AUXILIARY PRINCIPLE AND FUZZY VARIATIONAL-LIKE INEQUALITIES

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The purpose of this paper is to introduce the concept of fuzzy variational-like inequalities and to study the existence problem and the iterative approximation problem for solutions of certain kinds of fuzzy variational-like inequalities in Hilbert spaces. By using the general auxiliary principle technique, Ky Fan's KKM theorem, Nadler's fixed point theorem, and some new analytic techniques, some existence theorems and some iterative approximation schemes for solving this kind of fuzzy variational-like inequalities are established. The results presented in this paper are new and they generalize, improve, and unify a number of recent results.

1. Introduction

In recent years, the fuzzy set theory introduced by Zadeh [18] in 1965 has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of physical, mathematical and engineering sciences, see [2, 6, 20]. Equally important is variational inequality theory, which constitutes a significant and important extension of the variational principle. A useful and important generalization of variational inequalities is generalized mixed variational-like inequality. These kinds of variational inequalities have potential and significant applications in optimization theory [16, 17], structural analysis [14] and economics [5, 16]. Some special cases of mixed variational-like inequalities have been studied by Tian [16] and Parida and Sen [15] by using Berge maximum theorem in finite and infinite dimensional spaces. It is useful to remark that these methods are not constructive. Thus, the development of an efficient and implementable technique for solving variational-like inequalities is one of the most interesting and important problems in variational inequality theory. Although there exist many numerical methods (e.g., the projection method and its variant forms, linear approximation, descent and Newton's methods) for variational inequalities, there are very few methods for general variational-like inequalities. One method used in the literature is to develop an auxiliary technique for solving various mixed variational-like inequalities.

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The auxiliary principle technique was suggested by Glowinski et al. [9] in 1981. These days it is a useful and powerful tool for solving various mixed variational-like inequalities. Recently, Noor [12] extended the auxiliary principle technique to study the existence and uniqueness of a solution for a class of generalized mixed variational-like inequalities for set-valued mappings with compact values. However, the proof of the uniqueness part in [12, Theorem 3.1] is not quite right. Also the proof of the existence part is based on the assumption that the auxiliary problem has a solution. Unfortunately he did not show the existence of the solution for this auxiliary problem. Subsequently, Huang and Deng [10] introduced and studied a class of generalized set-valued strongly nonlinear mixed variational-like inequalities which includes the known class of mixed variational-like inequalities introduced by Noor [12] as a special case. On the other hand, in 2001, Ansari and Yao [1] considered and studied a class of mixed variational-like inequality problems for single-valued mappings by using the auxiliary principle technique.

Inspired and motivated by recent research in this interesting field, the purpose of this paper is to introduce the concept of fuzzy generalized set-valued mixed variational-like inequalities and to study the existence problem and the iterative approximation problem for solutions of certain kinds of fuzzy generalized set-valued mixed variational-like inequalities in Hilbert spaces. By using the general auxiliary principle technique, Ky Fan's KKM theorem, Nadler's fixed point theorem and some new analytic techniques, some existence theorems and some iterative approximation schemes for solving this kind of fuzzy variational-like inequalities are established. The results presented in this paper are new and they generalize, improve and unify a number of recent results in [1, 10, 19].

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$, $\|\cdot\|$, respectively. Let K be a nonempty convex subset of H and $CB(H)$ be the family of all nonempty bounded and closed subsets of H, $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$ defined by

$$D(A,B)=\max\left\{\sup_{x\in A} d(x,B),\sup_{y\in B} d(A,y)\right\},\quad A,B\in CB(H). \tag{2.1}$$

$D(T)$ and $R(T)$denote the domain and range of T, respectively.

In what follows, we denote the collection of all fuzzy sets on H by $\mathcal{F}(H)=\{A:H\to[0,1]\}$. A mapping T from H to $\mathcal{F}(H)$ is called a fuzzy mapping. If $T:H\to\mathcal{F}(H)$ is a fuzzy mapping, then the set $T(x)$, for $x\in H$, is a fuzzy set in $\mathcal{F}(H)$ (in the sequel we denote $T(x)$ by T_x) and $T_x(y)$, $y\in H$ is the degree of membership of y in T_x.

Definition 2.1. (1) A fuzzy mapping $T:H\to\mathcal{F}(H)$ is said to be closed, if for each $x\in H$, the function $y\mapsto T_x(y)$ is upper semicontinuous, that is, for any given net $\{y_\alpha\}\subset H$ satisfying $y_\alpha\to y_0\in H$, we have $\limsup_\alpha T_x(y_\alpha)\le T_x(y_0)$. For $A\in\mathcal{F}(H)$ and $\lambda\in[0,1]$, the set

$$(A)_\lambda=\{x\in H:A(x)\ge\lambda\} \tag{2.2}$$

is called a λ-cut set of A.

(2) A closed fuzzy mapping $T : H \to \mathcal{F}(H)$ is said to satisfy the condition $(*)$, if there exists a function $a : H \to [0,1]$ such that for each $x \in H$ the set

$$(T_x)_{a(x)} := \{y \in H : T_x(y) \geq a(x)\} \tag{2.3}$$

is a nonempty bounded subset of H.

Remark 2.2. It should be pointed out that if T is a closed fuzzy mapping satisfying condition $(*)$, then for each $x \in H$, the set $(T_x)_{a(x)} \in CB(H)$. To see this let $\{y_\alpha\}_{\alpha\in\Gamma} \subset (T_x)_{a(x)}$ be a net and $y_\alpha \to y_0 \in H$. Then $(T_x)(y_\alpha) \geq a(x)$ for each $\alpha \in \Gamma$ and since T is closed, we have

$$T_x(y_0) \geq \limsup_{\alpha\in\Gamma} T_x(y_\alpha) \geq a(x). \tag{2.4}$$

This implies that $y_0 \in (T_x)_{a(x)}$ and so $(T_x)_{a(x)} \in CB(H)$.

