EMBEDDING THEOREMS IN BANACH-VALUED *B*-SPACES AND MAXIMAL *B*-REGULAR DIFFERENTIAL-OPERATOR EQUATIONS

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The embedding theorems in anisotropic Besov-Lions type spaces $B^l_{p,\theta}(R^n;E_0,E)$ are studied; here E_0 and E are two Banach spaces. The most regular spaces E_α are found such that the mixed differential operators D^α are bounded from $B^l_{p,\theta}(R^n;E_0,E)$ to $B^s_{q,\theta}(R^n;E_\alpha)$, where E_α are interpolation spaces between E_0 and E depending on $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ and $l=(l_1,l_2,\dots,l_n)$. By using these results the separability of anisotropic differential-operator equations with dependent coefficients in principal part and the maximal B-regularity of parabolic Cauchy problem are obtained. In applications, the infinite systems of the quasielliptic partial differential equations and the parabolic Cauchy problems are studied.

1. Introduction

Embedding theorems in function spaces have been studied in [8, 35, 37, 38]. A comprehensive introduction to the theory of embedding of function spaces and historical references may be also found in [37]. In abstract function spaces embedding theorems have been investigated in [4, 5, 10, 17, 21, 27, 34, 40]. Lions and Peetre [21] showed that if

$$u\in L_2(0,T;H_0),\qquad u^{(m)}\in L_2(0,T;H),\tag{1.1}$$

then

$$u^{(i)}\in L_2(0,T;[H,H_0]_{i/m}),\quad i=1,2,\dots,m-1,\tag{1.2}$$

where H_0, H are Hilbert spaces, H_0 is continuously and densely embedded in H, where $[H_0,H]_\theta$ are interpolation spaces between H_0 and H for $0\le\theta\le 1$. The similar questions for anisotropic Sobolev spaces $W^l_p(\Omega;H_0,H)$, $\Omega\subset\mathbb{R}^n$ and for corresponding weighted

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spaces have been investigated in [28–31] and [23, 24], respectively. Embedding theorems in Banach-valued Besov spaces have been studied in [4, 5, 27, 32]. The solvability and spectrum of boundary value problems for elliptic differential-operator equations (DOE's) have been refined in [3–7, 13, 28–33, 39, 40]. A comprehensive introduction to DOE's and historical references may be found in [15, 18, 40]. In these works, Hilbert-valued function spaces essentially have been considered. The maximal L_p regularity and Fredholmness of partial elliptic equations in smooth regions have been studied, for example, in [1, 2, 20] and for nonsmooth domains studied, for example, in [16, 26]. For DOE's the similar problems have been investigated in [13, 28–32, 36, 39, 40].

Let E_0, E be Banach spaces such that E_0 is continuously and densely embedded in E. In the present paper, E-valued Besov spaces $B^{l+s}_{p,\theta}(R^n;E_0,E) = B^s_{p,\theta}(R^n;E_0) \cap B^{l+s}_{p,\theta}(R^n;E)$ are introduced and called Besov-Lions type spaces. The most regular interpolation class E_α between E_0 and E is found such that the appropriate mixed differential operators D^α are bounded from $B^{l+s}_{p,q}(R^n;E_0,E)$ to $B^s_{p,q}(R^n;E_\alpha)$. By applying these results the maximal regularity of certain class of anisotropic partial DOE with varying coefficients in Banach-valued Besov spaces is derived.

The paper is organized as follows. Section 2 collects notations and definitions. Section 3 presents the embedding theorems in Besov-Lions type spaces

$$B^{s+l}_{p,q}(R^n;E_0,E). \tag{1.3}$$

Section 4 contains applications of the underlying embedding theorem to vector-valued function spaces. Section 5 is devoted to the maximal regularity (in $B^s_{p,q}(R^n;E)$) of the certain class of anisotropic DOE with variable coefficients in principal part. Then by using these results the maximal B-regularity of the parabolic Cauchy problem is shown. In Section 6 these DOE are applied to BVP's and Cauchy problem for the finite and infinite systems of quasielliptic and parabolic PDEs, respectively.

2. Notations and definitions

Let E be a Banach space. Let $L_p(\Omega;E)$ denote the space of all strongly measurable E-valued functions that are defined on $\Omega \subset R^n$ with the norm

$$\begin{gathered} \|f\|_{L_p(\Omega;E)} = \left(\int \|f(x)\|_E^p\,dx\right)^{1/p}, \quad 1 \le p < \infty, \\ \|f\|_{L_\infty(\Omega;E)} = \operatorname*{ess\,sup}_{x\in\Omega}\left[\|f(x)\|_E\right], \quad x = (x_1,x_2,\dots,x_n). \end{gathered} \tag{2.1}$$

The Banach space E is said to be a ζ-convex space (see [9, 11, 12, 19]) if there exists on $E \times E$ a symmetric real-valued function $\zeta(u,v)$ which is convex with respect to each of the variables, and satisfies the conditions

$$\zeta(0,0) > 0, \qquad \zeta(u,v) \le \|u+v\|, \quad \text{for } \|u\| \le 1 \le \|v\|. \tag{2.2}$$

A ζ-convex space E is often called a UMD-space and written as $E \in \text{UMD}$. It is shown in [9] that the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy \tag{2.3}$$

is bounded in $L_p(R;E)$, $p \in (1,\infty)$ for those and only those spaces E, which possess the property of UMD spaces. The UMD spaces include, for example, L_p, l_p spaces and the Lorentz spaces L_{pq}, $p,q \in (1,\infty)$.

Let $\mathbf{C}$ be the set of complex numbers and let

$$S_\varphi = \{\lambda;\ \lambda \in \mathbf{C}, |\arg\lambda - \pi| \le \pi - \varphi\} \cup \{0\}, \quad 0 < \varphi \le \pi. \tag{2.4}$$

A linear operator A is said to be a φ-positive in a Banach space E, with bound $M > 0$ if $D(A)$ is dense on E and

$$\|(A - \lambda I)^{-1}\|_{L(E)} \le M(1 + |\lambda|)^{-1} \tag{2.5}$$

with $\lambda \in S_\varphi$, $\varphi \in (0,\pi]$, I is identity operator in E, and $L(E)$ is the space of all bounded linear operators in E. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and denoted by A_λ. It is known [37, Section 1.15.1] that there exist fractional powers A^θ of the positive operator A. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graphical norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \|A^\theta u\|^p\right)^{1/p}, \quad 1 \le p < \infty,\ -\infty < \theta < \infty. \tag{2.6}$$

Let E_0 and E be two Banach spaces. By $(E_0,E)_{\sigma,p}$, $0 < \sigma < 1$, $1 \le p \le \infty$ we will denote the interpolation spaces obtained from $\{E_0,E\}$ by the K-method (see, e.g., [37, Section 1.3.1] or [10]).

Let $S(R^n;E)$ denote a Schwartz class, that is, the space of all E-valued rapidly decreasing smooth functions φ on R^n. $E = \mathbf{C}$ will be denoted by $S(R^n)$. Let $S'(R^n;E)$ denote the space of E-valued tempered distributions, that is, the space of continuous linear operators from $S(R^n)$ to E.

Let $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$, α_i are integers. An E-values generalized function $D^\alpha f$ is called a generalized derivative in the sense of Schwartz distributions of the generalized function $f \in S'(R^n,E)$ if the equality

$$\langle D^\alpha f,\varphi\rangle = (-1)^{|\alpha|}\langle f, D^\alpha\varphi\rangle \tag{2.7}$$

holds for all $\varphi \in S(R^n)$.

By using (2.7) the following relations

$$F(D_x^\alpha f) = (i\xi_1)^{\alpha_1},\ldots,(i\xi_n)^{\alpha_n}\hat{f}, \qquad D_\xi^\alpha(F(f)) = F[(-ix_n)^{\alpha_1},\ldots,(-ix_n)^{\alpha_n} f] \tag{2.8}$$

are obtained for all $f \in S'(R^n;E)$.

Let $L_\theta^*(E)$ denote the space of all E-valued function spaces such that

$$\|u\|_{L_\theta^*(E)} = \left(\int_0^\infty \|u(t)\|_E^\theta \frac{dt}{t}\right)^{1/\theta} < \infty, \quad 1 \le \theta < \infty, \qquad \|u\|_{L_\infty^*(E)} = \sup_{0<t<\infty} \|u(t)\|_E. \tag{2.9}$$

Let $s=(s_1,s_2,\ldots,s_n)$ and $s_k>0$. Let F denote the Fourier transform. Fourier-analytic representation of E-valued Besov space on R^n is defined as

$$B^s_{p,\theta}(R^n;E)=\Bigg\{u\in S'(R^n;E),\ \|u\|_{B^s_{p,\theta}(R^n;E)} = \left\|F^{-1}\sum_{k=1}^{n}t^{\varkappa_k-s_k}\left(1+|\xi_k|^{\varkappa_k}\right)e^{-t|\xi|^2}Fu\right\|_{L^*_\theta(L_p(R^n;E))}, \quad p\in(1,\infty),\ \theta\in[1,\infty],\ \varkappa_k>s_k\Bigg\}. \tag{2.10}$$

It should be noted that the norm of Besov space do not depend on $\varkappa_k$. Sometimes we will write $\|u\|_{B^s_{p,\theta}}$ in place of $\|u\|_{B^s_{p,\theta}(R^n;E)}$.

