CONTINUITY PROPERTIES OF PROJECTION OPERATORS

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We prove that the projection operator on a nonempty closed convex subset C of a uniformly convex Banach spaces is uniformly continuous on bounded sets and we provide an estimate of its modulus of uniform continuity. We derive this result from a study of the dependence of the projection on C of a given point when C varies.

1. Introduction

Many optimization problems can be reformulated as best approximation problems with respect to an appropriate norm. Their solutions are thus given by projection operators. It follows that projection operators play a key role in several areas, such as mechanics, minimization algorithms, variational inequalities, complementary problems. Thus, it is of interest to study the properties of such operators. In particular, continuity properties of projection maps express important properties about the dependence on parameters of the solution maps of such problems.

It is well known that the projection operator onto a nonempty closed convex subset of a uniformly convex Banach space is continuous (see, e.g., [5, 19] and the references in the discussion closing the paper). We prove here that it is uniformly continuous on bounded sets and we provide a simple estimate of the modulus of uniform continuity. We also show that the projection of a given point onto a nonempty closed convex subset C is continuous with respect to C for the topology defined by the Hausdorff-Pompeiu metric, or, more precisely, the family of bounded hemi-metrics of Hausdorff-Pompeiu type used in [7, 8, 9, 10, 28, 29, 35] which defines a more realistic uniform structure. The associated convergence is metrizable and it has been the subject of many recent studies; see [36] for a recent survey and references.

It is known that even in Hilbert spaces this dependence is of Hölder type and cannot be of Lipschitz type [8, 13]. Note however, that any uniformly continuous mapping satisfies a Lipschitz type inequality for points which are sufficiently apart (see [32] for a recent proof).

Here we deal with the case of uniformly convex Banach spaces since this framework is adapted to questions of existence and uniqueness. We use a general duality mapping,

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since such a tool is often easier to deal with than the usual duality map in application to concrete spaces such as L_p-spaces. Our main results are presented in Sections 3 and 4 after some preliminary material has been recalled in the next section. This material concerns duality mappings, uniform monotonicity and uniform convexity of functions. Let us note that this last question is related to the study of totally convex functions which have recently been used for algorithms (see [14, 15, 16]) and which has an independent interest. The arguments of our proofs use a uniform monotonicity of the duality mapping and are rather simple and direct. This property can be related to results involving the modulus of uniform convexity of the space, but the links are rather sophisticated inequalities. A short comparison with related works is given in our conclusion.

2. Preliminaries

Throughout X denotes a normed vector space (n.v.s.) or a Banach space with closed unit ball B_X. For $w \in W$, $r > 0$, $B_x(w,r)$ stands for the open ball with center w and radius r. For a subset A of X one sets

$$d(x,A) = \inf\{d(x,a) : a \in A\}, \quad \text{with } d(a,b) := \|a-b\|. \tag{2.1}$$

The *excess* (or hemidistance) of A over a subset B of X is defined by

$$e(A,B) = \sup_{a\in A} d(a,B). \tag{2.2}$$

The *Pompeiu-Hausdorff distance* of A and B is given by

$$d(A,B) = \max\left(e(A,B), e(B,A)\right). \tag{2.3}$$

The use of such a metric for unbounded subsets is rather restrictive. A localized version is more amenable and realistic. It goes as follows: given a base point $w \in X$, the bounded (or local) excess $e_{w,r}$ is given for $r \geq 0$ by

$$e_{w,r}(A,B) := e\left(A \cap B_X(w,r), B\right) \tag{2.4}$$

and with it is associated the symmetric bounded hemi-metric $d_{w,r}$ given by

$$d_{w,r}(A,B) = \max\left(e\left(A \cap B_X(w,r), B\right), e\left(B \cap B_X(w,r), A\right)\right). \tag{2.5}$$

When w is the origin of X, we omit w in the notation. These families of "hemi-metrics" have been introduced in [27] (see also [20] for the case of vector subspaces and [7, 10, 26, 28]) in order to get quantitative properties for the convergence of sets as a substitute to convergence in terms of the Pompeiu-Hausdorff distance. In finite dimensional spaces, convergence with respect to the family $(d_r)_{r>0}$ is simply the classical Painlevé-Kuratowski convergence of sets (see [23] for instance). In infinite dimensions, the family of bounded hemi-metrics $(d_r)_{r>0}$ is effective to deal with the behavior of usual operations with respect to convergence of sequences of sets (see [7, 11, 28, 35, 36] for instance).