Problem 2.3. Let H be a real Hilbert space, K be a nonempty closed convex subset of H. Let $T, V : K \to \mathcal{F}(H)$ be two closed fuzzy mappings satisfying condition $(*)$ with functions $a, c : H \to [0,1]$, respectively. For given nonlinear mappings $N, \eta : H \times H \to H$ we consider the problem of finding $u \in K$, $w, y \in H$ such that

$$\begin{gathered} T_u(w) \geq a(u), \quad V_u(y) \geq c(u), \quad \text{i.e., } w \in (T_u)_{a(u)},\ y \in (V_u)_{c(u)}, \\ \langle N(w,y), \eta(v,u)\rangle + b(u,v) - b(u,u) \geq 0, \quad \forall v \in K; \end{gathered} \tag{2.5}$$

here $b(\cdot,\cdot) : H \times H \to R$ is a nondifferentiable function satisfying Assumption 2.4.

Assumption 2.4. (i) $b(\cdot,\cdot)$ is linear in the first argument.

(ii) For any $u, v, w \in H$ there exists a constant $\gamma > 0$ such that

(a) $|b(u,v)| \leq \gamma\|u\|\,\|v\|$;

(b) $|b(u,v) - b(u,w)| \leq \gamma\|u\|\,\|v - w\|$.

(iii) For any given $u \in H$, the function $v \mapsto b(u,v) : H \to R$ is convex.

Problem (2.5) is called the fuzzy variational-like inequality in Hilbert spaces.

Remark 2.5. It follows from property (i) that for any $u, v \in H$, $-b(u,v) = b(-u,v)$.

Now we consider some special cases of problem (2.5).

(1) Let $T, V : K \to CB(H)$ be two ordinary multivalued mappings and η, N, b be the mappings as in Problem 2.3. Now we define two fuzzy mappings $\widetilde{T}(\cdot), \widetilde{V}(\cdot) : K \to \mathcal{F}(H)$ as follows:

$$\widetilde{T}_x = \chi_{T(x)}, \qquad \widetilde{V}_x = \chi_{V(x)}, \tag{2.6}$$

where $\chi_{T(x)}$ and $\chi_{V(x)}$ are the characteristic functions of the sets $T(x)$ and $V(x)$, respectively. It is easy to see that $\widetilde{T}$ and $\widetilde{V}$ both are closed fuzzy mappings satisfying condition $(*)$

with constant functions $a(x) = 1$ and $c(x) = 1$, for all $x \in H$, respectively. Also

$$(\widetilde{T}_x)_{a(x)} = (\chi_{T(x)})_1 = \{y \in H : \chi_{T(x)}(y) = 1\} = T(x),$$
$$(\widetilde{V}_x)_{c(x)} = (\chi_{V(x)})_1 = \{y \in H : \chi_{V(x)}(y) = 1\} = V(x). \tag{2.7}$$

Then problem (2.5) is equivalent to finding $u \in K$, $w, y \in H$ such that

$$(\widetilde{T}_u)(w) = 1, \quad (\widetilde{V}_u)(y) = 1, \quad \text{i.e., } w \in T(u),\ y \in V(u),$$
$$\langle N(w,y), \eta(v,u)\rangle + b(u,v) - b(u,u) \geq 0, \quad \forall v \in K. \tag{2.8}$$

This kind of problem is called the set-valued strongly nonlinear mixed variational-like inequality and was introduced and studied by Noor [12] under the additional condition that $T, V : H \to CB(H)$ is compact-valued. It is also considered by Zeng [19].

(2) If $N(w, y) = w - y$, $w, y \in H$ and $b(u,v) = f(v)$ and $T, V : K \to H$ are single-valued, then problem (2.8) is equivalent to finding a $u \in K$ such that

$$\langle T(u) - V(u), \eta(v,u)\rangle + f(v) - f(u) \geq 0, \quad \forall v \in K. \tag{2.9}$$

This is called the mixed variational-like inequality problem and was studied by Ansari and Yao [1].

(3) If $K = H$ and $b(u,v) = 0$, then problem (2.5) is equivalent to finding $u, w, y \in H$ such that

$$T_u(w) \geq a(u), \quad V_u(y) \geq c(u), \quad \text{i.e., } w \in (T_u)_{a(u)},\ y \in (V_u)_{c(u)},$$
$$\langle N(w,y), \eta(v,u)\rangle \geq 0, \quad \forall v \in H. \tag{2.10}$$

This is also a class of special fuzzy variational-like inequalities. The case of ordinary set-valued mappings (i.e., in the nonfuzzy case) was considered by Noor [13].

As a result for a suitable choice of the fuzzy mappings T, V and mappings η, N, b, we can obtain a number of old and new classes of (fuzzy) variational inequalities, (fuzzy) variational inclusions and we can obtain corresponding (fuzzy) optimization problems from the fuzzy variational-like inequality (2.5).

Now we recall some definitions and notions which will be needed to prove our main results.

Definition 2.6. Let $T, V : H \to \mathcal{F}(H)$ be two closed fuzzy mappings satisfying condition $(*)$ with functions $a, c : H \to [0,1]$, respectively and let $N(\cdot,\cdot) : H \times H \to H$ be a nonlinear mapping.

(1) The mapping $x \mapsto N(x, y)$ is said to be β-Lipschitzian continuous with respect to the fuzzy mapping T if, for any $x_1, x_2 \in H$ and $w_1 \in (T_{x_1})_{a(x_1)}$, $w_2 \in (T_{x_2})_{a(x_2)}$,

$$||N(w_1, y) - N(w_2, y)|| \leq \beta ||x_1 - x_2||, \quad y \in H, \tag{2.11}$$

where $\beta > 0$ is a constant.

(2) The mapping $y \mapsto N(x,y)$ is said to be γ-Lipschitzian continuous with respect to the fuzzy mapping V if, for any $u_1,u_2 \in H$ and $v_1 \in (V_{u_1})_{c(u_1)}$, $v_2 \in (V_{u_2})_{c(u_2)}$,

$$||N(x,v_1) - N(x,v_2)|| \le \gamma ||u_1 - u_2||, \quad x \in H, \tag{2.12}$$

where $\gamma > 0$ is a constant.

(3) N is said to be α-strongly mixed monotone with respect to the fuzzy mappings T and V, if for any $u_1,u_2 \in H$

$$\begin{gathered}\langle N(w_1,y_1) - N(w_2,y_2), u_1 - u_2\rangle \ge \alpha ||u_1 - u_2||^2, \\ \forall w_1 \in (T_{u_1})_{a(u_1)},\ y_1 \in (V_{u_1})_{c(u_1)};\ w_2 \in (T_{u_2})_{a(u_2)},\ y_2 \in (V_{u_2})_{c(u_1)},\end{gathered} \tag{2.13}$$

where $\alpha > 0$ is a constant.

