A CLASS OF CONSERVATIVE FOUR-DIMENSIONAL MATRICES

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The concepts $P-\limsup$ and $P-\liminf$ for double sequences were introduced by Patterson in 1999. In this paper, we have studied some new inequalities related to these concepts by using the RH-conservative four-dimensional matrices.

1. Introduction

A double sequence $x = [x_{jk}]_{j,k=0}^{\infty}$ is said to be convergent to a number l in the Pringsheim sense or P-convergent if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, the set of natural numbers, such that $|x_{jk} - l| < \varepsilon$ whenever $j,k > N$, [5]. In this case, we write $P-\lim x = l$. In what follows, we will write $[x_{jk}]$ in place of $[x_{jk}]_{j,k=0}^{\infty}$.

A double sequence x is said to be bounded if there exists a positive number M such that $|x_{jk}| < M$ for all j, k, that is, if

$$\|x\| = \sup_{j,k} |x_{jk}| < \infty. \tag{1.1}$$

Let ℓ_∞^2 be the space of all real bounded double sequences. We should note that in contrast to the case for single sequences, a convergent double sequence need not be bounded. By c_2^∞, we mean the space of all P-convergent and bounded double sequences.

Let $A = [a_{jk}^{mn}]_{j,k=0}^{\infty}$ be a four-dimensional infinite matrix of real numbers for all $m,n = 0,1,\ldots$. The sums

$$y_{mn} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk} \tag{1.2}$$

are called the A-transforms of the double sequence x. We say that a sequence x is A-summable to the limit s if the A-transform of x exists for all $m,n = 0,1,\ldots$ and convergent

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in the Pringsheim sense, that is,

$$\lim_{p,q\to\infty}\sum_{j=0}^{p}\sum_{k=0}^{q}a_{jk}^{mn}x_{jk}=y_{mn},$$
$$\lim_{m,n\to\infty}y_{mn}=s. \tag{1.3}$$

A matrix $A=[a_{jk}^{mn}]$ is said to be RH-regular (see [1, 6]) if $Ax\in c_2^{\infty}$ and $P-\lim Ax=P-\lim x$ for each $x\in c_2^{\infty}$. If a matrix A is RH-regular, then we write $A\in(c_2^{\infty},c_2^{\infty})_{\text{reg}}$. It is shown that A is RH-regular if and only if

$$P-\lim_{m,n}a_{jk}^{mn}=0 \quad \text{for each } j,k, \tag{1.4}$$

$$P-\lim_{m,n}\sum_{j}\sum_{k}a_{jk}^{mn}=1, \tag{1.5}$$

$$P-\lim_{m,n}\sum_{j}\left|a_{jk}^{mn}\right|=0 \quad \text{for each } k, \tag{1.6}$$

$$P-\lim_{m,n}\sum_{k}\left|a_{jk}^{mn}\right|=0 \quad \text{for each } j, \tag{1.7}$$

$$\|A\|=\sup_{m,n}\sum_{j}\sum_{k}\left|a_{jk}^{mn}\right|<\infty. \tag{1.8}$$

A matrix $A=[a_{jk}^{mn}]$ is said to be RH-conservative if $Ax\in c_2^{\infty}$ for each $x\in c_2^{\infty}$. In this case, we write $A\in(c_2^{\infty},c_2^{\infty})$. One can prove that A is RH-conservative if and only if the condition (1.8) holds and

$$P-\lim_{m,n}a_{jk}^{mn}=v_{jk} \quad \text{for each } j,k, \tag{1.9}$$

$$P-\lim_{m,n}\sum_{j}\sum_{k}a_{jk}^{mn}=v \quad \text{exists}, \tag{1.10}$$

$$P-\lim_{m,n}\sum_{j}\left|a_{jk}^{mn}-v_{kl}\right|=0 \quad \text{for each } k, \tag{1.11}$$

$$P-\lim_{m,n}\sum_{k}\left|a_{jk}^{mn}-v_{kl}\right|=0 \quad \text{for each } k. \tag{1.12}$$

For an RH-conservative matrix A, we can define the functional

$$\Gamma(A)=v-\sum_{j}\sum_{k}v_{jk}, \tag{1.13}$$

where $\sum_j\sum_k|v_{jk}|<\infty$ which follows from (1.8) and (1.9). Note that $\Gamma(A)=1$, when A is an RH-regular matrix.