Let $l=(l_1,l_2,\ldots,l_n)$, $s=(s_1,s_2,\ldots,s_n)$, where l_k are integers and s_k are positive numbers. Let $W^lB^s_{p,\theta}(R^n;E)$ denote an E-valued Sobolev-Besov space of all functions $u\in B^s_{p,\theta}(R^n;E)$ such that they have the generalized derivatives $D^{l_k}_k u=\partial^{l_k}u/\partial x^{l_k}_k\in B^s_{p,\theta}(R^n;E)$, $k=1,2,\ldots,n$ with the norm

$$\|u\|_{W^lB^s_{p,\theta}(R^n;E)}=\|u\|_{B^s_{p,\theta}(R^n;E)}+\sum_{k=1}^{n}\|D^{l_k}_k u\|_{B^s_{p,\theta}(R^n;E)}<\infty. \tag{2.11}$$

Let E_0 is continuously and densely embedded into E. $W^lB^s_{p,\theta}(R^n;E_0,E)$ denotes a space of all functions $u\in B^s_{p,\theta}(R^n;E_0)\cap W^lB^s_{p,\theta}(R^n;E)$ with the norm

$$\|u\|_{W^lB^s_{p,\theta}}=\|u\|_{W^lB^s_{p,\theta}(R^n;E_0,E)}=\|u\|_{B^s_{p,\theta}(R^n;E_0)}+\sum_{k=1}^{n}\left\|D^{l_k}_k u\right\|_{B^s_{p,\theta}(R^n;E)}<\infty. \tag{2.12}$$

Let $l=(l_1,l_2,\ldots,l_n)$, $s=(s_1,s_2,\ldots,s_n)$, where s_k are real numbers and l_k are positive numbers. $B^{l+s}_{p,\theta}(R^n;E_0,E)$ denotes a space of all functions $u\in B^s_{p,\theta}(R^n;E_0)\cap B^{l+s}_{p,\theta}(R^n;E)$ with the norm

$$\|u\|_{B^{s+l}_{p,\theta}(R^n;E_0,E)}=\|u\|_{B^s_{p,\theta}(R^n;E_0)}+\|u\|_{B^{l+s}_{p,\theta}(R^n;E)}. \tag{2.13}$$

For $E_0=E$ the space $B^{l+s}_{p,\theta}(R^n;E_0,E)$ will be denoted by $B^{l+s}_{p,\theta}(R^n;E)$.

Let m be a positive integer. $C(\Omega;E)$ and $C^m(\Omega;E)$ will denote the spaces of all E-valued bounded continuous and m-times continuously differentiable functions on Ω, respectively. We set

$$C_b(\Omega;E)=\left\{u\in C(\Omega;E),\ \lim_{|x|\to\infty}u(x)\text{ exists}\right\}. \tag{2.14}$$

Let E_1 and E_2 be two Banach spaces. A function $\Psi\in C^m(R^n;L(E_1,E_2))$ is called a multiplier from $B^s_{p,\theta}(R^n;E_1)$ to $B^s_{q,\theta}(R^n;E_2)$ for $p\in(1,\infty)$ and $q\in[1,\infty]$ if the map $u\to Ku=F^{-1}\Psi(\xi)Fu$, $u\in S(R^n;E_1)$, is well defined and extends to a bounded linear operator

$$K:B^s_{p,\theta}(R^n;E_1)\longrightarrow B^s_{q,\theta}(R^n;E_2). \tag{2.15}$$

The set of all multipliers from $B^s_{p,\theta}(R^n;E_1)$ to $B^s_{q,\theta}(R^n;E_2)$ will be denoted by $M^{q,\theta}_{p,\theta}(s,E_1,E_2)$. $E_1 = E_2 = E$ will be denoted by $M^{q,\theta}_{p,\theta}(s,E)$. The multipliers and operator-valued multipliers in Banach-valued function spaces were studied, for example, by [25], [37, Section 2.2.2.], and [4, 11, 12, 14, 22], respectively.

Let

$$H_k = \{\Psi_h \in M^{q,\theta}_{p,\theta}(s,E_1,E_2),\ h = (h_1h_2,\dots,h_n) \in K\} \tag{2.16}$$

be a collection of multipliers in $M^{q,\theta}_{p,\theta}(s,E_1,E_2)$. We say that H_k is a uniform collection of multipliers if there exists a constant $M_0 > 0$, independent on $h \in K$, such that

$$||F^{-1}\Psi_h Fu||_{B^s_{p,\theta}(R^n;E_2)} \le M_0\|u\|_{B^s_{q,\theta}(R^n;E_1)} \tag{2.17}$$

for all $h \in K$ and $u \in S(R^n;E_1)$.

Let $\beta = (\beta_1,\beta_2,\dots,\beta_n)$ be multiindexes. We also define

$$\begin{gathered} V_n = \{\xi = (\xi_1,\xi_2,\dots,\xi_n) \in R^n,\ \xi_i \neq 0,\ i = 1,2,\dots,n\}, \\ U_n = \{\beta : |\beta| \le n\}, \qquad \xi^\beta = \xi_1^{\beta_1}\xi_2^{\beta_2},\dots,\xi_n^{\beta_n}, \qquad \nu = \frac{1}{p} - \frac{1}{q}. \end{gathered} \tag{2.18}$$

Definition 2.1. A Banach space E satisfies a B-multiplier condition with respect to p, q, θ, and s (or with respect to p, θ, and s for the case of $p = q$) when $\Psi \in C^n(R^n;L(E))$, $1 \le p \le q \le \infty$, $\beta \in U_n$, and $\xi \in V_n$ if the estimate

$$|\xi_1|^{\beta_1+\nu}|\xi_2|^{\beta_2+\nu},\dots,|\xi_n|^{\beta_n+\nu}||D^\beta\Psi(\xi)||_{L(E)} \le C \tag{2.19}$$

implies $\Psi \in M^{q,\theta}_{p,\theta}(s,E)$.

Remark 2.2. Definition 2.1 is a combined restriction to E, p, q, θ, and s. This condition is sufficient for our main aim. Nevertheless, it is well known that there are Banach spaces satisfying the B-multiplier condition for isotropic case and $p = q$, for example, the UMD spaces (see [4, 14]).

A Banach space E is said to have a local unconditional structure (l.u.st.) if there exists a constant $C < \infty$ such that for any finite-dimensional subspace E_0 of E there exists a finite-dimensional space F with an unconditional basis such that the natural embedding $E_0 \subset E$ factors as AB with $B : E_0 \to F$, $A : F \to E$, and $\|A\|\|B\| \le C$. All Banach lattices (e.g., L_p, $L_{p,q}$, Orlicz spaces, $C[0,1]$) have l.u.st.

The expression $\|u\|_{E_1} \sim \|u\|_{E_2}$ means that there exist the positive constants C_1 and C_2 such that

$$C_1\|u\|_{E_1} \le \|u\|_{E_2} \le C_2\|u\|_{E_1} \tag{2.20}$$

for all $u \in E_1 \cap E_2$.

Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be nonnegative and let $l_1,l_2,\ldots,l_n$ be positive integers and let

$$1 \le p \le q \le \infty, \quad 1 \le \theta \le \infty, \quad |\alpha : .l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k}, \quad \varkappa = \sum_{k=1}^{n} \frac{\alpha_k + 1/p - 1/q}{l_k},$$
$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2},\ldots,D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2},\ldots,\partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_{k=!}^{n} \alpha_k. \tag{2.21}$$

Consider in general, the anisotropic differential-operator equation

$$(L+\lambda)u = \sum_{|\alpha:.l|=1} a_\alpha(x) D^\alpha u + A_\lambda(x)u + \sum_{|\alpha:.l|<1} A_\alpha(x) D^\alpha u = f \tag{2.22}$$

in $B^s_{p,\theta}(R^n;E)$, where a_α are complex-valued functions and $A(x)$, $A_\alpha(x)$ are possibly unbounded operators in a Banach space E, here the domain definition $D(A) = D(A(x))$ of operator $A(x)$ does not depend on x. For $l_1 = l_2 =,\ldots,= l_n$ we obtain isotropic equations containing the elliptic class of DOE.

The function belonging to space $B^{s+l}_{p,\theta}(R^n;E(A),E)$ and satisfying (2.22) a.e. on R^n is said to be a solution of (2.22) on R^n.