(4) T is said to be ξ-Lipschitz continuous , if for any $x,y \in H$

$$D\big((T_x)_{a(x)}, (T_y)_{a(y)}\big) \le \xi \|x - y\|, \tag{2.14}$$

where $\xi > 0$ is a constant.

Definition 2.7. Let $\eta : H \times H \to H$ be a mapping.

(1) η is strongly monotone, if there exists constant $\sigma > 0$ such that

$$\langle \eta(u,v), u - v\rangle \ge \sigma \|u - v\|, \quad \forall u,v \in H; \tag{2.15}$$

(2) η is Lipschitz continuous, if there exists a constant $\delta > 0$ such that

$$||\eta(u,v)|| \le \delta \|u - v\|, \quad \forall u,v \in H. \tag{2.16}$$

Definition 2.8. Let D be a nonempty subset of H and $f : D \to (-\infty,+\infty)$ be a function. f is said to be lower semicontinuous in D, if for any $a \in (-\infty,+\infty)$, the set $\{u \in D : f(u) \le a\}$ is a closed set in D. f is said to be upper semicontinuous in D, if $-f$ is lower semicontinuous in D.

Definition 2.9. Let K be a nonempty subset of H and $G : K \to 2^H$ be a mapping. G is called a KKM mapping, if for any finite subset $\{x_1,x_2,\dots,x_n\} \subset K$ we have

$$co(\{x_1,x_2,\dots,x_n\}) \subset \bigcup_{i=1}^{n} G(x_i), \tag{2.17}$$

where $co\{A\}$ is the convex hull of the set $A \subset K$.

In the sequel, we assume that N and η satisfy the following assumption.

Assumption 2.10. Let $N,\eta : H \times H \to H$ be two mappings satisfying the following conditions:

(i) $\eta(v,u) = -\eta(u,v)$, $\forall u,v \in H$;

(ii) for any given $x,y,u \in H$ the mapping $v \mapsto \langle N(x,y), \eta(u,v)\rangle$ is concave and upper semicontinuous.

The following lemma due to K. Fan will be needed in proving our main result.

Lemma 2.11 (Fan [7]). *Let E be a topological vector space and K be a nonempty subset of E. Let $G : K \to 2^E$ be a KKM mapping with closed values and suppose there exists at least a point $x \in K$ such that $G(x)$ is compact. Then $\bigcap_{x\in K} G(x) \neq \emptyset$.*

In Section 3 the following result will be needed.

Theorem 2.12 [3, Theorem 1.4.7]. *Let E be a locally convex Hausdorff topological vector space and $f : E \to R \cup \{+\infty\}$ be a proper convex functional. Then f is lower semicontinuous on E, if and only if f is weakly lower semicontinuous on E.*

Remark 2.13. A functional $f : E \to R \cup \{+\infty\}$ is said to be *proper*, if $f(x) > -\infty$ for all $x \in E$ and $f \not\equiv +\infty$.

3. Auxiliary principle and algorithm

In this section, we extend the auxilary principle technique of Glowinski et al. [9] and then use it to study the fuzzy variational-like inequality (2.5) in Hilbert spaces. We first establish an existence theorem for the auxiliary problem for the fuzzy variational-like inequality (2.5). Based on this existence theorem, we construct the iterative algorithm for this kind of fuzzy variational-like inequality.

For given $u \in K$, $w \in (T_u)_{a(u)}$, $y \in (V_u)_{c(u)}$ the "so called" *auxiliary problem $P(u,w,y)$ for the fuzzy variational-like inequality (2.5) in Hilbert space H*, is to find $z \in K$ such that

$$\langle z, v - z\rangle \geq \langle u, v - z\rangle - \rho\langle N(w,y), \eta(v,z)\rangle + \rho b(u,z) - \rho b(u,v), \quad \forall v \in K, \tag{3.1}$$

where $\rho > 0$ is a constant.

Theorem 3.1. *Let H be a real Hilbert space, K be a nonempty bounded closed convex subset of H and $b(\cdot,\cdot)$ be a function satisfying Assumption 2.4. Let $\eta : K \times K \to H$ be a Lipschitz mapping with a Lipschitz constant δ. If Assumption 2.10 is satisfied, then the auxiliary problem $P(u,w,y)$ has a unique solution in K.*

Proof. For given $u \in K$, $w \in (T_u)_{a(u)}$, $y \in (V_u)_{c(u)}$ we define the mapping $G : K \to 2^H$ by

$$G(v) = \{x \in K : \langle x - u, v - x\rangle + \rho[\langle N(w,y), \eta(v,x)\rangle + b(u,v) - b(u,x)] \geq 0\}, \quad v \in K. \tag{3.2}$$

Note that for each $v \in K$, $G(v)$ is nonempty, since $v \in G(v)$.

Next we prove that $G : K \to 2^H$ is a KKM mapping. Suppose the contrary, that is, suppose G is not a KKM mapping. Then there exist a finite subset $\{v_1, v_2, \dots, v_k\}$ of K and constants $\alpha_i \geq 0$, $i = 1,2,\dots,k$ with $\sum_{i=1}^n \alpha_i = 1$ such that

$$x_* = \sum_{i=1}^k \alpha_i v_i \notin \bigcup_{i=1}^k G(v_i), \tag{3.3}$$

that is, $x_* \notin G(v_i)$, for all $i = 1, 2, \ldots, k$. Hence we have

$$\langle x_* - u, v_i - x_* \rangle + \rho\big[\langle N(w,y), \eta(v_i, x_*)\rangle + b(u, v_i) - b(u, x_*)\big] < 0. \tag{3.4}$$

From Assumption 2.10 we know $\eta(v,u) = -\eta(u,v)$ (and so $\eta(u,u) = 0$, $\forall u \in H$) and $v \mapsto \langle N(x,y), \eta(u,v)\rangle$ is concave and also from Assumption 2.4 we know $v \mapsto b(u,v)$ is convex, so the above inequality yields

$$\begin{aligned}
0 &> \sum_{i=1}^{k} \alpha_i \langle x_* - u, v_i - x_* \rangle + \rho \sum_{i=1}^{k} \alpha_i \big[\langle N(w,y), \eta(v_i, x_*)\rangle + b(u, v_i) - b(u, x_*)\big] \\
&\geq \langle x_* - u, x_* - x_* \rangle - \rho\big[\langle N(w,y), \eta(x_*, x_*)\rangle\big] + \rho \sum_{i=1}^{k} \alpha_i \big[b(u, v_i) - b(u, x_*)\big] \\
&= \rho \sum_{i=1}^{k} \alpha_i \big[b(u, v_i) - b(u, x_*)\big] \geq \rho\big[b(u, x_*) - b(u, x_*)\big] = 0,
\end{aligned} \tag{3.5}$$

a contradiction. This implies that G is a KKM mapping.