Móricz and Rhoades [2] have defined almost convergence of a double sequence as follows.

A double sequence $x = [x_{jk}]$ of real numbers is said to be almost convergent to a limit l if

$$\lim_{p,q\to\infty} \sup_{m,n\geq 0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} - l \right| = 0 \quad \text{uniformly in } m,n = 1,2,\ldots. \tag{1.14}$$

Note that a convergent single sequence is also almost convergent but for a double sequence this is not the case, that is, a convergent double sequence need not be almost convergent. However, every bounded convergent double sequence is almost convergent. By f_2 we denote the space of all almost convergent double sequences. A matrix $A \in (f_2, c_2^\infty)_{\text{reg}}$ is said to be strongly regular and the conditions of strong regularity are known [2].

For any real bounded double sequence x, the concepts $l(x) = P - \liminf x$ and $L(x) = P - \limsup x$ have been introduced in [4] and also an inequality related to the $P - \limsup$ has been studied as follows.

Lemma 1.1 [4, Theorem 3.2]. *For any real double sequence x, $P - \limsup Ax \leq P - \limsup x$ if and only if A is RH-regular and*

$$P - \lim_{m,n} \sum_j \sum_k \left| a_{jk}^{mn} \right| = 1. \tag{1.15}$$

Let us define the sublinear functionals $L^{\text{ast}}(x)$, $l^{\text{ast}}(x)$ on ℓ_∞^2 as follows:

$$\begin{aligned} L^{\text{ast}}(x) &= P - \limsup_{p,q\to\infty} \sup_{m,n\geq 0} \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk}, \\ l^{\text{ast}}(x) &= P - \liminf_{p,q\to\infty} \sup_{m,n\geq 0} \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk}. \end{aligned} \tag{1.16}$$

Then, the MR-core of a real bounded double sequence x is the closed interval $[l^{\text{ast}}(x), L^{\text{ast}}(x)]$, [3]. Also, it is proved in [3] that $L(Ax) \leq L^{\text{ast}}(x)$ for all $x \in \ell_\infty^2$ if and only if A is strongly regular and (1.15) holds.

In this paper, we have proved some new inequalities related to the $P - \limsup$ by using the RH-conservative matrices.

2. The main results

Firstly, we need two lemmas, the first can be obtained from [4, Lemma 3.1].

Lemma 2.1. *If $A = [a_{jk}^{mn}]$ is a matrix such that the conditions (1.4), (1.6), (1.7), and (1.8) hold, then for any $y \in \ell_\infty^2$ with $\|y\| \leq 1$,*

$$P - \limsup_{m,n} \sum_j \sum_k a_{jk}^{mn} y_{jk} = P - \limsup_{m,n} \sum_j \sum_k \left| a_{jk}^{mn} \right|. \tag{2.1}$$

Lemma 2.2. *Let $A = [a_{jk}^{mn}]$ be RH-conservative and $\lambda \in \mathbb{R}^+$. Then,*

$$P-\limsup_{m,n}\sum_j\sum_k |a_{jk}^{mn}-v_{jk}| \leq \lambda \tag{2.2}$$

if and only if

$$\begin{aligned} P-\limsup_{m,n}\sum_j\sum_k (a_{jk}^{mn}-v_{jk})^+ &\leq \frac{\lambda+\Gamma(A)}{2}, \\ P-\limsup_{m,n}\sum_j\sum_k (a_{jk}^{mn}-v_{jk})^- &\leq \frac{\lambda-\Gamma(A)}{2}, \end{aligned} \tag{2.3}$$

where for any $\gamma \in \mathbb{R}$, $\gamma^+ = \max\{0,\gamma\}$ and $\gamma^- = \max\{-\gamma,0\}$.