Definition 2.3. The problem (2.22) is said to be a B-separable (or $B^s_{p,\theta}(R^n;E)$-separable) if the problem (2.22) for all $f \in B^s_{p,\theta}(R^n;E)$ has a unique solution $u \in B^{s+l}_{p,\theta}(R^n;E(A),E)$ and

$$\|Au\|_{B^s_{p,\theta}(R^n;E)} + \sum_{|\alpha:l|=1} \|D^\alpha u\|_{B^s_{p,\theta}(R^n;E)} \le C\|f\|_{B^s_{p,\theta}(R^n;E)}. \tag{2.23}$$

Consider the following parabolic Cauchy problem

$$\frac{\partial u(y,x)}{\partial y} + (L+\lambda)u(y,x) = f(y,x), \quad u(0,x) = 0,\ y \in R_+,\ x \in R^n, \tag{2.24}$$

where L is a realization differential operator in $B^s_{p,\theta}(R^n;E)$ generated by problem (2.22), that is,

$$D(L) = B^{s+l}_{p,\theta}(R^n;E(A),E), \qquad Lu = \sum_{|\alpha:.l|=1} a_\alpha(x) D^\alpha u + A(x)u + \sum_{|\alpha:.l|<1} A_\alpha(x) D^\alpha u. \tag{2.25}$$

We say that the parabolic Cauchy problem (2.24) is said to be a maximal B-regular, if for all $f \in B^s_{p,\theta}(R^{n+1}_+;E)$ there exists a unique solution u satisfying (2.24) almost everywhere on R^{n+1}_+ and there exists a positive constant C independent on f, such that it has the estimate

$$\left\|\frac{\partial u(y,x)}{\partial y}\right\|_{B^s_{p,\theta}(R^{n+1}_+;E)} + \|Lu\|_{B^s_{p,\theta}(R^{n+1}_+;E)} \le C\|f\|_{B^s_{p,\theta}(R^{n+1}_+;E)}. \tag{2.26}$$

3. Embedding theorems

In this section we prove the boundedness of the mixed differential operators D^α in the Besov-Lions type spaces.

Lemma 3.1. *Let A be a positive operator in a Banach space E, let b be a positive number, $r=(r_1,r_2,\ldots,r_n)$, $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$, and $l=(l_1,l_2,\ldots,l_n)$, where $\varphi\in(0,\pi]$, $r_k\in[0,b]$, l_k are positive and α_k, $k=1,2,\ldots,n$, are nonnegative integers such that $\varkappa=|(\alpha+r):l|\leq 1$. For $0<h\leq h_0<\infty$ and $0\leq\mu\leq 1-\varkappa$ the operator-function*

$$\Psi(\xi)=\Psi_{h,\mu}(\xi)=|\xi_1|^{r_1}|\xi_2|^{r_2},\ldots,|\xi_n|^{r_n}(i\xi)^{\alpha}A^{1-\varkappa-\mu}h^{-\mu}\left[A+\eta(\xi)\right]^{-1} \tag{3.1}$$

is a bounded operator in E uniformly with respect to ξ and h, that is, there is a constant C_μ such that

$$\left\|\Psi_{h,\mu}(\xi)\right\|_{L(E)}\leq C_\mu \tag{3.2}$$

for all $\xi\in R^n$, where

$$\eta=\eta(\xi)=\sum_{k=1}^{n}|\xi_k|^{l_k}+h^{-1}. \tag{3.3}$$

Proof. Since $-\eta(\xi)\in S(\varphi)$, for all $\varphi\in(0,\pi]$ and A is a φ-positive in E, then the operator $A+\eta(\xi)$ is invertiable in E. Let

$$u=h^{-\mu}\left[A+\eta(\xi)\right]^{-1}f. \tag{3.4}$$

Then

$$\left\|\Psi(\xi)f\right\|_E=\left\|(hA)^{1-\varkappa-\mu}u\right\|_E h^{-(1-\mu)}|h^{1/l_1}\xi_1|^{\alpha_1+r_1},\ldots,|h^{1/l_n}\xi_n|^{\alpha_n+r_n}. \tag{3.5}$$

Using the moment inequality for powers of positive operators, we get a constant C_μ depending only on μ such that

$$\left\|\Psi(\xi)f\right\|_E\leq C_\mu h^{-(1-\mu)}\|hAu\|^{1-\varkappa-\mu}\|u\|^{\varkappa+\mu}|h^{1/l_1}\xi_1|^{\alpha_1+r_1},\ldots,|h^{1/l_n}\xi_n|^{\alpha_n+r_n}. \tag{3.6}$$

Now, we apply the Young inequality, which states that $ab\leq a^{k_1}/k_1+b^{k_2}/k_2$ for any positive real numbers a, b and k_1, k_2 with $1/k_1+1/k_2=1$ to the product

$$\|hAu\|^{1-\varkappa-\mu}\left[\|u\|^{\varkappa+\mu}|h^{1/l_1}\xi_1|^{\alpha_1+r_1},\ldots,|h^{1/l_n}\xi_n|^{\alpha_n+r_n}\right] \tag{3.7}$$

with $k_1=1/(1-\varkappa-\mu)$, $k_2=1/(\varkappa+\mu)$ to get

$$\begin{aligned}\left\|\Psi(\xi)f\right\|_E\leq C_\mu h^{-(1-\mu)}\Big\{&(1-\varkappa-\mu)\|hAu\|\\&+(\varkappa+\mu)\left[h^{1/l_1}|\xi_1|\right]^{(\alpha_1+r_1)/(\varkappa+\mu)},\ldots,\left[h^{1/l_n}|\xi_n|\right]^{(\alpha_n+r_n)/(\varkappa+\mu)}\|u\|\Big\}.\end{aligned} \tag{3.8}$$

Since

$$\sum_{i=1}^{n}\frac{\alpha_i+r_i}{(\varkappa+\mu)}=\frac{1}{\varkappa+\mu}\sum_{i=1}^{n}\frac{\alpha_i+r_i}{l_i}=\frac{\varkappa}{\varkappa+\mu}\leq 1, \tag{3.9}$$

there exists a constant M_0 independent on ξ, such that

$$|\xi_1|^{(\alpha_1+r_1)/(\varkappa+\mu)},\ldots,|\xi_n|^{(\alpha_n+r_n)/(\varkappa+\mu)} \le M_0\left(1+\sum_{k=1}^{n}|\xi_k|^{l_k}\right) \tag{3.10}$$

for all $\xi \in R^n$. Substituting this on the inequality (3.8) and absorbing the constant coefficients in C_μ, we obtain

$$\|\psi(\xi)f\| \le C_\mu\left[h^\mu\left(\|Au\|+\sum_{k=1}^{n}|\xi_k|^{l_k}\|u\|\right)+h^{-(1-\mu)}\|u\|\right]. \tag{3.11}$$

Substituting the value of u we get

$$\begin{aligned}\|\psi(\xi)f\| \le C_\mu h^\mu&\left[\|A[A+\eta(\xi)]^{-1}f\|+\sum_{k=1}^{n}|\xi_k|^{l_k}\|[A+\eta(\xi)]^{-1}f\|\right]\\ &+h^{-(1-\mu)}\left\|[A+\eta(\xi)]^{-1}f\right\|.\end{aligned} \tag{3.12}$$

By using the properties of the positive operator A for all $f \in E$ we obtain from (3.12)

$$\|\Psi(\xi)f\|_E \le C_\mu\|f\|_E. \tag{3.13}$$

□

Lemma 3.2. *Let E be a UMD space with l.u.st., $p \in (1,\infty)$, $\theta \in [1,\infty]$ and let for all $k, j \in (1,n)$*

$$\frac{s_k}{l_k+s_k}+\frac{s_j}{l_j+s_j} \le 1. \tag{3.14}$$

Then the spaces $B^{l+s}_{p,\theta}(R^n;E)$ and $W^lB^s_{p,\theta}(R^n;E)$ are coincided.

Proof. In the first step we show that the continuous embedding $W^lB^s_{p,\theta}(R^n;E) \subset B^{l+s}_{p,\theta}(R^n;E)$ holds, that is, there is a positive constant C such that

$$\|u\|_{B^{l+s}_{p,\theta}(R^n;E)} \le C\|u\|_{W^lB^s_{p,\theta}(R^n;E)} \tag{3.15}$$

for all $u \in W^lB^s_{p,\theta}(R^n;E)$. For this aim by using the Fourier-analytic definition of an E-valued Besov space and the space $W^lB^s_{p,\theta}(R^n;E)$ it is sufficient to prove the following estimate:

$$\left\|F^{-1}\sum_{k=1}^{n}t^{\varkappa_k-l_k-s_k}\left(1+|\xi_k|^{\varkappa_k}\right)e^{-t|\xi|^2}Fu\right\|_{L_{\theta p}} \le C\left\|F^{-1}\sum_{k=1}^{n}t^{\varkappa_k-s_k}\left(1+|\xi_k|^{\varkappa_k}\right)e^{-t|\xi|^2}Fv\right\|_{L_{\theta p}}, \tag{3.16}$$

where

$$L_{\theta p}=L^*_\theta\left(L_p(R^n;E)\right), \qquad v=F^{-1}\left(1+\sum_{k=1}^{n}\xi_k^{l_k}\right)Fu. \tag{3.17}$$

To see this, it is sufficient to show that the function

$$\phi(\xi)=\sum_{k=1}^{n}\left(1+|\xi_k|^{l_k+s_k+\delta}\right)\left(\sum_{k=1}^{n}\left(1+|\xi_k|^{s_k+\delta}\right)\right)^{-1}\left(1+\sum_{k=1}^{n}|\xi_k|^{l_k}\right)^{-1},\quad \delta>0 \tag{3.18}$$

is Fourier multiplier in $L_p(R^n;E)$. It is clear to see that for $\beta\in U_n$ and $\xi\in V_n$

$$|\xi_1|^{\beta_1}|\xi_2|^{\beta_2},\dots,|\xi_n|^{\beta_n}\|D^\beta\phi(\xi)\|_{L(E)}\leq C. \tag{3.19}$$

Then in view of [41, Proposition 3] we obtain that the function ϕ is Fourier multiplier in $L_p(R^n;E)$.