Now $\overline{G(v)}^{w}$ (the weak closure of $G(v)$) is a weakly closed subset of a bounded set K in H, so it is weakly compact. Hence by Lemma 2.11

$$\bigcap_{v \in K} \overline{G(v)}^{w} \neq \emptyset. \tag{3.6}$$

Let $z \in \bigcap_{v \in K} \overline{G(v)}^{w}$. Fix $v \in K$, then there exists a sequence $\{z_m\}$ in $G(v)$ such that $z_m \to z$ weakly. Therefore we have

$$\langle z_m - u, v - z_m \rangle + \rho\big[\langle N(w,y), \eta(v, z_m)\rangle + b(u, v) - b(u, z_m)\big] \geq 0. \tag{3.7}$$

Since the norm is weakly lower semicontinuous, we have

$$\begin{aligned}
\limsup_{m\to\infty} \langle z_m - u, v - z_m \rangle &= \limsup_{m\to\infty} \Big[\langle z_m - u, v \rangle + \langle u, z_m \rangle - ||z_m||^2\Big] \\
&= \lim_{m\to\infty} \langle z_m - u, v \rangle + \lim_{m\to\infty} \langle u, z_m \rangle - \liminf_{m\to\infty} ||z_m||^2 \\
&\leq \langle z - u, v - z \rangle.
\end{aligned} \tag{3.8}$$

Note from Assumption 2.4 that b is Lipschitz continuous and convex in the second argument and from Assumption 2.10 the mapping $v \mapsto \langle N(x,y), \eta(u,v)\rangle$ is upper semicontinuous and concave so (3.7) and (3.8) and Theorem 2.12 imply

$$\begin{aligned}
0 &\leq \limsup_{m\to\infty} \big\{\langle z_m - u, v - z_m \rangle + \rho\big[\langle N(w,y), \eta(v, z_m)\rangle + b(u,v) - b(u, z_m)\big]\big\} \\
&\leq \limsup_{m\to\infty} \langle z_m - u, v - z \rangle + \limsup_{m\to\infty} \big\{\rho\langle N(w,y), \eta(v, z_m)\rangle + b(u,v) - b(u, z_m)\big\} \\
&\leq \langle z - u, v - z \rangle + \rho\big[\langle N(w,y), \eta(v, z)\rangle + b(u,v) - b(u,z)\big].
\end{aligned} \tag{3.9}$$

This implies that

$$\langle z, v-z\rangle \geq \langle u, v-z\rangle - \rho\langle N(w,y),\eta(v,z)\rangle + \rho b(u,z) - \rho b(u,v), \quad \forall v\in K, \tag{3.10}$$

that is, z is a solution of the auxiliary problem $P(u,w,y)$ in K.

Finally we prove the uniqueness of solutions of $P(u,w,y)$.

In fact, if $z_1\in K$ is also a solution of the auxiliary problem $P(u,w,y)$, then we have

$$\langle z_1, v-z_1\rangle \geq \langle u, v-z_1\rangle - \rho\langle N(w,y),\eta(v,z_1)\rangle + \rho b(u,z_1) - \rho b(u,v), \quad \forall v\in K. \tag{3.11}$$

Taking $v=z_1$ in (3.10) and $v=z$ in (3.11), and adding the resultant inequalities gives

$$\langle z-z_1, z_1-z\rangle \geq 0, \tag{3.12}$$

hence $z=z_1$.

This completes the proof of Theorem 3.1. □

Notice that Theorem 3.1 suggests the following algorithms for solving the fuzzy variational-like inequality (2.5).

Algorithm 3.2. For any given $u_0\in K$, $w_0\in (T_{u_0})_{a(u_0)}$, $y_0\in (V_{u_0})_{c(u_0)}$, from Theorem 3.1 the auxiliary problem $P(u_0,w_0,y_0)$ has a unique solution $u_1\in K$, that is,

$$\langle u_1, v-u_1\rangle \geq \langle u_0, v-u_1\rangle - \rho\langle N(w_0,y_0),\eta(v,u_1)\rangle + \rho b(u_0,u_1) - \rho b(u_0,v), \quad \forall v\in K. \tag{3.13}$$

Now since $w_0\in (T_{u_0})_{a(u_0)}\in CB(H)$, $y_0\in (V_{u_0})_{c(u_0)}\in CB(H)$, by Nadler's theorem [11], there exist $w_1\in (T_{u_1})_{a(u_1)}$, $y_1\in (V_{u_1})_{c(u_1)}$ such that

$$\begin{aligned} ||w_0-w_1|| &\leq (1+1)D\Big(\big(T_{u_0}\big)_{a(u_0)},\big(T_{u_1}\big)_{a(u_1)}\Big),\\ ||y_0-y_1|| &\leq (1+1)D\Big(\big(V_{u_0}\big)_{c(u_0)},\big(V_{u_1}\big)_{c(u_1)}\Big), \end{aligned} \tag{3.14}$$

where D is the Hausdorff metric on $CB(H)$. Again by Theorem 3.1, the auxiliary problem $P(u_1,w_1,y_1)$ has a unique solution $u_2\in K$, that is,

$$\langle u_2, v-u_2\rangle \geq \langle u_1, v-u_2\rangle - \rho\langle N(w_1,y_1),\eta(v,u_2)\rangle + \rho b(u_1,u_2) - \rho b(u_1,v) \quad \forall v\in K. \tag{3.15}$$