Proof. Since A is RH-conservative, we have

$$P-\limsup_{m,n}\sum_j\sum_k (a_{jk}^{mn}-v_{jk}) = \Gamma(A). \tag{2.4}$$

Therefore, the results follow from the relations

$$\begin{aligned} \sum_j\sum_k (a_{jk}^{mn}-v_{jk}) &= \sum_j\sum_k (a_{jk}^{mn}-v_{jk})^+ - \sum_j\sum_k (a_{jk}^{mn}-v_{jk})^-, \\ \sum_j\sum_k |a_{jk}^{mn}-v_{jk}| &= \sum_j\sum_k (a_{jk}^{mn}-v_{jk})^+ + \sum_j\sum_k (a_{jk}^{mn}-v_{jk})^-. \end{aligned} \tag{2.5}$$

□

Theorem 2.3. *Let $A = [a_{jk}^{mn}]$ be RH-conservative. Then, for some constant $\lambda \geq |\Gamma(A)|$ and for all $x \in \ell_\infty^2$, one has*

$$P-\limsup_{m,n}\sum_j\sum_k (a_{jk}^{mn}-v_{jk})x_{jk} \leq \frac{\lambda+\Gamma(A)}{2}L(x) - \frac{\lambda-\Gamma(A)}{2}l(x) \tag{2.6}$$

if and only if (2.2) holds.

Proof. Suppose that (2.6) holds. Define the matrix $B = [b_{jk}^{mn}]$ by $b_{jk}^{mn} = (a_{jk}^{mn}-v_{jk})$ for all $m,n,j,k \in \mathbb{N}$. Then, since A is RH-conservative, the matrix B satisfies the hypothesis of Lemma 2.1. Hence, for a $y \in \ell_\infty^2$ such that $\|y\| \leq 1$, we have (2.1) with b_{jk}^{mn} in place of a_{jk}^{mn}. So, from (2.6), we get that

$$\begin{aligned} P-\limsup_{m,n}\sum_j\sum_k |b_{jk}^{mn}| &\leq \frac{\lambda+\Gamma(A)}{2}L(y) - \frac{\lambda-\Gamma(A)}{2}l(y) \\ &\leq \left[\frac{\lambda+\Gamma(A)}{2} + \frac{\lambda-\Gamma(A)}{2}\right]\|y\| \leq \lambda \end{aligned} \tag{2.7}$$

which is (2.2).

Conversely, suppose that (2.2) holds and $x \in \ell_\infty^2$. Then, for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$l(x) - \varepsilon < x_{jk} < L(x) + \varepsilon \tag{2.8}$$

whenever $j, k > N$. Now, we can write

$$\begin{aligned}\sum_j \sum_k b_{jk}^{mn} x_{jk} = {}& \sum_{j \le N} \sum_{k \le N} b_{jk}^{mn} x_{jk} + \sum_{j \le N} \sum_{k > N} b_{jk}^{mn} x_{jk} + \sum_{j > N} \sum_{k \le N} b_{jk}^{mn} x_{jk} \\ &+ \sum_{j > N} \sum_{k > N} \left(b_{jk}^{mn}\right)^+ x_{jk} - \sum_{j > N} \sum_{k > N} \left(b_{jk}^{mn}\right)^- x_{jk},\end{aligned} \tag{2.9}$$

where b_{jk}^{mn} is defined as above. Hence, by the RH-conservativeness of A and Lemma 2.2, we obtain

$$\begin{aligned}P - \limsup_{m,n} \sum_j \sum_k b_{jk}^{mn} x_{jk} &\le (L(x) + \varepsilon)\left(\frac{\lambda + \Gamma(A)}{2}\right) - (l(x) - \varepsilon)\left(\frac{\lambda - \Gamma(A)}{2}\right) \\ &= \frac{\lambda + \Gamma(A)}{2} L(x) - \frac{\lambda - \Gamma(A)}{2} l(x) + \lambda\varepsilon.\end{aligned} \tag{2.10}$$

Since ε is arbitrary, this completes the proof. □

In the case $\Gamma(A) > 0$ and $\lambda = \Gamma(A)$, we have the following result.