In the sequel, we say that a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a *gage* (or a firm function, or a forcing function or an admissible function) if it is nondecreasing and such that $\gamma(0) = 0$,

$\gamma(t) > 0$ for $t > 0$. The term "gage" coined in [11] evokes the idea of a penalization or the guarantee that some quantity will remain positive if the data are moved a little bit. For a nondecreasing function γ, we set

$$\gamma^{-1}(s) = \sup\{r \in \mathbb{R}_+ : \gamma(r) \le s\}. \tag{2.6}$$

In [30, 34], the function γ^{-1} is denoted by γ^h and is called the *upper quasi-inverse* of γ (the lower quasi-inverse will not be used here). It satisfies the obvious property that $r \le \gamma^{-1}(s)$ whenever r and s are such that $\gamma(r) \le s$. When γ is a gage, the function $\mu := \gamma^{-1}$ is a *modulus* in the sense that μ is nondecreasing, $\mu(0) = 0$ and $\mu(s) \to 0$ as $s \to 0_+$. Thus it can serve as a modulus of uniform continuity. Here we say that a modulus μ is a *modulus of uniform continuity* of a mapping $f : X \to Y$ between two metric spaces if

$$d(f(x), f(x')) \le \mu(d(x,x')) \tag{2.7}$$

for any $x,x' \in X$. Then the function $\gamma := \mu^{-1}$ is a gage [34] and enables one to write the implication

$$\forall \varepsilon > 0\ \forall x,x' \in X, \quad d(x,x') < \gamma(\varepsilon) \Longrightarrow d(f(x), f(x')) \le \varepsilon. \tag{2.8}$$

As mentioned above, when X is a convex subset of a normed vector space and f is uniformly continuous, for any $\delta > 0$ one can find $c > 0$ such that

$$d(f(x), f(x')) \le c d(x,x') \quad \text{whenever } x,x' \in X \text{ satisfy } d(x,x') \ge \delta. \tag{2.9}$$

Gages can also serve to give a measure of rotundity of sets or functions and a measure of monotonicity of operators, as in the following definition.

Definition 2.1. A multifunction $M : X \rightrightarrows X^*$ from a normed vector space X to its dual X^* is said to be *uniformly monotone* on a subset B of X at some $\overline{x} \in X$ with gage γ if for any $x \in B$, $x^* \in M(x)$, $\overline{x}^* \in M(\overline{x})$ one has

$$\gamma(\|x - \overline{x}\|) \le \langle x^* - \overline{x}^*, x - \overline{x}\rangle. \tag{2.10}$$

It is said to be uniformly monotone on B with gage γ if it is uniformly monotone on B at each point of B with gage γ.

Such a property arises when dealing with the subdifferential ∂f of a uniformly convex function f on X, which is given by

$$\partial f(x) := \{x^* \in X^* : f(w) \ge f(x) + \langle x^*, w - x\rangle\ \forall w \in X\}. \tag{2.11}$$

Let us give a precise definition taken from [11, 37, 41, 42] and close to the one in [17, page 50].

Definition 2.2. A function $f : X \to \mathbb{R}^{\cdot} = \mathbb{R} \cup \{\infty\}$ on a normed vector space X is said to be uniformly convex on some convex subset B of X at some $\overline{x} \in X$ with gage γ if for any $x \in B$, $t \in [0,1]$ one has

$$f((1-t)\overline{x} + tx) \le (1-t)f(\overline{x}) + tf(x) - t(1-t)\gamma(\|x - \overline{x}\|). \tag{2.12}$$

It is said to be uniformly convex on B with gage γ if for each $\overline{x} \in B$ it is uniformly convex on B at $\overline{x}$ with gage γ.