In the second step we prove that the embedding $B^{l+s}_{p,\theta}(R^n;E)\subset W^lB^s_{p,\theta}(R^n;E)$ is continuous. In a similar way as in the first step we show that for $s_k/(l_k+s_k)+s_j/(l_j+s_j)\leq 1$ the function

$$\psi(\xi)=\left(\sum_{k=1}^{n}\left(1+|\xi_k|^{s_k+\delta}\right)\right)\left(1+\sum_{k=1}^{n}|\xi_k|^{l_k}\right)\left[\sum_{k=1}^{n}\left(1+|\xi_k|^{l_k+s_k+\delta}\right)\right]^{-1} \tag{3.20}$$

is Fourier multiplier in $L_p(R^n;E)$. So, we obtain for all $u\in B^{l+s}_{p,\theta}(R^n;E)$ the estimate

$$\begin{aligned}&\left\|F^{-1}\sum_{k=1}^{n}t^{\varkappa_k-s_k}\left(1+|\xi_k|^{\varkappa_k}\right)\left(1+\sum_{k=1}^{n}\xi_k^{l_k}\right)e^{-t|\xi|^2}Fu\right\|_{L_\theta p}\\&\quad\leq C\left\|F^{-1}\sum_{k=1}^{n}t^{\varkappa_k-l_k-s_k}\left(1+|\xi_k|^{\varkappa_k}\right)e^{-t|\xi|^2}Fu\right\|_{L_\theta p}.\end{aligned} \tag{3.21}$$

It implies the second embedding. This completes the prove of Lemma 3.2. □

Theorem 3.3. *Suppose the following conditions hold:*

(1) *E is a UMD space with l.u.st. satisfying the B-multiplier condition with respect to $p,q\in(1,\infty)$, $\theta\in[1,\infty]$, and $s=(s_1,s_2,\dots,s_n)$, where s_k are positive numbers;*

(2) *$\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$, $l=(l_1,l_2,\dots,l_n)$, where α_k are nonnegative, l_k are positive integers, and s_k such that $s_k/(l_k+s_k)+s_j/(l_j+s_j)\leq 1$ for $k,j=1,2,\dots,n$ and $0\leq\mu\leq 1-\varkappa$, $\varkappa=|(\alpha+1/p-1/q):l|$;*

(3) *A is a φ-positive operator in E, where $\varphi\in(0,\pi]$ and $0<h\leq h_0<\infty$.*

Then the following embedding

$$D^\alpha B^{l+s}_{p,\theta}(R^n;E(A),E)\subset B^s_{q,\theta}(R^n;E(A^{1-\varkappa-\mu})) \tag{3.22}$$

is continuous and there exists a positive constant C_μ depending only on μ, such that

$$\|D^\alpha u\|_{B^s_{q,\theta}(R^n;E(A^{1-\varkappa-\mu}))}\leq C_\mu\left[h^\mu\|u\|_{B^{l+s}_{p,\theta}(R^n;E(A),E)}+h^{-(1-\mu)}\|u\|_{B^s_{p,\theta}(R^n;E)}\right] \tag{3.23}$$

for all $u\in B^{l+s}_{p,\theta}(R^n;E(A),E)$.

Proof. We have

$$\|D^\alpha u\|_{B^s_{q,\theta}(R^n;E(A^{1-\varkappa-\mu}))} = \|A^{1-\varkappa-\mu}D^\alpha u\|_{B^s_{q,\theta}(R^n;E)} \tag{3.24}$$

for all u such that

$$\|D^\alpha u\|_{B^s_{q,\theta}(R^n;E(A^{1-\varkappa-\mu}))} < \infty. \tag{3.25}$$

On the other hand by using the relation (2.8) we have

$$A^{1-\alpha-\mu}D^\alpha u = F^{-1}FA^{1-\varkappa-\mu}D^\alpha u = F^{-1}(i\xi)^\alpha A^{1-\varkappa-\mu}Fu. \tag{3.26}$$

Since the operator A is closure and does not depend on $\xi \in R^n$ hence denoting Fu by $\hat{u}$, from the relations (3.24), (3.26) and by definition of the space $W^lB^s_{p,\theta}(R^n;E_0,E)$ we have

$$\begin{gathered}\|D^\alpha u\|_{B^s_{q,\theta}(R^n;E(A^{1-\varkappa-\mu}))} \backsim \|F^{-1}(i\xi)^\alpha A^{1-\varkappa-\mu}\hat{u}\|_{B^s_{q,\theta}(R^n;E)},\\ \|u\|_{W^lB^s_{p,\theta}(R^n;E_0,E)} \sim \|Au\|_{B^s_{p,\theta}(R^n;E)} + \sum_{k=1}^n \|F^{-1}\xi_k^{l_k}\hat{u}\|_{B^s_{p,\theta}(R^n;E)}.\end{gathered} \tag{3.27}$$

By virtue of Lemma 3.2 and by the above relations it is sufficient to prove that

$$\begin{aligned}&\|F^{-1}(i\xi)^\alpha A^{1-\varkappa-\mu}\hat{u}\|_{B^s_{q,\theta}(R^n;E)}\\ &\quad\le C_\mu\left[h^\mu\left(\|F^{-1}A\hat{u}\|_{B^s_{p,\theta}(R^n;E)} + \sum_{k=1}^n \|F^{-1}(\xi_k^{l_k}\hat{u})\|_{B^s_{p,\theta}(R^n;E)}\right) + h^{-(1-\mu)}\|F^{-1}\hat{u}\|_{B^s_{p,\theta}(R^n;E)}\right].\end{aligned} \tag{3.28}$$

The inequality (3.23) will be followed if we prove the following inequality

$$\|F^{-1}[(i\xi)^\alpha A^{1-\varkappa-\mu}\hat{u}]\|_{B^s_{p,\theta}(R^n;E)} \le C_\mu\|F^{-1}[h^\mu(A+\eta)]\hat{u}\|_{B^s_{p,\theta}(R^n;E)} \tag{3.29}$$

for a suitable C_μ and for all $u \in B^{s+l}_{p,\theta}(R^n;E(A),E)$, where

$$\eta = \eta(\xi) = \sum_{k=1}^n |\xi_k|^{l_k} + h^{-1}. \tag{3.30}$$

Let us express the left-hand side of (3.29) as follows:

$$\|F^{-1}[(i\xi)^\alpha A^{1-\varkappa-\mu}\hat{u}]\|_{B^s_{q,\theta}(R^n;E)} \tag{3.31}$$

$$= \|F^{-1}(i\xi)^\alpha A^{1-\varkappa-\mu}[h^\mu(A+\eta)]^{-1}[h^\mu(A+\eta)]\hat{u}\|_{B^s_{q,\theta}(R^n;E)}. \tag{3.32}$$

(Since A is the positive operator in E and $-\eta(\xi) \in S(\varphi)$ so it is possible). By virtue of Definition 2.1 it is clear that the inequality (3.23) will follow immediately from (3.31) if we can prove that the operator-function $\Psi = (i\xi)^\alpha A^{1-\varkappa-\mu}[h^\mu(A+\eta)]^{-1}$ is a multiplier in

$M_{p,\theta}^{q,\theta}(s,E)$, which is uniform with respect to h. Since E satisfies the multiplier condition with respect to p, q, θ, and s, then by Definition 2.1 in order to show that $\Psi \in M_{p,\theta}^{q,\theta}(s,E)$, it suffices to show that there exists a constant $M_\mu > 0$ with

$$|\xi_1|^{\beta_1+\nu}|\xi_2|^{\beta_2+\nu},\dots,|\xi_n|^{\beta_n+\nu}\|D_\xi^\beta \Psi(\xi)\|_{L(E)} \le M_\mu \tag{3.33}$$

for all $\beta \in U_n$, $\xi \in V_n$, and $0 < h \le h_0 < \infty$. To see this, we apply Lemma 3.1 and get a constant $M_\mu > 0$ depending only on μ such that

$$|\xi_1|^{\nu}|\xi_2|^{\nu},\dots,|\xi_n|^{\nu}\|\Psi(\xi)\|_{L(E)} \le M_\mu \tag{3.34}$$

for all $\xi \in R^n$ and $\nu = 1/p - 1/q$. This shows that the inequality (3.33) is satisfied for $\beta = (0,\dots,0)$. We next consider (3.33) for $\beta = (\beta_1,\dots,\beta_n)$ where $\beta_k = 1$ and $\beta_j = 0$ for $j \ne k$. By differentiation of the operator-function $\Psi(\xi)$, by virtue of the positivity of A, and by using (3.34) we have

$$\left\|\frac{\partial}{\partial \xi_k}\Psi(\xi)\right\|_{L(E)} \le M_\mu |\xi_k|^{-(1+\nu)}, \quad k = 1,2\dots,n. \tag{3.35}$$