Theorem 2.4. *Let A be RH-conservative and $x \in \ell_\infty^2$. Then,*

$$P - \limsup_{m,n} \sum_j \sum_k \left(a_{jk}^{mn} - v_{jk}\right) x_{jk} \le \Gamma(A) L(x) \tag{2.11}$$

if and only if

$$P - \lim_{m,n} \sum_j \sum_k \left| a_{jk}^{mn} - v_{jk} \right| = \Gamma(A). \tag{2.12}$$

Also, we should note that when A is RH-regular, Theorem 2.4 is reduced to Lemma 1.1.

Theorem 2.5. *Let $A = [a_{jk}^{mn}]$ be RH-conservative. Then, for some constant $\lambda \ge |\Gamma(A)|$ and for all $x \in \ell_\infty^2$, one has*

$$P - \limsup_{m,n} \sum_j \sum_k \left(a_{jk}^{mn} - v_{jk}\right) x_{jk} \le \frac{\lambda + \Gamma(A)}{2} L^{\mathrm{ast}}(x) + \frac{\lambda - \Gamma(A)}{2} l^{\mathrm{ast}}(-x) \tag{2.13}$$

if and only if (2.2) holds and

$$P - \lim_{m,n} \sum_j \sum_k \left| \Delta_{10} a_{jk}^{mn} \right| = 0, \tag{2.14}$$

$$P - \lim_{m,n} \sum_j \sum_k \left| \Delta_{01} a_{jk}^{mn} \right| = 0, \tag{2.15}$$

where

$$\Delta_{10} a_{jk}^{mn} = a_{jk}^{mn} - a_{j+1,k}^{mn} - (v_{jk} - v_{j+1,k}), \qquad \Delta_{01} a_{jk}^{mn} = a_{jk}^{mn} - a_{j,k+1}^{mn} - (v_{jk} - v_{j,k+1}). \tag{2.16}$$

Proof. Suppose that (2.13) holds. Then, since $L^{\text{ast}}(x) \le L(x)$ and $l^{\text{ast}}(-x) \le -l(x)$ for all $x \in \ell_\infty^2$ (see [3]), the necessity of (2.2) follows from Theorem 2.3.

Define a matrix $C = [c_{jk}^{mn}]$ by $c_{jk}^{mn} = (b_{jk}^{mn} - b_{j+1,k}^{mn})$ for all $m, n, j, k \in \mathbb{N}$; where b_{jk}^{mn} is as in Theorem 2.3. Then, we have from Lemma 2.1, a $y \in \ell_\infty^2$ such that $\|y\| \le 1$ and (2.1) holds with c_{jk}^{mn} in place of a_{jk}^{mn}. Also, for the same y, we can write

$$\sum_j \sum_k c_{jk}^{mn} y_{j+1,k} = \sum_j \sum_k b_{jk}^{mn} (y_{jk} - y_{j+1,k}). \tag{2.17}$$

So, we have from (2.13) that

$$\begin{aligned} P - \limsup_{m,n} \sum_j \sum_k |c_{jk}^{mn}| &= P - \limsup_{m,n} \sum_j \sum_k c_{jk}^{mn} y_{j+1,k} \\ &= P - \limsup_{m,n} \sum_j \sum_k b_{jk}^{mn} (y_{jk} - y_{j+1,k}) \\ &\le \frac{\lambda + \Gamma(A)}{2} L^{\text{ast}}(y_{jk} - y_{j+1,k}) + \frac{\lambda - \Gamma(A)}{2} l^{\text{ast}}(y_{j+1} - y_{jk}). \end{aligned} \tag{2.18}$$

Now, let $y = [y_{jk}] = 1$ for all $j, k \in \mathbb{N}$. Then, since $(y_{jk} - y_{j+1,k}) \in f_2^{\infty,0}$, the space of all double almost null sequences

$$L^{\text{ast}}(y_{jk} - y_{j+1,k}) = l^{\text{ast}}(y_{j+1} - y_{jk}) = 0. \tag{2.19}$$

This implies the necessity of (2.14). By the same argument one can prove the necessity of (2.15).