Then the *canonical gage of convexity* $\gamma_{f,B}$ of f on B is given by

$$\gamma_{f,B}(s) = \inf_{\substack{t\in]0,1[\\ x,y\in B}} \left\{ \frac{(1-t)f(x)+tf(y)-f((1-t)x+ty)}{t(1-t)} : \|x-y\| \geq s \right\}. \tag{2.13}$$

The following lemma can be proved as in [37, Lemma 1] where $B = X$; see also [42, Proposition 3.5.1].

Lemma 2.3. *If f is uniformly convex on the convex subset B of X, then its canonical gage of convexity $\gamma_{f,B}$ on B is hyper-starshaped in the following sense: for any $r \in \mathbb{R}_+$, $c \in]0,1]$, one has $c^{-2}\gamma_{f,B}(cr) \leq \gamma_{f,B}(r)$.*

The passage from uniform convexity of a function to uniform monotonicity of its subdifferential is described in the following lemma of general interest.

Lemma 2.4 (see [11, 37, 41], [42, Theorem 3.5.10]). *Let B be a convex subset of X, let $z \in B$ and let $f : X \to \mathbb{R}^{\cdot} = \mathbb{R}\cup\{\infty\}$ be uniformly convex on B at z with gage γ. Then for each $x \in B$ one has*

(a) $f(x) \geq f(z)+f'(z,x-z)+\gamma(\|x-z\|)$;

(b) $f(x) \geq f(z)+\langle z^*,x-z\rangle+\gamma(\|x-z\|)$ *for each* $z^* \in \partial f(z)$;

(c) $\langle x^*-z^*,x-z\rangle \geq \gamma(\|x-z\|)$ *for any* $x^* \in \partial f(x)$, $z^* \in \partial f(z)$.

If moreover f is uniformly convex on B with gage γ then one has

(d) $\langle x^*-y^*,x-y\rangle \geq 2\gamma(\|x-y\|)$ *for any* $x,y \in B$, $x^* \in \partial f(x)$, $y^* \in \partial f(y)$.

One could also introduce a canonical gage of uniform monotonicity of ∂f and relate it to the gage of uniform convexity of f.

The geometry of the space X is reflected by properties of the function $x \mapsto (1/2)\|x\|^2$, or, more generally, by the properties of $j_p : x \mapsto (1/p)\|x\|^p$ for $p > 1$ or $j_h : x \mapsto H(\|x\|)$, where $H : \mathbb{R}_+ \to \mathbb{R}_+$ is the primitive

$$H(t) = \int_0^t h(s)ds \tag{2.14}$$

of a weight h, a *weight* being a continuous and increasing function h such that $h(0) = 0$. The functions j_h may be easier to study, for instance in L_p spaces. This fact leads us to make use of duality mappings associated with weights; they are defined as follows. Given a weight h, the multimapping $J_h : X \rightrightarrows X^*$ given by

$$J_h(x) = \{x^* \in X^* : \|x^*\| = h(\|x\|),\ \langle x^*,x\rangle = \|x^*\| \cdot \|x\|\} \tag{2.15}$$

is called the *duality mapping of weight* (or gauge) h. It is known (see [17, page 26]) that J_h is the subdifferential of the continuous convex function $j_h = H \circ \|\cdot\|$, and thus has nonempty values. When X is a $L_p(\Omega)$ space one may like to take $h(t) = h_p(t) := t^{p-1}$, in which case $J_h(x) = (1/p)\partial\|\cdot\|^p(x)$. Then, for $x \in L_p(\Omega)$, $\omega \in \Omega$ one has

$$J_h(x)(\omega) = |x(\omega)|^{p-2}x(\omega), \tag{2.16}$$

so that J_h is more convenient to use than the normalized duality mapping J corresponding to $p = 2$ which involves integrals whereas $J_h(x)(\omega)$ only depends on the value of $x(\cdot)$ at $\omega \in \Omega$. This fact is exploited in [33].