Repeating the above process we obtain the estimate (3.33). Thus the operator-function $\Psi_{h,\mu}(\xi)$ is a uniform multiplier with respect to h, that is,

$$\Psi_{h,\mu} \in H_K \subset M_{p,\theta}^{q,\theta}(s,E), \quad K = \mathbb{R}_+. \tag{3.36}$$

This completes the proof of Theorem 3.3. □

Result 3.4. Let all conditions of Theorem 3.3 hold. Then for all $u \in B_{p,\theta}^{l+s}(R^n;E(A),E)$ we have a multiplicative estimate

$$\|D^\alpha u\|_{B_{q,\theta}^s(R^n;E(A^{1-\varkappa-\mu}))} \le C_\mu \|u\|_{B_{p,\theta}^{l+s}(R^n;E(A),E)}^{1-\mu}\|u\|_{B_{p,\theta}^s(R^n;E)}^{\mu}. \tag{3.37}$$

Indeed setting $h = \|u\|_{B_{p,\theta}^s(R^n;E)} \cdot \|u\|_{B_{p,\theta}^{l+s}(R^n;E(A),E)}^{-1}$ in the estimate (3.23) we obtain the above estimate.

Remark 3.5. It seems from the proof of Theorem 3.3 that the extra condition to space E (E is UMD space with l.u.st.) and the condition $s_k/(l_k+s_k)+s_j/(l_j+s_j) \le 1$ for $k,j = 1,2,\dots,n$ are due to Lemma 3.2 (here the l.u.st. condition for the space E is required due to using of Marcinkiewicz-Lizorkin type multiplier theorem [41] in $L_p(R^n;E)$ space). Therefore, the proof of Theorem 3.3 implies the following.

Result 3.6. Suppose the following conditions hold:

(1) E is a Banach space satisfying the B-multiplier condition with respect to $p,q \in (1,\infty)$, $\theta \in [1,\infty]$ and $s = (s_1,s_2,\dots,s_n)$, where s_k are positive numbers;

(2) $\alpha = t(\alpha_1,\alpha_2,\dots,\alpha_n)$, $l = (l_1,l_2,\dots,l_n)$, where α_k are nonnegative and l_k are positive integers such that $\varkappa = |(\alpha+1/p-1/q):l| \le 1$ and let $0 \le \mu \le 1-\varkappa$;

(3) A is a φ-positive operator in E, where $\varphi \in (0,\pi]$ and $0 < h \le h_0 < \infty$.

Then the following embedding

$$D^{\alpha} W^{l} B^{s}_{p,\theta}(R^n;E(A),E) \subset B^{s}_{q,\theta}(R^n;E(A^{1-\varkappa-\mu})) \tag{3.38}$$

is continuous and there exists a positive constant C_μ depending only on μ such that

$$\|D^{\alpha}u\|_{B^{s}_{q,\theta}(R^n;E(A^{1-\varkappa-\mu}))} \leq C_{\mu}\left[h^{\mu}\|u\|_{W^{l}B^{s}_{p,\theta}(R^n;E(A),E)} + h^{-(1-\mu)}\|u\|_{B^{s}_{p,\theta}(R^n;E)}\right] \tag{3.39}$$

for all $u \in W^{l}B^{s}_{p,\theta}(R^n;E(A),E)$.

Remark 3.7. The condition $s_k/(l_k+s_k)+s_j/(l_j+s_j) \leq 1$ for $k,j=1,2,\ldots,n$ in Theorem 3.3 arise due to anisotropic nature of space $B^{s}_{p,\theta}$. For an isotropic case the above conditions hold without any assumptions.

4. Application to vector-valued function spaces

By virtue of Theorem 3.3 we obtain the following.

Result 4.1. For $A = I$ we obtain the continuous embedding $D^{\alpha}B^{l+s}_{p,\theta}(R^n;E) \subset B^{s}_{p,\theta}(R^n;E)$ and corresponding estimate (3.23) for $0 \leq \mu \leq 1-\varkappa$ in space $B^{s+l}_{p,\theta}(R^n;E)$.

Result 4.2. For $E=R^m$, $A=I$ we obtain the following embedding $D^{\alpha}B^{l+s}_{p,\theta}(R^n;R^m) \subset B^{s}_{q,\theta}(R^n;R^m)$ for $0 \leq \mu \leq 1-\varkappa$ and a corresponding estimate (3.23). For $E = R$, $A = I$ we get the embedding $D^{\alpha}B^{l+s}_{p,\theta}(R^n) \subset B^{s}_{q,\theta}(R^n)$ proved in [8, Section 18] for the numerical Besov spaces.

Result 4.3. Let $l_1 = l_2 = \cdots = l_n = m$, $s_1 = s_2 = \cdots = s_n = \sigma$, and $p = q$. Then for all $E \in$ UMD and $|\alpha| \leq m$ we obtain that the continuous embedding $D^{\alpha}B^{\sigma+m}_{p,\theta}(R^n;E(A),E) \subset B^{\sigma}_{p,\theta}(R^n;E(A^{1-|\alpha|/m}))$ and a corresponding estimate (3.23) for the isotropic Besov-Lions spaces $B^{\sigma+m}_{p,\theta}(R^n;E(A),E)$.

Result 4.4. Let σ be a positive number. Consider the following space [37, Section 1.18.2]:

$$l^{\sigma}_{q} = \{u;\ u=\{u_i\},\ i=1,2,\ldots,\infty,\ u_i \in \mathbf{C}\} \tag{4.1}$$

with the norm

$$\|u\|_{l^{\sigma}_{q}} = \left(\sum_{i=1}^{\infty} 2^{iq\sigma}\,|u_i|^{q}\right)^{1/q} < \infty. \tag{4.2}$$

Note that $l^{0}_{q} = l_q$. Let A be an infinite matrix defined in l_q such that

$$D(A) = l^{\sigma}_{q}, \quad A = [\delta_{ij}2^{si}], \tag{4.3}$$

where $\delta_{ij} = 0$, when $i \neq j$, $\delta_{ij} = 1$, when $i = j$, $i,j = 1,2,\ldots,\infty$. It is clear to see that this operator A is positive in l_q. Then from Theorem 3.3 for $s_k/(l_k+s_k)+s_j/(l_j+s_j) \leq 1$, $k,j=1,2,\ldots,n$ and $0 \leq \mu \leq 1-\varkappa$, $\varkappa = \sum_{k=1}^{n}(\alpha_k+1/p_1-1/p_2)/l_k$ we obtain the continuous embedding $D^{\alpha}B^{l+s}_{p_1,\theta}(\Omega;l^{\sigma}_{q},l_q) \subset B^{s}_{p_2,\theta}(\Omega;l^{\sigma(1-\varkappa-\mu)}_{q})$ and the corresponding estimate (3.23).

It should not be that the above embedding has not been obtained with a classical method until now.

5. Maximal B-regular DOE in R^n

Consider the following differential-operator equation

$$(L+\lambda)u=\sum_{|\alpha:.l|=1}a_\alpha(x)D^\alpha u+A_\lambda(x)u+\sum_{|\alpha:.l|<1}A_\alpha(x)D^\alpha u=f \tag{5.1}$$

in $B^s_{p,q}(R^n;E)$, where $A(x)$, $A_\alpha(x)$ are possible unbounded operators in a Banach space E, a_k are complex-valued functions, $l=(l_1,l_2,\dots,l_n)$ and l_i are positive integers. The maximal regularity for DOE was investigated, for example, in [12, 14, 30]. Let us consider DOE with constant coefficients

$$(L_0+\lambda)u=\sum_{|\alpha:.l|=1}b_\alpha D^\alpha u+A_\lambda u=f, \tag{5.2}$$

where A is a possible unbounded operator in E, $A_\lambda=A+\lambda$ and b_α are complex numbers.

Theorem 5.1. *Suppose the following conditions hold:*

(1) *E is UMD space with l.u.st. satisfying B-multiplier condition with respect to $p\in(1,\infty)$, $q\in[1,\infty]$, and $s=(s_1,s_2,\dots,s_n)$, where s_k are positive numbers;*

(2) *A is a φ-positive operator in E with $\varphi\in(0,\pi]$ and*

$$K(\xi)=-\sum_{|\alpha:.l|=1}b_\alpha(i\xi_1)^{\alpha_1}\cdot(i\xi_2)^{\alpha_2},\dots,(i\xi_n)^{\alpha_n}\in S(\varphi),\quad |K(\xi)|\geq C\sum_{k=1}^n|\xi_k|^{l_k},\quad \xi\in R^n; \tag{5.3}$$

(3) $s_k/(l_k+s_k)+s_j/(l_j+s_j)\leq 1$ *for* $k,j=1,2,\dots,n$.

