilon}_\gamma)}{\varepsilon - \hat{\varepsilon}_\gamma} = \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \frac{x_\gamma(\varepsilon) + s_\gamma(\varepsilon)}{(\theta'(x_\gamma(\varepsilon)) - \varepsilon)(\varepsilon - \hat{\varepsilon}_\gamma)}. \quad (5.83)$$

Using L'Hospital rule and property (3), we find

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \left(1 - \frac{\gamma - 1}{2} x_\gamma^2(\varepsilon) \right)^{(2-\gamma)/(\gamma-1)} (\varepsilon - \hat{\varepsilon}_\gamma) \\
&= \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \frac{\varepsilon - \hat{\varepsilon}_\gamma}{(1 - ((\gamma - 1)/2)x_\gamma^2(\varepsilon))^{(\gamma-2)/(\gamma-1)}} \\
&= \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \frac{1}{-(\gamma - 2)x_\gamma(\varepsilon)x'_\gamma(\varepsilon)(1 - ((\gamma - 1)/2)x_\gamma^2(\varepsilon))^{-1/(\gamma-1)}} \\
&= -\frac{1}{\gamma - 2} \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \frac{\sigma(x_\gamma(\varepsilon))(\theta'(x_\gamma(\varepsilon)) - \varepsilon)}{x_\gamma(\varepsilon)(x_\gamma(\varepsilon) + s_\gamma(\varepsilon))} \\
&= -\frac{1}{\gamma - 2} \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \frac{\sigma(x_\gamma(\varepsilon))\theta'(x_\gamma(\varepsilon))}{x_\gamma(\varepsilon)(x_\gamma(\varepsilon) + s_\gamma(\varepsilon))} \\
&= -\frac{1}{\gamma - 2} \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \frac{(1 - ((\gamma + 1)/2)x_\gamma^2(\varepsilon))(1 - ((\gamma - 1)/2)x_\gamma^2(\varepsilon))^{(3-\gamma)/(\gamma-1)}}{x_\gamma(\varepsilon)(x_\gamma(\varepsilon) + s_\gamma(\varepsilon))}.
\end{aligned} \tag{5.84}$$

Then

$$\begin{aligned}
B_\gamma &= \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \frac{x_\gamma(\varepsilon) + s_\gamma(\varepsilon)}{\theta'(x_\gamma(\varepsilon))(\varepsilon - \hat{\varepsilon}_\gamma)} \\
&= -(\gamma - 2) \lim_{\varepsilon \rightarrow \hat{\varepsilon}_\gamma + 0} \frac{x_\gamma(\varepsilon)(x_\gamma(\varepsilon) + s_\gamma(\varepsilon))^2}{(1 - ((\gamma + 1)/2)x_\gamma^2(\varepsilon))^2(1 - ((\gamma - 1)/2)x_\gamma^2(\varepsilon))^{(3-\gamma)/(\gamma-1)}}.
\end{aligned} \tag{5.85}$$

By property (3) we find

$$\begin{aligned}
B_\gamma &= -\infty \quad \text{for } \gamma \in (2, 3), \\
B_\gamma &= -(1 + s_\gamma(\hat{\varepsilon}_\gamma))^2 < 0 \quad \text{for } \gamma = 3, \\
B_\gamma &= 0 \quad \text{for } \gamma \in (3, +\infty).
\end{aligned} \tag{5.86}$$

Therefore the function $x_\gamma(\varepsilon)$ is not doubly differentiable at the point $\hat{\varepsilon}_\gamma$ for $\gamma \in (2, 3]$ and it has second continuous derivative at the point $\hat{\varepsilon}_\gamma$ for $\gamma \in (3, +\infty)$. Thus property (5) is proved. \square