Our use of uniformly convex functions is justified by the following result (see [37, Theorem 6], [11, Lemma 5.1] for instance).

LEMMA 2.5. *A Banach space is uniformly convex if and only if for some (resp., any) $p > 1$ the function $j_p(\cdot) = (1/p)\|\cdot\|^p$ is uniformly convex on the unit ball. More precisely, if δ_p is the function given by*

$$\delta_p(s) := \inf\left\{\frac{\|x\|^p + \|y\|^p - 2^{1-p}\|x+y\|^p}{\|x\|^p + \|y\|^p} : x, y \in B_X,\ \|x - y\| \geq s\right\} \tag{2.17}$$

then j_p is uniformly convex on B_X with gage γ_p given by $\gamma_p(t) := 2^{1-p}\delta_p(t)t^p/p$.

It follows from the previous lemma or from a direct homogeneity argument that in such a case the function $j_p(\cdot)$ is uniformly convex on rB_X with gage $t \mapsto r^p\gamma_p(t/r) = 2^{1-p}\delta_p(t/r)t^p/p$. Adopting the choice $h = h_p$, with $h_p(t) := t^{p-1}$ as above, Lemma 2.4 ensures that a gage of uniform monotonicity of J_h on rB_X is given by $t \mapsto 2r^p\gamma_p(t/r) = 2^{2-p}\delta_p(t/r)t^p/p$.

A convenient criteria for obtaining uniform convexity is the following one (see [5, 24, 25, 41, 42]): if f is *uniformly mid-convex* on a convex subset B of X with gage of mid-convexity $\tilde{\gamma}$ in the sense that for any $x, y \in B$

$$f\left(\frac{1}{2}(x+y)\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) - \frac{1}{4}\tilde{\gamma}(\|x - y\|) \tag{2.18}$$

then f is uniformly convex on B with gage of uniform convexity $\gamma = (1/2)\tilde{\gamma}$.

This criteria enables one to use Clarkson's inequality for $L_p(\Omega)$ with $p \geq 2$

$$\left\|\frac{1}{2}(x+y)\right\|^p + \left\|\frac{1}{2}(x-y)\right\|^p \leq \frac{1}{2}\|x\|^p + \frac{1}{2}\|y\|^p \quad \forall x, y \in L_p(\Omega) \tag{2.19}$$

to get that $L_p(\Omega)$ is uniformly convex and that a gage of uniform convexity of $j_p = (1/p)\|\cdot\|^p$ on $L_p(\Omega)$ is $\gamma_p : s \mapsto 2^{1-p}(1/p)s^p$. It follows from Lemma 2.4 that a gage of uniform monotonicity of $J_p := \partial j_p$ is $s \mapsto 2^{2-p}(1/p)s^p$. For $p \in]1,2]$, the "characteristic inequality" of [38] on $L_p(\Omega)$, which can be written

$$\frac{1}{2}\|(1-t)x + ty\|^2 + (p-1)t(1-t)\frac{1}{2}\|x - y\|^2 \leq (1-t)\frac{1}{2}\|x\|^2 + t\frac{1}{2}\|y\|^2 \tag{2.20}$$

for any $t \in [0,1]$, $x, y \in L_p(\Omega)$, shows that j_2 is uniformly convex on $L_p(\Omega)$ with gage of uniform convexity $s \mapsto (1/2)(p-1)s^2$. Thus, the normalized duality mapping $J = \partial j_2$ is uniformly monotone with gage $s \mapsto (p-1)s^2$.

In turn, uniform convexity of a function f at a point can be established with the help of a differentiability property of its Fenchel conjugate function f^* given by $f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$. Let us recall that μ is an l.s.c. convex *hyper-modulus* or a *remainder* (i.e., $\mu(t) = t\nu(t)$ where ν is a modulus) if and only if its conjugate $\gamma = \mu^*$ is a gage (see [6], [11, Lemma 2.1]). In the next statement we say that two n.v.s. are in *metric duality* if they are paired by a coupling function $\langle\cdot,\cdot\rangle$ in such a way that the norm on each space is induced by the norm which is dual to the norm of the other space.