Then for all $f\in B^s_{p,q}(R^n;E)$, for $|\arg\lambda|\leq\pi-\varphi$ and sufficiently large $|\lambda|>0$ (5.2) has a unique solution $u(x)$ that belongs to space $B^{l+s}_{p,q}(R^n;E(A),E)$, and the coercive uniform estimate

$$\sum_{|\alpha:.l|\leq 1}|\lambda|^{1-|\alpha:.l|}\|D^\alpha u\|_{B^s_{p,q}}+\|Au\|_{B^s_{p,q}}\leq C\|f\|_{B^s_{p,q}} \tag{5.4}$$

holds with respect to the parameter λ.

Proof. By applying the Fourier transform to (5.2) we obtain

$$[K(\xi)+A_\lambda]\hat{u}(\xi)=\hat{f}(\xi). \tag{5.5}$$

Since $K(\xi)\in S(\varphi)$ for all $\xi\in R^n$, the operator $A+[\lambda+K(\xi)]$ is invertible in E. So, we obtain that the solution of (5.5) can be represented in the form

$$u(x)=F^{-1}[A+\lambda+K(\xi)]^{-1}\hat{f}. \tag{5.6}$$

By using (5.6) we have

$$\|Au\|_{B^s_{p,q}} = \left\|F^{-1}A\left[A+(\lambda+K(\xi))\right]^{-1}\hat{f}\right\|_{B^s_{p,q}},$$
$$\left\|D^\alpha u\right\|_{B^s_{p,q}} = \left\|F^{-1}(i\xi_1)^{\alpha_1}\cdot(i\xi_2)^{\alpha_2},\dots,(i\xi_n)^{\alpha_n}\left[A+(\lambda+K(\xi))\right]^{-1}\hat{f}\right\|_{B^s_{p,q}}. \tag{5.7}$$

Hence, it is suffices to show that the operator-functions

$$\sigma_{1\lambda}(\xi) = \left[A+(\lambda+K(\xi))\right]^{-1},$$
$$\sigma_{2\lambda}(\xi) = |\lambda|^{1-|\alpha:.l|}(i\xi_1)^{\alpha_1}\cdot(i\xi_2)^{\alpha_2},\dots,(i\xi_n)^{\alpha_n}\left[A+(\lambda+K(\xi))\right]^{-1} \tag{5.8}$$

are multipliers in $B^s_{p,q}(R^n;E)$ uniformly with respect to λ. Firstly, by using the positivity properties of operator A we obtain that the operator function $\sigma_\lambda(\xi)$ is bounded uniformly with respect to λ. That is,

$$\left\|\sigma_{j\lambda}(\xi)\right\|_{B(E)} \le C, \quad j=1,2. \tag{5.9}$$

Then by virtue of the same properties of the operator A we obtain from (5.9)

$$\left\|\xi^\beta D^\beta_\xi \sigma_{j\lambda}(\xi)\right\|_{L(E)} \le M_j, \quad \beta\in U_n,\ \xi\in V_n,\ j=1,2. \tag{5.10}$$

Then in view of (5.10) we obtain that the operator-valued functions $\sigma_{j\lambda}(\xi)$ are the uniform collection of multipliers from $B^s_{p,q}(R^n;E)$ to $B^s_{p,q}(R^n;E)$. So we get that for all $f\in B^s_{p,q}(R^n;E)$ there is a unique solution of (5.2) in the form $u(x)=F^{-1}[A+(\lambda+K(\xi))]^{-1}\hat{f}$ and the estimate (5.4) holds.

Consider the problem (5.1). Let L_0 and L operators in $B^s_{p,q}(R^n;E)$ be generated by problems (5.2) and (5.1), respectively, that is,

$$D(L_0)=D(L)=B^{l+s}_{p,q}(R^n,E(A),E),$$
$$L_0u=\sum_{|\alpha:.l|=1}a_\alpha(x)D^\alpha u+Au, \tag{5.11}$$
$$Lu=L_0u+L_1u, \quad L_1u=\sum_{|\alpha:l|<1}A_\alpha(x)D^\alpha u.$$

□

Theorem 5.2. *Suppose condition (1) of Theorem 5.1 holds and let*

(1) *$A(x)$ be a φ positive in E uniformly with respect to x, $A(x)A^{-1}(x_0)\in C_b(R;B(E))\exists x_0\in(-\infty,\infty)$, $a_\alpha\in C_b(R)$, where $\varphi\in(0,\pi]$;*

(2) *$A_\alpha(x)A^{-(1-|\alpha:l|-\mu)}\in L_\infty(R^n;L(E))$, $0<\mu<1-|\alpha:l|$;*

(3) *$K(x,\xi)=-\sum_{|\alpha:.l|=1}b_\alpha(i\xi_1)^{\alpha_1}\cdot(i\xi_2)^{\alpha_2},\dots,(i\xi_n)^{\alpha_n}\in S(\varphi)$, $|K(x,\xi)|\ge C\sum_{k=1}^n|\xi_k|^{l_k}$, $\xi\in R^n$, $x\in R^n$.*

Then for all $f \in B^{s}_{p,q}(R^n;E)$, $|\arg\lambda| \le \pi - \varphi$ and for sufficiently large $|\lambda|$ (5.1) has a unique solution $u(x)$ that belongs to space $B^{l+s}_{p,q}(R^n;E(A),E)$, and the coercive uniform estimate

$$\sum_{\|\alpha:.l\|\le 1} |\lambda|^{1-|\alpha:.l|} \|D^{\alpha}u\|_{B^{s}_{p,q}} + \|Au\|_{B^{s}_{p,q}} \le C\|f\|_{B^{s}_{p,q}} \tag{5.12}$$

holds with respect to λ.

Proof. Let $\varphi_j \in C_0^{\infty}(R^n)$, $j = 1,2,\dots,\infty$, be a partition of unity such that $0 \le \varphi_j \le 1$ and $\operatorname{supp}\varphi_j \subset G_j$, $\sum_j \varphi_j(x) = 1$. Let $g_j \in C^{\infty}(R^n)$ such that $g_j(x) \equiv 1$ on $\operatorname{supp}\varphi_j$. Then for all $u \in B^{l+s}_{p,q}(R^n;E(A),E)$ we have $u(x) = \sum_j u_j(x)$, where $u_j(x) = u(x)\varphi_j(x)$. From the equality (5.1) for $u \in B^{l+s}_{p,q}(R^n;E(A),E)$ we obtain

$$(L+\lambda)u_j = \sum_{|\alpha:.l|=1} a_{\alpha}(x)D^{\alpha}u_j + A_{\lambda}(y)u_j(y) = f_j(y), \tag{5.13}$$

where

$$f_j = f\varphi_j - \sum_{|\alpha:.l|<1} b_{\alpha j}(x)D^{\alpha}u - \sum_{|\alpha:.l|<1} A_{\alpha}(x)D^{\alpha}u_j \tag{5.14}$$

and $b_{\alpha j}(x)$ are continuous and uniformly bounded functions containing derivatives of φ_j. Choose a large ball $B_{r_0}(0)$ such that $|a_{\alpha}(x) - a_{\alpha}(\infty)| \le \delta$ for all $|x| \ge r_0$ and $G_0 = R^n \setminus \overline{B}_{r_0}(0)$. Cover $\overline{B}_{r_0}(0)$ by finitely many balls $G_j = B_{r_j}(x_j)$ such that $|a_{\alpha}(x) - a_{\alpha}(x_j)| \le \delta$ for all $|x - x_j| \le r_j$, $j = 1,2,\dots,N$. Define coefficients of the local operators L_j as in [12, Theorem 5.7], that is,

$$a^{0}_{\alpha}(x) = \begin{cases} a_{\alpha}(x), & x \notin \overline{B}_{r_0}(0), \\ a_{\alpha}\left(r_0^2 \dfrac{x}{|x|^2}\right), & x \in \overline{B}_{r_0}(0), \end{cases}$$
$$a^{j}_{\alpha}(x) = \begin{cases} a_{\alpha}(x), & x \in \overline{B}_{r_j}(x_j), \\ a_{\alpha}\left(x_j + r_0^2 \dfrac{x - x_j}{|x - x_j|^2}\right), & x \notin \overline{B}_{r_j}(x_j) \end{cases} \tag{5.15}$$

for each $j = 1,2,\dots,N$. Then $|a_{\alpha}(x) - a_{\alpha}(x_j)| \le \delta$ for all $x \in R^n$ and $j = 0,1,2,\dots,N$. Freezing the coefficients in (5.13) we obtain that

$$\sum_{|\alpha:.l|=1} a_{\alpha}(x_j)D^{\alpha}u_j + A_{\lambda}(x_j)u_j(x) = F_j(x), \tag{5.16}$$

where

$$F_j = f_j + \sum_{|\alpha:.l|=1} [a_{\alpha}(x_j) - a_{\alpha}(x)]D^{\alpha}u_j + [A(x_j) - A(x)]u_j. \tag{5.17}$$