Since $w_1\in (T_{u_1})_{a(u_1)}\in CB(H)$, $y_1\in (V_{u_1})_{c(u_1)}\in CB(H)$, again by Nadler's theorem, there exist $w_2\in (T_{u_2})_{a(u_2)}$, $y_2\in (V_{u_2})_{c(u_2)}$, such that

$$\begin{aligned} ||w_1-w_2|| &\leq \Big(1+\frac{1}{2}\Big)D\Big(\big(T_{u_1}\big)_{a(u_1)},\big(T_{u_2}\big)_{a(u_2)}\Big),\\ ||y_1-y_2|| &\leq \Big(1+\frac{1}{2}\Big)D\Big(\big(V_{u_1}\big)_{c(u_1)},\big(V_{u_2}\big)_{c(u_2)}\Big). \end{aligned} \tag{3.16}$$

Continuing in this way, we can obtain sequences $\{u_n\} \subset K$, $\{w_n\},\{y_n\} \subset H$ such that

(i) $w_n \in (T_{u_n})_{a(u_n)}$,

$$\|w_n - w_{n+1}\| \le \left(1+\frac{1}{n+1}\right) D\Big(\big(T_{u_n}\big)_{a(u_n)}, \big(T_{u_{n+1}}\big)_{a(u_{n+1})}\Big), \tag{3.17}$$

(ii) $y_n \in (V_{u_n})_{c(u_n)}$,

$$\|y_n - y_{n+1}\| \le \left(1+\frac{1}{n+1}\right) D\Big(\big(V_{u_n}\big)_{c(u_n)}, \big(V_{u_{n+1}}\big)_{c(u_{n+1})}\Big), \tag{3.18}$$

(iii)

$$\begin{aligned} \langle u_{n+1}, v - u_{n+1}\rangle \ge{}& \langle u_n, v-u_{n+1}\rangle - \rho\langle N(w_n,y_n), \eta(v,u_{n+1})\rangle \\ &+\rho b(u_n,u_{n+1}) - \rho b(u_n,v) \quad \forall v \in K,\ \forall n \ge 0, \end{aligned} \tag{3.19}$$

where $\rho > 0$ is a constant.

4. Main results

THEOREM 4.1. *Let H be a real Hilbert space, K be a nonempty bounded closed convex subset of H, $T,V : K \to \mathcal{F}(H)$ be two closed fuzzy mappings satisfying condition $(*)$ with functions $a,c : H \to [0,1]$, respectively, $N(\cdot,\cdot) : H\times H \to H$ be a nonlinear single-valued continuous mappings and $b(\cdot,\cdot) : H \times H \to H$ be a nondifferentiable function satisfying the following conditions:*

(i) *$T,V : K \to \mathcal{F}(H)$ are two Lipschitzian continuous fuzzy mappings with Lipschitzian constants ν and μ, respectively,*

(ii) *the mapping $x \mapsto N(x,y)$ is β-Lipschitzian continuous with respect to the fuzzy mapping T for any given $y \in H$,*

(iii) *the mapping $y \mapsto N(x,y)$ is ξ-Lipschitzian continuous with respect to the fuzzy mapping V for any given $x \in H$,*

(iv) *N is α-strongly mixed monotone with respect to the fuzzy mappings T and V,*

(v) *$\eta : K\times K \to H$ is σ-strongly monotone and δ-Lipschitz continuous,*

here μ, ν, β, ξ, σ, δ all are positive constants.

If Assumptions 2.4 and 2.10 are satisfied and if there exists a constant $\rho > 0$ satisfies the following condition:

$$\begin{gathered} k = \frac{\sqrt{1-2\sigma+\delta^2}+\delta-1}{\delta}, \qquad \frac{\rho\gamma}{\delta}+k<1, \\ \alpha > \frac{\gamma}{\delta} + \sqrt{\left((\beta+\xi)^2 - \frac{\gamma^2}{\delta^2}\right)2k(1-k)}, \\ \left|\rho - \frac{\alpha-\gamma/\delta}{(\beta+\xi)^2-\gamma^2/\delta^2}\right| \le \frac{\sqrt{(\alpha-\gamma/\delta)^2 - ((\beta+\xi)^2-\gamma^2/\delta^2)2k(1-k)}}{(\beta+\xi)^2-\gamma^2/\delta^2}, \end{gathered} \tag{4.1}$$

then there exist $u \in K$, $w \in (T_u)_{a(u)}$, $y \in (V_u)_{c(u)}$ which is a solution of the fuzzy variational-like inequality (2.5) and the iterative sequences $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ generated by (3.18) converge strongly to u, w, y in H, respectively.

Proof. It follows from (3.18) that for each $v \in K$ and each $n \geq 1$ we have

$$\begin{aligned}\langle u_n, v - u_n\rangle &\geq \langle u_{n-1}, v - u_n\rangle - \rho\langle N(w_{n-1}, y_{n-1}), \eta(v, u_n)\rangle \\ &\quad + \rho b(u_{n-1}, u_n) - \rho b(u_{n-1}, v), \quad \forall v \in K,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\langle u_{n+1}, v - u_{n+1}\rangle &\geq \langle u_n, v - u_{n+1}\rangle - \rho\langle N(w_n, y_n), \eta(v, u_{n+1})\rangle \\ &\quad + \rho b(u_n, u_{n+1}) - \rho b(u_n, v), \quad \forall v \in K.\end{aligned} \tag{4.3}$$

Taking $v = u_{n+1}$ in (4.2) and $v = u_n$ in (4.3) and then adding the resultant inequalities gives

$$\begin{aligned}\langle u_{n+1} - u_n, u_n - u_{n+1}\rangle &\geq \langle u_n - u_{n-1}, u_n - u_{n+1}\rangle \\ &\quad - \rho\langle N(w_n, y_n) - N(w_{n-1}, y_{n-1}), \eta(u_n, u_{n+1})\rangle \\ &\quad + \rho b(u_{n-1} - u_n, u_n) + \rho b(u_n - u_{n-1}, u_{n+1}).\end{aligned} \tag{4.4}$$