Lemma 2.6. *Let X, Y be two Banach spaces in metric duality. Let $f : X \to \dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be a $\sigma(X,Y)$-l.s.c. convex function finite at some z for which $Z^* := \partial f(z) \cap Y$ is nonempty and let μ be an hypermodulus, $\gamma = \mu^*$. Then the following assertions are equivalent:*

(a) *for any $y \in Y$, $z^* \in Z^*$, $f^*(y) \le f^*(z^*) + \langle y - z^*, z\rangle + \mu(\|y - z^*\|)$;*

(b) *for each $x \in X$ $f(x) \ge f(z) + \sup_{z^* \in Z^*}\langle z^*, x - z\rangle + \gamma(\|x - z\|)$.*

Both assertions imply that for any $x \in X$, $y \in \partial f(x)$, $z^* \in \partial f(z)$

$$\langle y - z^*, x - z\rangle \ge \gamma(\|x - z\|). \tag{2.21}$$

This criteria can be used in connection with differentiability results for norms. For example it is known that the supremum norm on the space $C = C(T)$ of continuous functions on a compact metric space T is Fréchet differentiable at x if and only if $M(x) = \{t \in T : |x(t)| = \|x\|_\infty\}$ is a singleton $\{t_0\}$, t_0 being an isolated point of T [18, page 5].

3. Dependence on the convex set

Let us give an estimate of the variation of the projection of a given point w when the convex set varies. The result is close to the one of [5, Theorem 2]. However, we use here the bounded hemi-metric $d_{w,r}$ instead of the Hausdorff distance, a general duality mapping J_h and the estimate is given in terms of a gage of uniform monotonicity instead of the gage of uniform convexity of the space. Note that the condition of uniform monotonicity can be ensured by a differentiability assumption on the conjugate function of j_h.

Theorem 3.1. *Let C be a nonempty closed convex subset of a n.v.s. X and let x be a nearest point in C to a point $w \in X$. Suppose that for some $r > d(w,C)$ the duality mapping J_h associated with a weight h is uniformly monotone at $z = w - x$ on rB_X with gage γ. Then for any nonempty closed convex subset C' of X such that $d(w,C') < r$ and any $x' \in P_{C'}(w)$ one has, with $s = r + \|w\|$,*

$$\|x' - x\| \le \gamma^{-1}\big(2h(r)d_{w,r}(C,C')\big) \le \gamma^{-1}\big(2h(r)d_s(C,C')\big). \tag{3.1}$$

Taking $C = C'$, this estimate shows that $P_C(w)$ is a singleton if nonempty.

Proof. An easy adaptation of the characterization of best approximations (see, e.g., [31]) yields some $z^* \in J_h(w - x)$, $z'^* \in J_h(w - x')$ such that

$$\langle z^*, y - x\rangle \le 0 \quad \forall y \in C, \tag{3.2}$$

$$\langle z'^*, y' - x'\rangle \le 0 \quad \forall y' \in C'. \tag{3.3}$$

Since $w - x \in rB_X$, $w - x' \in rB_X$, and since $(w-x)-(w-x') = x' - x$, we have

$$\gamma(\|x - x'\|) \le \langle z^* - z'^*, x' - x\rangle. \tag{3.4}$$

Since $\|x' - w\| = d(w, C') < r$, for any $\delta > d_{w,r}(C, C')$ we can find $y \in C$ such that $\|x' - y\| < \delta$. It follows that

$$\langle z^*, x' - x\rangle = \langle z^*, x' - y\rangle + \langle z^*, y - x\rangle \le h(r)\delta \tag{3.5}$$

by (2.15) and the fact that $\|z^*\| = h(\|w - x\|) \le h(r)$. Similarly

$$\langle z'^*, x - x'\rangle \le h(r)\delta. \tag{3.6}$$

Gathering relations (3.4), (3.5), and (3.6), we obtain

$$\gamma(\|x - x'\|) \le 2h(r)\delta. \tag{3.7}$$

Since δ is arbitrarily close to $d_{w,r}(C, C')$ we get

$$\gamma(\|x - x'\|) \le 2h(r)d_{w,r}(C, C'), \tag{3.8}$$

and the result follows. □

Taking $h(t) = t$ we get the following consequence (here for the empty subset of $\mathbb{R}_+$ we set $\sup\varnothing = 0$).