By virtue of Theorem 5.1 we obtain that the problem (5.16) has a unique solution u_j, and for $|\arg\lambda| \leq \pi - \varphi$ and sufficiently large $|\lambda|$ we get

$$\sum_{|\alpha:.l|\leq 1} |\lambda|^{1-|\alpha:.l|} ||D^{\alpha}u_j||_{B^s_{p,q}(G_j;E)} + ||Au_j||_{B^s_{p,q}(G_j;E)} \leq C||F_j||_{B^s_{p,q}\left(G_j;E\right)}. \tag{5.18}$$

Whence, using properties of the smoothness of coefficients of (5.14), (5.17) and choosing diameters of G_j sufficiently small, we get that

$$||F_j||_{B^s_{p,q}(G_j;E)} \leq \varepsilon||u_j||_{B^{s+l}_{p,q}(G_j;E(A),E)} + C(\varepsilon)||u_j||_{B^s_{p,q}(G_j;E)}, \tag{5.19}$$

where ε is a sufficiently small function and $C(\delta)$ is a continuous function. Consequently, from (5.18) and (5.19) we get

$$\sum_{|\alpha:.l|\leq 1} |\lambda|^{1-|\alpha:.l|} ||D^{\alpha}u_j||_{B^s_{p,q}(G_j;E)} \leq C\|f\|_{B^s_{p,q}(G_j;E)} + \delta||u_j||_{B^{s+l}_{p,q}} + C(\delta)||u_j||_{B^s_{p,q}(G_j;E)}. \tag{5.20}$$

Choosing $\delta < 1$ from the above inequality we have

$$\sum_{|\alpha:.l|\leq 1} |\lambda|^{1-|\alpha:.l|} ||D^{\alpha}u_j||_{B^s_{p,q}(G_j;E)} \leq C\Big[\|f\|_{G_j} + ||u_j||_{B^s_{p,q}(G_j;E)}\Big]. \tag{5.21}$$

Then by using the equality $u(x) = \sum_j u_j(x)$ and by virtue of the estimate (5.21) for $u \in B^{s+l}_{p,q}(R^n;E(A),E)$ we have

$$\sum_{|\alpha:.l|\leq 1} |\lambda|^{1-|\alpha:.l|} ||D^{\alpha}u_j||_{B^s_{p,q}(G_j;E)} \leq C\Big[||(L+\lambda)u||_{B^s_{p,q}} + \|u\|_{B^s_{p,q}}\Big]. \tag{5.22}$$

Let $u \in B^{s+l}_{p,q}(R^n;E(A),E)$ be a solution of the problem (5.1). Then for $|\arg\lambda| \leq \pi - \varphi$ we have

$$\|u\|_{B^s_{p,q}} = ||(L+\lambda)u - Lu||_{B^s_{p,q}} \leq \frac{1}{\lambda}\Big[||(L+\lambda)u||_{B^s_{p,q}} + \|u\|_{B^{s+l}_{p,q}}\Big]. \tag{5.23}$$

Then by Theorem 3.3 and by virtue of (5.21)–(5.23), for sufficiently large $|\lambda|$ we have

$$\sum_{|\alpha:.l|\leq 1} |\lambda|^{1-|\alpha:.l|} ||D^{\alpha}u_j||_{B^s_{p,q}} \leq C||(L+\lambda)u||_{B^s_{p,q}}. \tag{5.24}$$

The above estimate implies that the problem (5.1) has a unique solution and the operator $(L+\lambda)$ has an invertible operator in its rank space. We need to show that this rank space coincide with the space $B^s_{p,q}(R^n;E)$. Let us construct for all j the function u_j, that is defined on the regions G_j and satisfying the problem (5.1). The problem (5.1) can be expressed in the form

$$\begin{aligned} &\sum_{|\alpha:.l|=1} a_{\alpha}(x_j)D^{\alpha}u_j + A_{\lambda}(x_j)u_j(x) \\ &\quad = \Big\{g_j f + [A(x_j) - A(x)]u_j - \sum_{|\alpha:.l|<1} A_{\alpha}(x)D^{\alpha}u_j\Big\}, \quad j = 1,2,\ldots. \end{aligned} \tag{5.25}$$

Consider operators $O_{j\lambda}$ in $B^s_{p,q}(G_j;E)$ generated by problems (5.25). By virtue of Theorem 5.1 for all $f \in B^s_{p,q}(G_j;E)$, for $|\arg\lambda| \le \pi - \varphi$ and sufficiently large $|\lambda|$ we obtain

$$\sum_{|\alpha:.l|\le 1} |\lambda|^{1-|\alpha:.l|} \left\| D^\alpha O^{-1}_{j\lambda} f \right\|_{B^s_{p,q}} + \left\| A O^{-1}_{j\lambda} f \right\|_{B^s_{p,q}} \le C\|f\|_{B^s_{p,q}}. \tag{5.26}$$

Extending u_j zero on the outside of $\operatorname{supp}\varphi_j$ in equalities (5.25) and passing substitutions $u_j = O^{-1}_{j\lambda} v_j$ we obtain operator equations with respect to v_j:

$$v_j = K_{j\lambda} v_j + g_j f, \quad j = 1,2,\dots,N. \tag{5.27}$$

By virtue of Theorem 3.3 and the estimate (5.26), in view of the smoothness of the coefficients of the expression $K_{j\lambda}$ for $|\arg\lambda| \le \pi - \varphi$ and sufficiently large $|\lambda|$ we have $\|K_{j\lambda}\| < \varepsilon$, where ε is sufficiently small. Consequently, (5.27) has a unique solution $v_j = [I - K_{j\lambda}]^{-1} g_j f$ and we get

$$\left\| v_j \right\|_{B^s_{p,q}} = \left\| [I - K_{j\lambda}]^{-1} g_j f \right\|_{B^s_{p,q}} \le \|f\|_{B^s_{p,q}}. \tag{5.28}$$

Whence, $[I - K_{j\lambda}]^{-1} g_j$ are the bounded linear operators from $B^s_{p,q}(R^n;E)$ to $B^s_{p,q}(G_j;E)$. Thus, we obtain that the functions $u_j = U_{j\lambda} f = O^{-1}_{j\lambda}[I - K_{j\lambda}]^{-1} g_j f$ are the solutions of (5.25). Consider a linear operator $(U + \lambda I) = \sum_j \varphi_j(y) U_{j\lambda} f$ in $B^s_{p,q}(R^n;E)$. It is clear from the constructions U_j and the estimate (5.26) that the operators $U_{j\lambda}$ are bounded linear from $B^s_{p,q}(R^n;E)$ to $B^{s+l}_{p,q}(R^n;E(A),E)$ and

$$\sum_{|\alpha:.l|\le 1} |\lambda|^{1-|\alpha:.l|} \left\| D^\alpha U^{-1}_{j\lambda} f \right\|_{B^s_{p,q}} + \left\| A U^{-1}_{j\lambda} f \right\|_{B^s_{p,q}} \le C\|f\|_{B^s_{p,q}}, \tag{5.29}$$

for $|\arg\lambda| \le \pi - \varphi$ and sufficiently large $|\lambda|$. Therefore, $(U + \lambda I)$ is a bounded linear operator from $B^s_{p,q}$ to $B^s_{p,q}$. Then the act of $(L + \lambda)$ to $u = \sum_j \varphi_j U_{j\lambda} f$ gives $(L + \lambda)u = f + \sum_j \Phi_{j\lambda} f$, where $\Phi_{j\lambda}$ are linear combinations of $U_{j\lambda}$ and $(d/dy)U_{j\lambda}$. By virtue of Theorem 3.3, by estimate (5.29), and from the expression $\Phi_{j\lambda}$ we obtain that operators $\Phi_{j\lambda}$ are bounded linear from $B^s_{p,q}(R^n;E)$ to $B^s_{p,q}(G_j;E)$ and $\|\Phi_{j\lambda}\| < \delta$. Therefore, there exists a bounded linear invertible operator

$$\left(I + \sum_j \Phi_{j\lambda} \right)^{-1}. \tag{5.30}$$

Whence, we obtain that for all $f \in B^s_{p,q}(R^n;E)$ the problem (5.1) has a unique solution

$$u = (U + \lambda I)\left(I + \sum_j \Phi_{j\lambda} \right)^{-1} f, \tag{5.31}$$

that is, we obtain the assertion of Theorem 5.2. □

Result 5.3. Theorem 5.2 implies that the differential operator L has a resolvent operator $(L+\lambda)^{-1}$ for $|\arg\lambda| \leq \pi - \varphi$, and for sufficiently large $|\lambda|$ it has the estimate

$$\sum_{|\alpha:.l|\leq 1} |\lambda|^{1-|\alpha:.l|} \|D^{\alpha}(L+\lambda)^{-1}\|_{L(B^s_{p,q}(R^n;E))} + \|A(L+\lambda)^{-1}\|_{L(B^s_{p,q}(R^n;E))} \leq C. \tag{5.32}$$

Remark 3.5 and Theorem 5.2 imply the following.