Proof of property (6). By assertion (8) of Section 2,

$$0 < s_\gamma(\varepsilon) < r_\gamma(\varepsilon) \tag{5.87}$$

for every ε and γ satisfying (5.8) or (5.9). Letting $\varepsilon \rightarrow 1 - 0$ we obtain

$$\lim_{\varepsilon \rightarrow 1-0} s_\gamma(\varepsilon) = 0. \tag{5.88}$$

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Show that

$$\lim_{\varepsilon \rightarrow 1-0} x_\gamma(\varepsilon) = 0. \quad (5.89)$$

Indeed, suppose that this is not true, that is, there will be a number $\varepsilon_0 \in (0, 1)$ and a sequence $\varepsilon_i \rightarrow 1$ ($\varepsilon_0 < \varepsilon_i < 1$) such that the inequalities

$$m \leq x_\gamma(\varepsilon_i) \leq x_\gamma(\varepsilon_0) \quad (5.90)$$

hold with some constant $m > 0$. We can consider that

$$\lim_{\varepsilon_i \rightarrow 1} x_\gamma(\varepsilon_i) = a \in [m, x_\gamma(\varepsilon_0)]. \quad (5.91)$$

Using property (2), we have

$$1 = \lim_{\varepsilon_i \rightarrow 1} \varepsilon_i = \lim_{\varepsilon_i \rightarrow 1} I_\gamma^+(x_\gamma(\varepsilon_i), s_\gamma(\varepsilon_i)) = I_\gamma^+(a, 0) = \sigma(a). \quad (5.92)$$

Then $a = 0 < m$ and we arrive at a contradiction. \square

Proof of property (7). Letting $\varepsilon \rightarrow 0+$ in the inequality $x_\gamma(\varepsilon) \geq r_\gamma(\varepsilon)$, we obtain (5.7). \square

Proof of property (8). (a) Let $\gamma < -1$. By assertion (8) of Section 2,

$$0 < s_\gamma(\varepsilon) < r_\gamma(\varepsilon) \quad \forall \varepsilon \in (0, 1). \quad (5.93)$$

From this

$$\lim_{\varepsilon \rightarrow 0+} s_\gamma(\varepsilon)\varepsilon^{-\alpha} = 0 \quad \text{for every } \alpha < \frac{\gamma-1}{2}. \quad (5.94)$$

We set

$$\vartheta(t) = \left(1 - \frac{\gamma+1}{2}t^2\right) \left(1 - \frac{\gamma-1}{2}t^2\right)^{-1}. \quad (5.95)$$

Obviously

$$\lim_{t \rightarrow +\infty} \vartheta(t) = \frac{\gamma+1}{\gamma-1}. \quad (5.96)$$

It is easy to see that the function $\vartheta(t)$ is strictly decreasing on $[0, +\infty)$. Therefore

$$\vartheta(t) > \frac{\gamma+1}{\gamma-1} \quad \forall t \geq 0. \quad (5.97)$$

Next, for all $\varepsilon \in (0, 1)$,

$$\varepsilon = \theta'(s_\gamma(\varepsilon)) = \left(1 - \frac{\gamma-1}{2}s_\gamma^2(\varepsilon)\right)^{(1/(\gamma-1))-1} \left(1 - \frac{\gamma+1}{2}s_\gamma^2(\varepsilon)\right) > \sigma(s_\gamma(\varepsilon)) \frac{\gamma+1}{\gamma-1}. \quad (5.98)$$

From this

$$1 < \frac{\sigma(s_\gamma(\varepsilon))}{\varepsilon} < \frac{\gamma - 1}{\gamma + 1} \quad (5.99)$$

for all $\varepsilon \in (0, 1)$. We note that the equality $I_\gamma^+(x, y) = \varepsilon$ can be written as

$$x \left(\frac{\sigma(x)}{\varepsilon} - 1 \right) = y \left(1 - \frac{\sigma(y)}{\varepsilon} \right). \quad (5.100)$$

By (5.94) and (5.99) we obtain

$$0 = \lim_{\varepsilon \rightarrow 0^+} s_\gamma(\varepsilon) \varepsilon^{-\alpha} \left(\frac{\sigma(s_\gamma(\varepsilon))}{\varepsilon} - 1 \right) = \lim_{\varepsilon \rightarrow 0^+} x_\gamma(\varepsilon) \varepsilon^{-\alpha} \left(1 - \frac{\sigma(x_\gamma(\varepsilon))}{\varepsilon} \right) \quad (5.101)$$

for each $\alpha < (\gamma - 1)/2$.