Conversely, suppose that the conditions (2.2), (2.14), and (2.15) hold. For any given $\varepsilon > 0$, we can find integers $p, q \ge 2$ such that

$$l^{\text{ast}}(-x) - \varepsilon < \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} < L^{\text{ast}}(x) + \varepsilon \tag{2.20}$$

whenever $j, k \ge N$. Now, one can write

$$\sum_j \sum_k b_{jk}^{mn} x_{jk} = \sum_1 + \sum_2 + \sum_3 + \sum_4, \tag{2.21}$$

where

$$\begin{aligned}
\sum_1 &= \sum_j \sum_k b_{jk}^{mn} \frac{1}{pq} \sum_{s=j}^{j+p-1} \sum_{t=k}^{k+q-1} x_{st},\\
\sum_2 &= -\sum_{s=0}^{p-2} \sum_{t=0}^{q-2} \frac{1}{pq} \sum_{j=0}^{s} \sum_{k=0}^{t} b_{jk}^{mn} x_{st},\\
\sum_3 &= -\sum_{j=p-1}^{\infty} \sum_{t=q-1}^{\infty} \left(\frac{1}{pq} \sum_{j=s-p+1}^{s} \sum_{k=t-q+1}^{t} b_{jk}^{mn} - b_{jk}^{mn} \right) x_{st},\\
\sum_4 &= \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} b_{jk}^{mn} x_{jk},
\end{aligned} \tag{2.22}$$

and b_{jk}^{mn} is defined as in Theorem 2.3. Then, since

$$\left| \sum_2 \right| \le \|x\| \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} \left| b_{jk}^{mn} \right|, \qquad \left| \sum_4 \right| \le \|x\| \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} \left| b_{jk}^{mn} \right|, \tag{2.23}$$

using the condition (1.9), we observe that $\sum_2 \to 0$, $\sum_4 \to 0$ $(m,n \to \infty)$. On the other hand, since

$$\left| \sum_3 \right| \le \frac{\|x\|}{pq} \sum_{s=0}^{p-1} \sum_{t=0}^{q-1} \left((p-s-1) \sum_j \sum_k \left| \Delta_{10} a_{jk}^{mn} \right| + (q-t-1) \sum_j \sum_k \left| \Delta_{01} a_{jk}^{mn} \right| \right), \tag{2.24}$$

by the conditions (2.14)-(2.15), $\sum_3 \to 0$ $(m,n \to \infty)$. Thus, we can write

$$\begin{aligned}
\sum_1 &= \sum_{j\le N} \sum_{k\le N} b_{jk}^{mn} \frac{1}{pq} \sum_{s=j}^{j+p-1} \sum_{t=k}^{k+q-1} x_{st} + \sum_{j\ge N} \sum_{k\ge N} b_{jk}^{mn} \frac{1}{pq} \sum_{s=j}^{j+p-1} \sum_{t=k}^{k+q-1} x_{st}\\
&\quad - \sum_{j\ge N} \sum_{k\ge N} b_{jk}^{mn} \frac{1}{pq} \sum_{s=j}^{j+p-1} \sum_{t=k}^{k+q-1} x_{st}.
\end{aligned} \tag{2.25}$$

By (1.9), (2.20) and Lemma 2.2, we get that

$$\begin{aligned}
P - \limsup_{m,n} \sum_j \sum_k b_{jk}^{mn} x_{jk} &\le \left(L^{\text{ast}}(x) + \varepsilon\right) \frac{\lambda + \Gamma(A)}{2} + \left(l^{\text{ast}}(-x) + \varepsilon\right) \frac{\lambda - \Gamma(A)}{2}\\
&= \frac{\lambda + \Gamma(A)}{2} L^{\text{ast}}(x) + \frac{\lambda - \Gamma(A)}{2} l^{\text{ast}}(-x) + \lambda\varepsilon
\end{aligned} \tag{2.26}$$

which is (2.13), since ε is arbitrary. □

In the case $\Gamma(A) > 0$ and $\lambda = \Gamma(A)$, we have the following.

THEOREM 2.6. *Let A be RH-conservative and $x \in \ell_\infty^2$. Then,*

$$P-\limsup_{m,n}\sum_j\sum_k\left(a_{jk}^{mn}-v_{jk}\right)x_{jk}\leq\Gamma(A)L^{\text{ast}}(x) \tag{2.27}$$

if and only if (2.12), (2.14), and (2.15) hold.

We should state that when A is strongly regular, Theorem 2.6 is reduced to [3, Theorem 3.1].

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