Result 5.4. Suppose the following conditions hold:

(1) E is a Banach space satisfying B-multiplier condition with respect to $p \in (1,\infty)$ and $q \in [1,\infty]$;

(2) A is a φ-positive operator in E with $\varphi \in (0,\pi]$ and

$$K(\xi) = -\sum_{|\alpha:.l|=1} b_{\alpha}(i\xi_1)^{\alpha_1} \cdot (i\xi_2)^{\alpha_2},\dots,(i\xi_n)^{\alpha_n} \in S(\varphi),$$
$$|K(x,\xi)| \geq C\sum_{k=1}^{n} |\xi_k|^{l_k}, \quad \xi \in R^n,\ x \in R^n; \tag{5.33}$$

(3) $A(x)$ is a φ positive in E uniformly with respect to x, $A(x)A^{-1}(x_0) \in C_b(R;B(E))$, $x_0 \in (-\infty,\infty)$, $a_{\alpha} \in C_b(R)$, where $\varphi \in (0,\pi]$;

(4) $A_{\alpha}(x)A^{-(1-|\alpha:l|-\mu)} \in L_{\infty}(R^n;L(E))$, $0<\mu<1-|\alpha:l|$.

Then for all $f \in B^s_{p,q}(R^n;E)$, $|\arg\lambda| \leq \pi-\varphi$ and for sufficiently large $|\lambda|$ (5.1) has a unique solution $u(x)$ that belongs to space $W^l B^s_{p,\theta}(R^n;E(A),E)$, and the coercive uniform estimate

$$\sum_{|\alpha:.l|\leq 1} |\lambda|^{1-|\alpha:.l|} \|D^{\alpha}u\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} \leq C\|f\|_{B^s_{p,q}} \tag{5.34}$$

holds with respect to λ.

Theorem 5.5. *Let all conditions of Theorem 5.2 hold for $\varphi \in (0,\pi/2)$. Then the parabolic Cauchy problem (2.24) for $|\arg\lambda| \leq \pi-\varphi$ and sufficiently large $|\lambda|$ is maximal B-regular.*

Proof. The problem (2.24) can be expressed in $B^s_{p,\theta}(R_+;F)$ in the following form:

$$\frac{du(y)}{dy} + (L+\lambda)u(y) = f(t), \quad u(0)=0,\ y>0, \tag{5.35}$$

where $F = L_p(G;E)$ and L is the differential operator in $B^s_{p,\theta}(R^n;E)$ generated by the problem (5.1). In view of Result 4.3 the operator L is positive in $B^s_{p,\theta}(R^n;E)$ for $\varphi \in (0,\pi/2)$. Then by virtue of [4, Corollary 8.9] we obtain the assertion. □

Remark 5.6. There are lots of positive operators in concrete Banach spaces. Therefore, putting concrete Banach spaces instead of E and concrete positive differential, pseudo differential operators, or finite, infinite matrices, and so forth, instead of operator A on DOE (5.1), by virtue of Theorem 5.2 we can obtain the maximal regularity of different class of BVP's for partial differential equations or system of equations. Here we give some of its applications.

6. Applications

6.1. Infinite systems of quasielliptic equations. Consider the following infinity systems of boundary value problem:

$$(L+\lambda)u_m(x) = \sum_{|\alpha:.l|=1} a_\alpha(x)D^\alpha u_m(x) + \big(d_m(x)+\lambda\big)u_m(x) \\ + \sum_{|\alpha:.l|<1}\sum_{k=1}^{\infty} d_{\alpha km}(x)D^\alpha u_k(x) = f_m(x), \quad x\in R^n,\ m=1,2,\dots,\infty. \tag{6.1}$$

Let

$$D(x) = \{d_m(x)\}, \quad d_m>0, \quad u=\{u_m\}, \quad Du=\{d_m u_m\}, \quad m=1,2,\dots,\infty,$$
$$l_q(D) = \left\{u : u\in l_q,\ \|u\|_{l_q(D)} = \|Du\|_{l_q} = \left(\sum_{m=1}^{\infty} |d_m u_m|^q\right)^{1/q} < \infty\right\}, \tag{6.2}$$
$$x\in G, \quad 1<q<\infty, \quad l=(l_1,l_2,\dots,l_n), \quad s=(s_1,s_2,\dots,s_n), \quad s_k>0, \quad l_k\in\mathbb{N}.$$

Let O denote a differential operator in $B^s_{p,\theta}(R^n;l_q)$ generated by problem (6.1). Let

$$B = L\big(B^s_{p,\theta}(R^n;l_q)\big). \tag{6.3}$$

Theorem 6.1. *Let $a_\alpha\in C_b(R^n)$, $d_m\in C_b(R^n)$, $d_{\alpha km}\in L_\infty(R^n)$, and s_k, l_k such that*

$$\frac{s_k}{l_k+s_k}+\frac{s_j}{l_j+s_j}\le 1, \quad j=1,2,\dots,n, \\ \sum_{k,m=1}^{\infty} d_{\alpha km}^{q_1} d_m^{-q_1(1-|\alpha:l|-\mu)} < \infty, \quad \frac{1}{q}+\frac{1}{q_1}=1, \tag{6.4}$$

where $p,q\in(1,\infty)$, $\theta\in[1,\infty]$.

Then

(a) *for all $f(x)=\{f_m(x)\}_1^\infty\in B^s_{p,\theta}(R^n;l_q)$, $|\arg\lambda|\le\pi-\varphi$ and for sufficiently large $|\lambda|$ the problem (6.1) has a unique solution $u=\{u_m(x)\}_1^\infty$ that belongs to space $B^{s+l}_{p,\theta}(R^n,l_q(D),l_q)$, and the coercive estimate*

$$\sum_{|\alpha:l|\le 1} \|D^\alpha u\|_{B^s_{p,\theta}(R^n;l_q)} + \|du\|_{B^s_{p,\theta}(R^n;l_q)} \le C\|f\|_{B^s_{p,\theta}(R^n;l_q)} \tag{6.5}$$

holds for the solution of the problem (6.1);

(b) *for $|\arg\lambda|\le\pi-\varphi$ and for sufficiently large $|\lambda|$ there exists a resolvent $(O+\lambda)^{-1}$ of operator O and*

$$\sum_{|\alpha:l|\le 1} (1+|\lambda|)^{1-|\alpha:l|}\|D^\alpha(O+\lambda)^{-1}\|_B + \|d(O+\lambda)^{-1}\|_B \le M. \tag{6.6}$$

Proof. Really, let $E = l_q$, $A(x)$, and $A_\alpha(x)$ be infinite matrices, such that

$$A = [d_m(x)\delta_{km}], \quad A_\alpha(x) = [d_{\alpha km}(x)], \quad k,m = 1,2,\dots,\infty. \tag{6.7}$$

It is clear to see that operator A is positive in l_q. Therefore, by virtue of Theorem 5.2 we obtain that the problem (6.1) for all $f \in B^s_{p,\theta}(R^n;l_q)$, $|\arg\lambda| \le \pi - \varphi$, and sufficiently large $|\lambda|$ has a unique solution u that belongs to space $B^{s+l}_{p,\theta}(R^n;l_q(D),l_q)$ and the estimate (6.5) holds. By virtue of estimate (6.5) we obtain (6.6). □

6.2. Cauchy problems for infinite systems of parabolic equations. Consider the following infinity systems of parabolic Cauchy problem:

$$\frac{\partial u_m(y,x)}{\partial y} + \sum_{|\alpha:.l|=1} a_\alpha(x)D^\alpha u_m(y,x) + (d_m(x)+\lambda)u_m(y,x) + \sum_{|\alpha:.l|<1}\sum_{k=1}^{\infty} d_{\alpha km}(x)D^\alpha u_k(y,x)$$
$$= f_m(y,x), \quad u_m(0,x) = 0,\ m = 1,2,\dots,\infty,\ y \in R_+,\ x \in R^n. \tag{6.8}$$

Theorem 6.2. *Let all conditions of Theorem 6.1 hold. Then the parabolic systems (6.8) for $|\arg\lambda| \le \pi - \varphi$ and for sufficiently large $|\lambda|$ are maximal B-regular.*

Proof. Really, let $E = l_q$, A, and $A_k(x)$ be the infinite matrices, such that

$$A = [d_m(x)\delta_{km}], \quad A_\alpha(x) = [d_{\alpha km}(x)], \quad k,m = 1,2,\dots,\infty. \tag{6.9}$$

Then the problem (6.8) can be expressed as the problem (2.24), where

$$A = [d_m(x)\delta_{km}], \quad A_\alpha(x) = [d_{\alpha km}(x)], \quad k,m = 1,2,\dots,\infty. \tag{6.10}$$

Then by virtue of Theorems 5.2 and 5.5 we obtain the assertion. □

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