Corollary 3.2. *Let C, C', w, x, x' be as in the preceding statement. Suppose the normalized mapping J is uniformly monotone at $w - x$ on rB_X with gage γ. Let $s = \|w\| + r$. Then*

$$\sup_{x' \in P_{C'}(w)} \|x - x'\| \le \gamma^{-1}(2rd_s(C, C')) \le \gamma^{-1}(2rd(C, C')). \tag{3.9}$$

In the next corollary we use the *remoteness* of a subset C which is $d(0, C)$.

Corollary 3.3. *Suppose the norm is uniformly convex and let γ be a gage of monotonicity of J_h on rB_X for some $r > 0$. Then for any $q, t > 0$ with $q + t \le r$ and any $w \in qB_X$, the mapping $C \mapsto p_C(w)$ is uniformly continuous on the set $\mathscr{C}_t(X)$ of closed nonempty convex subsets of X with remoteness at most t, endowed with the hemi-metric d_{2q+t}:*

$$\|p_C(w) - p_{C'}(w)\| \le \gamma^{-1}(2rd_{2q+t}(C, C')). \tag{3.10}$$

Proof. For $w \in qB_X$, $C, C' \in \mathscr{C}_t(X)$ we have $d(w, C) \le q + t \le r$ and similarly $d(w, C') \le r$. Then we may take $s = 2q + t$. □

4. Uniform continuity of the projection map

The invariance of the distance under translations yields the following consequence for the projection onto a fixed convex subset.

Theorem 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let γ be a gage of monotonicity of the duality mapping J_h on the ball rB_X, with*

$r > d(0,C)$. Let $q > 0$ be such that $3q + d(0,C) < r$. Then, the projection mapping p_C is uniformly continuous on qB_X with

$$\|p_C(w) - p_C(w')\| \le \gamma^{-1}(2h(r)\|w - w'\|) + \|w - w'\|, \tag{4.1}$$

$$\|(I - p_C)(w) - (I - p_C)(w')\| \le \gamma^{-1}(2h(r)\|w - w'\|). \tag{4.2}$$

Proof. Let $t := 2q + d(0,C)$, so that $q + t < r$. Given $w, w' \in qB_X$, let us set

$$C' := C + w - w', \qquad u := p_C(w'). \tag{4.3}$$

Then, observing that, for any $y \in C$,

$$\|(u + w - w') - w\| = \|u - w'\| \le \|y - w'\| = \|(y + w - w') - w\|, \tag{4.4}$$

we have

$$p_{C'}(w) = p_C(w') + w - w'. \tag{4.5}$$

Since by definition of C' we have $d(C, C') \le \|w - w'\|$, we note that

$$d(0, C') \le d(0,C) + d(C, C') \le t \tag{4.6}$$

and $C, C' \in \mathscr{C}_t(X)$. Thus, Theorem 3.1 yields

$$\begin{aligned} \|p_C(w) - p_C(w')\| &\le \|p_C(w) - p_{C'}(w)\| + \|p_{C'}(w) - p_C(w')\| \\ &\le \gamma^{-1}(2h(r)\|w - w'\|) + \|w - w'\|. \end{aligned} \tag{4.7}$$