Assume that there exists $\alpha < (\gamma - 1)/2$ such that

$$\lim_{\varepsilon \rightarrow 0^+} x_\gamma(\varepsilon) \varepsilon^{-\alpha} \neq 0. \quad (5.102)$$

Then for some sequence $\varepsilon_i \rightarrow 0$ of positive numbers the inequality

$$x_\gamma(\varepsilon_i) \varepsilon_i^{-\alpha} \geq m \quad (5.103)$$

holds with some constant $m > 0$. By (5.101) we find

$$\lim_{\varepsilon_i \rightarrow 0^+} \frac{\sigma(x_\gamma(\varepsilon_i))}{\varepsilon_i} = 1. \quad (5.104)$$

By (5.103),

$$\lim_{\varepsilon_i \rightarrow 0^+} \frac{\sigma(x_\gamma(\varepsilon_i))}{\varepsilon_i} \leq \lim_{\varepsilon_i \rightarrow 0^+} \frac{\sigma(m \varepsilon_i^\alpha)}{\varepsilon_i} = 0 \quad (5.105)$$

and we arrive at a contradiction.

(b) Let $\gamma = -1$. We have

$$\varepsilon = \theta'(s_\gamma(\varepsilon)) = (1 + s_\gamma^2(\varepsilon))^{-3/2}. \quad (5.106)$$

Then

$$s_\gamma(\varepsilon) = \sqrt{\varepsilon^{-2/3} - 1}, \quad \sigma(s_\gamma(\varepsilon)) = \varepsilon^{1/3}. \quad (5.107)$$

By property (7),

$$\lim_{\varepsilon \rightarrow 0^+} \theta(x_\gamma(\varepsilon)) = 1. \quad (5.108)$$

We have

$$1 = \lim_{\varepsilon \rightarrow 0^+} (\theta(s_\gamma(\varepsilon)) - s_\gamma(\varepsilon)\varepsilon) = \lim_{\varepsilon \rightarrow 0^+} (x_\gamma(\varepsilon)\varepsilon - \theta(x_\gamma(\varepsilon))). \quad (5.109)$$

From this

$$\lim_{\varepsilon \rightarrow 0^+} x_\gamma(\varepsilon)\varepsilon = 2. \quad (5.110)$$

(c) Let $\gamma \in (-1, 1)$. By property (7),

$$\lim_{\varepsilon \rightarrow 0^+} \theta(x_\gamma(\varepsilon)) = 0. \quad (5.111)$$

Assertions (7) and (8) of Section 2 yield

$$\lim_{\varepsilon \rightarrow 0^+} s_\gamma(\varepsilon) = \sqrt{\frac{2}{\gamma+1}}. \quad (5.112)$$

Then, using (5.111), we obtain

$$\begin{aligned} \left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/(2\gamma-2)} &= \theta\left(\sqrt{\frac{2}{\gamma+1}}\right) = \lim_{\varepsilon \rightarrow 0^+} (\theta(s_\gamma(\varepsilon)) - s_\gamma(\varepsilon)\varepsilon) \\ &= \lim_{\varepsilon \rightarrow 0^+} (x_\gamma(\varepsilon)\varepsilon - \theta(x_\gamma(\varepsilon))) = \lim_{\varepsilon \rightarrow 0^+} x_\gamma(\varepsilon)\varepsilon. \end{aligned} \quad (5.113)$$

(d) The proof is analogous. \square

Proof of property (9). Assume that $\gamma > 1$. Then

$$x_\gamma(\varepsilon) \geq r_\gamma(\varepsilon) = \sqrt{\frac{2(1-\varepsilon^{\gamma-1})}{\gamma-1}}. \quad (5.114)$$

Using L'Hospital rule, we find

$$\lim_{\varepsilon \rightarrow 1-0} \frac{1-\varepsilon^{\gamma-1}}{(1-\varepsilon)^{2\alpha}} = \frac{\gamma-1}{2\alpha} \lim_{\varepsilon \rightarrow 1-0} \frac{\varepsilon^{\gamma-2}}{(1-\varepsilon)^{2\alpha-1}} = +\infty \quad (5.115)$$

for every $\alpha > 1/2$. From this we obtain desired result. The case $\gamma \leq 1$ is analogous. \square

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