Therefore we have

$$\begin{aligned}\langle u_n - u_{n+1}, u_n - u_{n+1}\rangle &\leq \langle u_{n-1} - u_n, u_n - u_{n+1}\rangle \\ &\quad - \rho\langle (N(w_{n-1}, y_{n-1}) - N(w_n, y_n)), \eta(u_n, u_{n+1})\rangle \\ &\quad + \rho b(u_n - u_{n-1}, u_n) - \rho b(u_n - u_{n-1}, u_{n+1}) \\ &= \langle u_{n-1} - u_n, u_n - u_{n+1} - \eta(u_n, u_{n+1})\rangle \\ &\quad + \langle u_{n-1} - u_n - \rho(N(w_{n-1}, y_{n-1}) - N(w_n, y_n)), \eta(u_n, u_{n+1})\rangle \\ &\quad + \rho\{b(u_n - u_{n-1}, u_n) - b(u_n - u_{n-1}, u_{n+1})\} \\ &\leq ||u_{n-1} - u_n||\,||u_n - u_{n+1} - \eta(u_n, u_{n+1})|| \\ &\quad + ||u_{n-1} - u_n - \rho(N(w_{n-1}, y_{n-1}) - N(w_n, y_n))||\,||\eta(u_n, u_{n+1})|| \\ &\quad + \rho\gamma||u_n - u_{n-1}||\,||u_n - u_{n+1}||.\end{aligned} \tag{4.5}$$

Now we consider the first term on the right-hand side of (4.5). Since $\eta(\cdot,\cdot)$ is σ-strongly monotone and δ-Lipschitzian continuous, we have

$$\begin{aligned}||u_n - u_{n+1} - \eta(u_n, u_{n+1})||^2 &= ||u_n - u_{n+1}||^2 - 2\langle u_n - u_{n+1}, \eta(u_n, u_{n+1})\rangle + ||\eta(u_n, u_{n+1})||^2 \\ &\leq (1 - 2\sigma + \delta^2)||u_n - u_{n+1}||^2.\end{aligned} \tag{4.6}$$

This implies that

$$||u_{n-1}-u_n||\,||u_n-u_{n+1}-\eta(u_n,u_{n+1})|| \le \sqrt{(1-2\sigma+\delta^2)}\,||u_{n-1}-u_n||\,||u_n-u_{n+1}||. \tag{4.7}$$

Now we consider the second term on the right-hand side of (4.5). By condition (iv), N is α-strongly mixed monotone with respect to the fuzzy mappings T and V, so we have

$$\begin{aligned}
&||u_{n-1}-u_n-\rho(N(w_{n-1},y_{n-1})-N(w_n,y_n))||^2\\
&\quad=||u_{n-1}-u_n||^2-2\rho\langle N(w_{n-1},y_{n-1})-N(w_n,y_n),u_{n-1}-u_n\rangle\\
&\qquad+\rho^2||N(w_{n-1},y_{n-1})-N(w_n,y_n)||^2\\
&\quad\le||u_{n-1}-u_n||^2-2\rho\alpha||u_{n-1}-u_n||^2\\
&\qquad+\rho^2||N(w_{n-1},y_{n-1})-N(w_n,y_{n-1})+N(w_n,y_{n-1})-N(w_n,y_n)||^2\\
&\quad\le(1-2\rho\alpha)||u_{n-1}-u_n||^2+\rho^2[\beta||u_{n-1}-u_n||+\xi||u_{n-1}-u_n||]^2\\
&\qquad\text{(by conditions (ii) and (iii))}\\
&\quad=[1-2\rho\alpha+\rho^2(\beta+\xi)^2]||u_{n-1}-u_n||^2.
\end{aligned} \tag{4.8}$$

Again since $\eta: K\times K\to H$ is δ-Lipschitz continuous, from (4.8) we have

$$\begin{aligned}
&||u_{n-1}-u_n-\rho(N(w_{n-1},y_{n-1})-N(w_n,y_n))||\,||\eta(u_n,u_{n+1})||\\
&\qquad\le\delta\sqrt{1-2\rho\alpha+\rho^2(\beta+\xi)^2}\,||u_{n-1}-u_n||\,||u_n-u_{n+1}||.
\end{aligned} \tag{4.9}$$

Hence from (4.5), (4.8), and (4.9) we have

$$\begin{aligned}
||u_n-u_{n+1}||^2&\le\sqrt{1-2\sigma+\delta^2}\,||u_{n-1}-u_n||\,||u_n-u_{n+1}||\\
&\quad+\Big\{\delta\sqrt{1-2\rho\alpha+\rho^2(\beta+\xi)^2}+\rho\gamma\Big\}||u_{n-1}-u_n||\,||u_n-u_{n+1}||,
\end{aligned} \tag{4.10}$$

that is,

$$||u_n-u_{n+1}||\le\theta||u_{n-1}-u_n||, \tag{4.11}$$

where

$$\theta=\delta\left\{t(\rho)+\rho\cdot\frac{\gamma}{\delta}+\frac{\sqrt{1-2\sigma+\delta^2}}{\delta}\right\},\qquad t(\rho)=\sqrt{1-2\rho\alpha+\rho^2(\beta+\xi)^2}. \tag{4.12}$$

Next we prove that $\theta<1$. Letting

$$k=\frac{\sqrt{1-2\sigma+\delta^2}+\delta-1}{\delta} \tag{4.13}$$

it is easy to see that

$$\theta < 1 \quad \text{iff } t(\rho) + \frac{\rho\gamma}{\delta} + k < 1. \tag{4.14}$$

Therefore

$$\frac{\rho\gamma}{\delta} + k < 1, \qquad t(\rho)^2 < \left(1 - \frac{\rho\gamma}{\delta} - k\right)^2. \tag{4.15}$$

Simplify it to obtain

$$\rho^2\left[(\beta+\xi)^2 - \left(\frac{\gamma}{\delta}\right)^2\right] - 2\rho\left(\alpha - \frac{\gamma}{\delta}\right) + 2k(1-k) < 2k\rho\frac{\gamma}{\delta}. \tag{4.16}$$

It follows from (4.16) and condition (4.1) that $\theta < 1$. Hence from (4.11) we know that $\{u_n\}$ is a Cauchy sequence in K. By the closedness of K, without loss of generality, we can assume that $u_n \to u \in K$.