Since

$$(w - p_C(w)) - (w' - p_C(w')) = p_{C'}(w) - p_C(w), \tag{4.8}$$

we get the second inequality. □

A more direct proof goes as follows. Let $w, w' \in qB_X$ and let $x = p_C(w)$, $x' = p_C(w')$, $v = w - x$, $v' = w' - x'$. Using the characterization of projections in terms of j_h mentioned above, we pick $v^* \in J_h(v)$, $v'^* \in J_h(v')$ such that for each $y \in C$

$$\langle v^*, y - x\rangle \le 0, \qquad \langle v'^*, y - x'\rangle \le 0. \tag{4.9}$$

Thus, taking $y = x'$ and then $y = x$, so that $\langle v^* - v'^*, x' - x\rangle \le 0$, we get

$$\begin{aligned} \gamma(\|v - v'\|) &\le \langle v^* - v'^*, v - v'\rangle \\ &\le \langle v^* - v'^*, w - w'\rangle \\ &\le 2h(r)\|w - w'\| \end{aligned} \tag{4.10}$$

since $\|v\|, \|v'\| \le r$, hence $\|v^*\|, \|v'^*\| \le h(r)$. Relation (4.2) follows by inverting γ; then (4.1) ensues.

5. Comparison with other works

Since the preceding results have been obtained in a preliminary version (1996), a number of works dealing with continuity properties of projection operators have come to our attention (in particular thanks to Al'ber). Most of the results of these papers such as [1], [3, Theorem 5.2], [2, Theorems 3.1, 3.4], [5, Theorem 1], [32, 40] cannot be compared with our contribution since they assume the space X is both uniformly smooth and uniformly convex, while we just assume X is uniformly convex, an assumption which seems to be natural as it ensures existence and uniqueness of projections. As mentioned above, however our Theorem 3.1 is close to [5, Theorem 2]. There, the space X is not assumed to be smooth but the estimate is given in terms of the Hausdorff distance; this rules out most of unbounded sets such as vector subspaces. More important is the nature of our estimate. Besides the fact it involves a more realistic distance than the Hausdorff distance, it is formulated in a more versatile way, since it uses a gage of uniform monotonicity γ of J_h which can be computed in different ways. In the preliminaries we have presented a general means to compute it. In the special case of h_p with $p = 2$ (i.e., $h(t) = t$) one can use [4, Theorem 2] stating that

$$\frac{1}{4}\|x\|^2 + \frac{1}{4}\|y\|^2 - \frac{1}{2}\left\|\frac{1}{2}(x+y)\right\|^2 \geq \frac{1}{8L}\delta_X\left(\frac{\|x-y\|}{C(x,y)}\right), \tag{5.1}$$

where $C(x,y) = \max(2,(2\|x\|^2 + 2\|y\|^2)^{1/2})$, L is the Figiel's constant ($1 < L < 3.18$), and δ_X is the so-called modulus of uniform convexity of the space X given by $\delta_X(s) := \gamma_X(s/2)$ for $s \in [0,2]$, where

$$\gamma_X(t) := \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : x, y \in S_X,\ \left\|\frac{1}{2}(x-y)\right\| \geq t\right\} \tag{5.2}$$

for $t \in [0,1]$ (which in fact is a gage and not a modulus). Then, comparing relations (5.1) and (2.18) with $f := j_2 = (1/2)\|\cdot\|^2$ and using the observation about gages of midconvexity following relation (2.18) we deduce from Lemma 2.4 that one can take

$$\gamma(t) = \frac{1}{4L}\delta_X\left(\frac{t}{2\max(1,r)}\right) \tag{5.3}$$

as a gage of uniform convexity of j_2 which differs from our general gage given by $t \mapsto 2^{1-p}\delta_p(t/r)t^p/p$. A third means is provided by [39, Theorem 1 and Remark 1] which yield a gage of monotonicity of $J_p = \partial j_p$ (for $p > 1$) on rB_X also using δ_X which allows to take

$$\gamma(t) = K_p(t/2)^p\delta_X\left(\frac{t}{2r}\right) \tag{5.4}$$

with K_p a constant determined by relation (2.13) of [39] (here, again we use the fact that $\|x\|^p \vee \|y\|^p \geq (t/2)^p$ when $\|x-y\| \geq t$). Finally, in a space for which the function $j_p : x \mapsto p^{-1}\|x\|^p$ (corresponding to the function $h := h_p : t \mapsto t^{p-1}$) is uniformly convex with gage of uniform convexity $t \mapsto 2^{1-p}t^p/p$ (as it is the case in L_p spaces for $p \geq 2$), the gage of uniform monotonicity of J_p on X is simply $t \mapsto 2^{2-p}t^p/p$ and the inequality (3.8) of the proof of Theorem 3.1 takes the form

$$\frac{1}{p}2^{2-p}\|x-x'\|^p \le 2r^{p-1}d_s(C,C') \tag{5.5}$$

or

$$\|x-x'\| \le p^{1/p}r^{1-1/p}d_s(C,C')^{1/p}. \tag{5.6}$$

Thus, we see that our result can be applied with various estimates without entering into the computations of these estimates which are often sophisticated or tedious. It is known that even in Hilbert spaces, the behavior of $C \mapsto p_C$ is of Hölder type and not of Lipschitz type. Let us note that in that case, the normalized duality mapping is just the identity mapping and one can take $\gamma(t) = t^2$ in Theorem 3.1.

While the estimate of Theorem 3.1 is sharp, let us note, however, that our estimate in Theorem 4.1 is rougher than the estimates obtained in [2, 3, 4, 5], [12, Theorem 2.8, page 43], [32] under the assumption that X is both uniformly convex and uniformly smooth. In particular, it does not yield the Lipschitz property of the projection operator in a Hilbert space whereas the estimates of [32] (which rely on different techniques from [11, 37, 41]) enable one to recover this classical case. This fact is due to the method we used to obtain Theorem 4.1 as a consequence of Theorem 3.1 which is a general result for arbitrary perturbations, while in Theorem 4.1 we use only translations of convex sets. On the other hand, the estimate of Theorem 4.1 avoids the use of the modulus of smoothness of the space, even when the space is uniformly smooth.

In [21, 22], Kien uses related techniques in order to deal with the projections on the values of a pseudo-Lipschitz multifunction. In [12, Lemma 2.5, page 40] a uniform continuity result on bounded neighborhoods of C is given in terms of the modulus of uniform convexity of the space X. It reads as follows: for $w \in X$ with $d(w,C) \le r$, for $w' \in B(w,r')$ with $r' \in]0,r[$ one has

$$||p_C(w) - p_C(w')|| \le (r+r')\delta_X^{-1}(2r'/(r+r')). \tag{5.7}$$

In order to compare this estimate with the one in Theorem 4.1 one has to use the links between δ_X and the gage of monotonicity of the duality mapping we have displayed above (and which are rather complex, whereas both proofs are simple).

In [30], the author obtained an estimate of the modulus of uniform continuity of the projection mapping in a uniformly convex Banach space X in terms of the dependence on C, using the canonical modulus of uniform convexity of X given by

$$\mu_X(t) := \sup\left\{\left\|\frac{1}{2}(u-v)\right\| : u,v \in B_X,\ 1 - \left\|\frac{1}{2}(u+v)\right\| \le t\right\} \tag{5.8}$$

(note that μ_X is a quasi-inverse in the sense of [34] of the gage γ_X of uniform convexity of X and is different from it in general; more precisely, $\mu_X = \gamma_X^{-1}$ when γ_X is increasing). Namely, given $w \in X$ and nonempty closed convex subsets C, C' such that $d(w,C) \le r$,

$d(w,C) \geq b > 0$ it is shown in [30] that

$$\|p_C(w) - p_{C'}(w)\| \leq \varphi(2d(C,C')) + d(C,C'), \tag{5.9}$$

with $\varphi(t) = 2(r-t)\mu_X(t/b)$ provided $d(C,C')$ is small enough. We observe that such an estimate only involves the canonical modulus of uniform convexity μ_X of X. On the other hand, its domain of validity is limited by the restriction $d(w,C) \geq b$ which is not present here.

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