On the other hand, by condition (i) we know $T, V : K \to \mathcal{F}(H)$ are two Lipschitzian continuous fuzzy mappings with Lipschitzian constants ν and μ respectively, so from (3.18) we have

$$\begin{aligned}
||w_n - w_{n+1}|| &\le \left(1 + \frac{1}{n+1}\right) D\Big(\big(T_{u_n}\big)_{a(u_n)}, \big(T_{u_{n+1}}\big)_{a(u_{n+1})}\Big) \\
&\le \left(1 + \frac{1}{n+1}\right)\nu||u_n - u_{n+1}||, \\
||y_n - y_{n+1}|| &\le \left(1 + \frac{1}{n+1}\right) D\Big(\big(V_{u_n}\big)_{c(u_n)}, \big(V_{u_{n+1}}\big)_{c(u_{n+1})}\Big) \\
&\le \left(1 + \frac{1}{n+1}\right)\mu||u_n - u_{n+1}||.
\end{aligned} \tag{4.17}$$

This implies that $\{w_n\}$, $\{y_n\}$ are both Cauchy sequences in H. We can assume that $w_n \to w$ and $y_n \to y$ (as $n \to \infty$). Note $w_n \in (T_{u_n})_{a(u_n)}$ and $y_n \in (V_{u_n})_{c(u_n)}$, so we have

$$\begin{aligned}
d\Big(w, \big(T_u\big)_{a(u)}\Big) &\le ||w - w_n|| + d\Big(w_n, \big(T_{u_n}\big)_{a(u_n)}\Big) + D\Big(\big(T_{u_n}\big)_{a(u_n)}, \big(T_u\big)_{a(u)}\Big) \\
&\le ||w - w_n|| + 0 + \nu||u_n - u|| \longrightarrow 0 \quad (n \longrightarrow \infty),
\end{aligned} \tag{4.18}$$

that is, $w \in (T_u)_{a(u)}$.

Essentially the same reasoning yields $y \in (V_u)_{c(u)}$.

Now we rewrite (3.18)(iii) as follows:

$$\langle u_{n+1} - u_n, v - u_{n+1}\rangle + \rho\langle N(w_n, y_n), \eta(v, u_{n+1})\rangle + \rho\big[b(u_n, v) - b(u_n, u_{n+1})\big] \ge 0. \tag{4.19}$$

Note $u_n \to u$ (as $n \to \infty$), $\langle u_{n+1} - u_n, v - u_{n+1}\rangle \to 0$ (as $n \to \infty$) and Assumption 2.10 guarantee the mapping $v \mapsto \langle N(w, y), \eta(u, v)\rangle$ is upper semicontinuous, so we have

$$\langle N(w, y), \eta(v, u)\rangle \ge \limsup_{n\to\infty} \langle N(w, y), \eta(v, u_{n+1})\rangle. \tag{4.20}$$

On the other hand, by condition (v) we have

$$||\eta(v,u_{n+1})|| \leq \delta||v-u_{n+1}||. \tag{4.21}$$

This implies that the sequence $\{\eta(v,u_{n+1})\}$ is bounded. Again since N is continuous, we have

$$\lim_{n\to\infty}\langle N(w_n,y_n)-N(w,y),\eta(v,u_{n+1})\rangle = 0. \tag{4.22}$$

Therefore from (4.20) and (4.22), we have

$$\begin{aligned}
0 &\leq \langle N(w,y),\eta(v,u)\rangle - \limsup_{n\to\infty}\langle N(w,y),\eta(v,u_{n+1})\rangle \\
&= \liminf_{n\to\infty}\{\langle N(w,y),\eta(v,u)\rangle - \langle N(w,y),\eta(v,u_{n+1})\rangle\} \\
&= \liminf_{n\to\infty}\{\langle N(w,y),\eta(v,u)\rangle - \langle N(w_n,y_n),\eta(v,u_{n+1})\rangle \\
&\qquad + \langle N(w_n,y_n)-N(w,y),\eta(v,u_{n+1})\rangle\} \\
&= \liminf_{n\to\infty}\{\langle N(w,y),\eta(v,u)\rangle - \langle N(w_n,y_n),\eta(v,u_{n+1})\rangle \\
&\qquad + \lim_{n\to\infty}\langle N(w_n,y_n)-N(w,y),\eta(v,u_{n+1})\rangle\} \\
&= \liminf_{n\to\infty}\{\langle N(w,y),\eta(v,u)\rangle - \langle N(w_n,y_n),\eta(v,u_{n+1})\rangle\}.
\end{aligned} \tag{4.23}$$

This implies that

$$\langle N(w,y),\eta(v,u)\rangle \geq \limsup_{n\to\infty}\langle N(w_n,y_n),\eta(v,u_{n+1})\rangle. \tag{4.24}$$

Now using Assumption 2.4, we get

$$\begin{aligned}
|b(u_n,u_{n+1})-b(u,u)| &\leq |b(u_n,u_{n+1})-b(u_n,u)| + |b(u_n,u)-b(u,u)| \\
&\leq \gamma||u_n||\,||u_{n+1}-u|| + \gamma||u_n-u||\,\|u\| \longrightarrow 0 \quad (\text{as } n\longrightarrow\infty),
\end{aligned} \tag{4.25}$$

and so

$$b(u_n,u_{n+1}) \longrightarrow b(u,u), \quad b(u_n,v)\longrightarrow b(u,v) \quad (\text{as } n\longrightarrow\infty). \tag{4.26}$$

Therefore, we have

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty}\{\langle u_{n+1}-u_n,v-u_{n+1}\rangle + \rho\langle N(w_n,y_n),\eta(v,u_{n+1})\rangle + \rho[b(u_n,v)-b(u_n,u_{n+1})]\} \\
&\leq \rho\langle N(w,y),\eta(v,u)\rangle + \rho[b(u,v)-b(u,u)],
\end{aligned} \tag{4.27}$$

that is,

$$\langle N(w,y),\eta(v,u)\rangle + b(u,v) - b(u,u) \geq 0, \quad \forall v \in K. \tag{4.28}$$

This completes the proof of Theorem 4.1. □

In the case of ordinary set-valued mappings (i.e., in the nonfuzzy case), from Theorem 4.1 we can obtain the following results.

THEOREM 4.2. *Let H be a real Hilbert space, K be a nonempty bounded closed convex subset of H, $T,V : K \to CB(H)$ be two mappings, $N(\cdot,\cdot) : H \times H \to H$ be a nonlinear single-valued continuous mappings and $b : H \times H \to R$ be a nondifferentiable function satisfying the following conditions:*

(i) *$T,V : K \to CB(H)$ are two Lipschitzian continuous mappings with Lipschitzian constants ν and μ, respectively,*
(ii) *the mapping $x \mapsto N(x,y)$ is β-Lipschitzian continuous with respect to the mapping T for any given $y \in H$,*
(iii) *the mapping $y \mapsto N(x,y)$ is ξ-Lipschitzian continuous with respect to the mapping V for any given $x \in H$,*
(iv) *N is α-strongly mixed monotone with respect to mappings T and V,*
(v) *$\eta : K \times K \to H$ is σ-strongly monotone and δ-Lipschitz continuous,*

here μ, ν, β, ξ, σ, δ all are positive constants.

If Assumptions 2.4 and 2.10 are satisfied and if there exists a constant $\rho > 0$ satisfies the following condition:

$$\begin{gathered} k = \frac{\sqrt{1-2\sigma+\delta^2}+\delta-1}{\delta}, \qquad \frac{\rho\gamma}{\delta}+k<1, \\ \alpha > \frac{\gamma}{\delta} + \sqrt{\left((\beta+\xi)^2-\gamma^2/\delta^2\right)2k(1-k)}, \\ \left|\rho - \frac{\alpha-\gamma/\delta}{(\beta+\xi)^2-\gamma^2/\delta^2}\right| \leq \frac{\sqrt{(\alpha-\gamma/\delta)^2-\left((\beta+\xi)^2-\gamma^2/\delta^2\right)2k(1-k)}}{(\beta+\xi)^2-\gamma^2/\delta^2}, \end{gathered} \tag{4.29}$$

then there exist $u \in K$, $w \in T(u)$, $y \in V(u)$ which is a solution of the set-valued strongly linear mixed variational-like inequality (2.8) and the iterative sequences $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ generated by (3.18) converge strongly to u, w, y in H, respectively.

Proof. By using the set-valued mapping T, $V : K \to CB(H)$ we define two fuzzy mappings $\widetilde{T}(\cdot)$, $\widetilde{V}(\cdot) : K \to \mathcal{F}(H)$ as follows:

$$\widetilde{T}_x = \chi_{T(x)}, \qquad \widetilde{V}_x = \chi_{V(x)}, \tag{4.30}$$

where $\chi_{T(x)}$ and $\chi_{V(x)}$ are the characteristic functions of the sets $T(x)$ and $V(x)$, respectively. It is easy to see that $\widetilde{T}$ and $\widetilde{V}$ both are closed fuzzy mappings satisfying condition $(*)$

with constant functions $a(x) = 1$ and $c(x) = 1$, $\forall x \in H$, respectively. Also

$$\begin{aligned}(\widetilde{T}_x)_{a(x)} &= (\chi_{T(x)})_1 = \{y \in H : \chi_{T(x)}(y) = 1\} = T(x),\\ (\widetilde{V}_x)_{c(x)} &= (\chi_{V(x)})_1 = \{y \in H : \chi_{V(x)}(y) = 1\} = V(x).\end{aligned} \tag{4.31}$$

Then the problem (2.5) is equivalent to finding $u \in K$, $w, y \in H$ such that

$$\begin{gathered}(\widetilde{T}_u)(w) = 1, \quad (\widetilde{V}_u)(y) = 1, \quad \text{i.e., } w \in T(u),\ y \in V(u),\\ \langle N(w,y), \eta(v,u)\rangle + b(u,v) - b(u,u) \geq 0, \quad \forall v \in K.\end{gathered} \tag{4.32}$$

Therefore the conclusion of Theorem 4.2 can be obtained from Theorem 4.1 immediately. □

References

[1] Q. H. Ansari and J. C. Yao, *Iterative schemes for solving mixed variational-like inequalities*, J. Optim. Theory Appl. **108** (2001), no. 3, 527–541.

[2] J.-P. Aubin, *Mathematical Methods of Game and Economic Theory*, Studies in Mathematics and Its Applications, vol. 7, North-Holland Publishing, Amsterdam, 1979.

[3] S. S. Chang, *Variational Inequality and Complementarity Problems Theory and Applications*, Shanghai Scientific and Technological Literature Publishing House, Shanghai, 1991.

[4] S. S. Chang and S. W. Xiang, *On the existence of solutions for a class of quasi-bilinear variational inequalities*, J. Systems Sci. Math. Sci. **16** (1996), 136–140.

[5] P. Cubiotti, *Existence of solutions for lower semicontinuous quasi-equilibrium problems*, Comput. Math. Appl. **30** (1995), no. 12, 11–22.

[6] D. Dubois and H. Prade, *Fuzzy Sets and Systems. Theory and Applications*, Mathematics in Science and Engineering, vol. 144, Academic Press, New York, 1980.

[7] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1960/1961), 305–310.

[8] Y. P. Fang and N. J. Huang, *Variational-like inequalities with generalized monotone mappings in Banach spaces*, J. Optim. Theory Appl. **118** (2003), no. 2, 327–338.

[9] R. Glowinski, J.-L. Lions, and R. Trémolières, *Numerical Analysis of Variational Inequalities*, Studies in Mathematics and Its Applications, vol. 8, North-Holland Publishing, Amsterdam, 1981.

[10] N.-J. Huang and C.-X. Deng, *Auxiliary principle and iterative algorithms for generalized set-valued strongly nonlinear mixed variational-like inequalities*, J. Math. Anal. Appl. **256** (2001), no. 2, 345–359.

[11] S. B. Nadler Jr., *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.

[12] M. A. Noor, *Auxiliary principle for generalized mixed variational-like inequalities*, J. Math. Anal. Appl. **215** (1997), no. 1, 75–85.

[13] ———, *Generalized variational-like inequalities*, Math. Comput. Modelling **27** (1998), no. 3, 93–101.

[14] P. D. Panagiotopoulos and G. E. Stavroulakis, *New types of variational principles based on the notion of quasidifferentiability*, Acta Mech. **94** (1992), no. 3-4, 171–194.

[15] J. Parida and A. Sen, *A variational-like inequality for multifunctions with applications*, J. Math. Anal. Appl. **124** (1987), no. 1, 73–81.

[16] G. Q. Tian, *Generalized quasi-variational-like inequality problem*, Math. Oper. Res. **18** (1993), no. 3, 752–764.

[17] J. C. Yao, *The generalized quasi-variational inequality problem with applications*, J. Math. Anal. Appl. **158** (1991), no. 1, 139–160.
[18] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.
[19] L. C. Zeng, *Iterative approximation of solutions to generalized set-valued strongly nonlinear mixed variational-like inequalities*, Acta Math. Sin. **48** (2005), no. 5, 879–888.
[20] H. I. Zimmermann, *Fuzzy Set Theory and Its Applications*, Kluwer Academic, Massachusetts, 1965.

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