HYERS-ULAM-RASSIAS STABILITY OF JORDAN HOMOMORPHISMS ON BANACH ALGEBRAS

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We prove that a Jordan homomorphism from a Banach algebra into a semisimple commutative Banach algebra is a ring homomorphism. Using a signum effectively, we can give a simple proof of the Hyers-Ulam-Rassias stability of a Jordan homomorphism between Banach algebras. As a direct corollary, we show that to each approximate Jordan homomorphism f from a Banach algebra into a semisimple commutative Banach algebra there corresponds a unique ring homomorphism near to f.

1. Introduction and statement of results

It seems that the stability problem of functional equations had been first raised by Ulam (cf. [11, Chapter VI] and [12]): For what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism?

An answer to the above problem has been given as follows. Suppose E_1 and E_2 are two real Banach spaces and $f : E_1 \to E_2$ is a mapping. If there exist $\delta \geq 0$ and $p \geq 0$, $p \neq 1$ such that

$$\left|\left|f(x+y) - f(x) - f(y)\right|\right| \leq \delta\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

for all $x, y \in E_1$, then there is a unique additive mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq 2\delta\|x\|^p/|2-2^p|$ for every $x \in E_1$. This result is called the Hyers-Ulam-Rassias stability of the additive Cauchy equation $g(x+y) = g(x)+g(y)$. Indeed, Hyers [5] obtained the result for $p = 0$. Then Rassias [8] generalized the above result of Hyers to the case where $0 \leq p < 1$. Gajda [4] solved the problem for $1 < p$, which was raised by Rassias; In the same paper, Gajda also gave an example that a similar result to the above does not hold for $p = 1$ (cf. [9]). If $p < 0$, then $\|x\|^p$ is meaningless for $x = 0$; In this case, if we assume that $\|0\|^p$ means ∞, then the proof given in [8] also works for $x \neq 0$. Moreover, with minor changes in the proof, the result is also valid for $p < 0$. Thus, the Hyers-Ulam-Rassias stability of the additive Cauchy equation holds for $p \in \mathbb{R} \setminus \{1\}$. Here and after, the letter $\mathbb{R}$ denotes the real number field and $\mathbb{C}$ stands for the complex number field.

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Suppose A and B are two Banach algebras. We say that a mapping $\tau : A \to B$ is a Jordan homomorphism if

$$\begin{gathered}\tau(a+b) = \tau(a) + \tau(b) \quad (a, b \in A), \\ \tau(a^2) = \tau(a)^2 \quad (a \in A).\end{gathered} \tag{1.2}$$

If, in addition, τ is multiplicative, that is

$$\tau(ab) = \tau(a)\tau(b) \quad (a, b \in A), \tag{1.3}$$

we say that τ is a ring homomorphism. The study of ring homomorphisms between Banach algebras A and B is of interest even if $A = B = \mathbb{C}$. For example, the zero mapping, the identity and the complex conjugate are ring homomorphisms on $\mathbb{C}$, which are all continuous. On the other hand, the existence of a discontinuous ring homomorphism on $\mathbb{C}$ is well-known (cf. [6]). More explicitly, if G is the set of all surjective ring homomorphisms on $\mathbb{C}$, then $\sharp G = 2^{\sharp \mathbb{C}}$, where $\sharp S$ denotes the cardinal number of a set S. In fact, Charnow [3, Theorem 3] proved that there exist $2^{\sharp k}$ automorphisms for every algebraically closed field k; It is also known that if $\mathscr{A}$ is a uniform algebra on a compact metric space, then there are exactly $2^{\sharp \mathbb{C}}$ complex-valued ring homomorphisms on $\mathscr{A}$ whose kernels are non-maximal prime ideals (see [7, Corollary 2.4]).

By definition, it is obvious that ring homomorphisms are Jordan homomorphisms. Conversely, under a certain condition, Jordan homomorphisms are ring homomorphisms. For example, each Jordan homomorphism τ from a commutative Banach algebra $\mathscr{B}$ into $\mathbb{C}$ is a ring homomorphism: Fix $a, b \in \mathscr{B}$ arbitrarily. Since $\tau((a+b)^2) = \tau(a+b)^2$, a simple calculation shows that $\tau(ab+ba) = 2\tau(a)\tau(b)$. The commutativity of $\mathscr{B}$ implies $\tau(ab) = \tau(a)\tau(b)$, and hence τ is a ring homomorphism. This simple example leads us to the following general result.

Theorem 1.1. *Suppose A is a Banach algebra, which need not be commutative, and suppose B is a semisimple commutative Banach algebra. If $\tau : A \to B$ is a Jordan homomorphism, then $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in A$, that is, τ is a ring homomorphism.*

Next, we consider the stability, in the sense of Hyers-Ulam-Rassias, of Jordan homomorphisms. Bourgin [2] proved the following stability result of ring homomorphisms between two unital Banach algebras.

Theorem 1.2. *Suppose A and B are unital Banach algebras. If $f : A \to B$ is a surjective mapping such that*

$$\begin{gathered}\|f(a+b) - f(a) - f(b)\| \le \varepsilon \quad (a, b \in A), \\ \|f(ab) - f(a)f(b)\| \le \delta \quad (a, b \in A)\end{gathered} \tag{1.4}$$

for some $\varepsilon \ge 0$ and $\delta \ge 0$, then f is a ring homomorphism.

Applying a theorem of Hyers [5], Rassias [8] and Gajda [4], Badora [1] proved the Hyers-Ulam-Rassias stability of ring homomorphisms, which generalizes the above result of Bourgin. We will prove the Hyers-Ulam-Rassias stability of Jordan homomorphisms.

We emphasize that the introduction of the signum $s = |1-p|/(1-p)$ made it possible to give a simple proof of our stability results.

Theorem 1.3. *Suppose A and B are Banach algebras. If $f : A \to B$ is a mapping such that*

$$||f(a+b) - f(a) - f(b)|| \leq \delta(||a||^p + ||b||^p) \quad (a,b \in A), \tag{1.5}$$

$$||f(a^2) - f(a)^2|| \leq \delta\|a\|^{2p} \quad (a \in A) \tag{1.6}$$

for some $\delta \geq 0$ and $p \geq 0, p \neq 1$, then there is a unique Jordan homomorphism $\tau : A \to B$ such that

$$||f(a) - \tau(a)|| \leq \frac{2\delta}{|2-2^p|}\|a\|^p \quad (a \in A). \tag{1.7}$$

For $p < 0$, we can also give a similar result to Theorem 1.3, under an additional condition that $f(0) = 0$. The hypothesis $f(0) = 0$ seems to be natural. It follows from (1.5) that $f(0) = 0$ whenever $p > 0$; On the other hand, if $p < 0$ then the inequalities (1.5) and (1.6) give no information for $f(0)$.

Theorem 1.4. *Suppose A and B are Banach algebras. If $f : A \to B$ is a mapping, with $f(0) = 0$, such that the inequalities (1.5) and (1.6) are valid for some $\delta \geq 0$ and $p < 0$, then there is a unique Jordan homomorphism $\tau : A \to B$ such that*

$$||f(a) - \tau(a)|| \leq \frac{2\delta}{|2-2^p|}\|a\|^p \quad (a \in A). \tag{1.8}$$

As an easy corollary to Theorems 1.1, 1.3, and 1.4, we obtain the following stability result.

Corollary 1.5. *Suppose A is a Banach algebra and suppose B is a semisimple commutative Banach algebra. If $f : A \to B$ is a mapping such that*

$$\begin{gathered} ||f(a+b) - f(a) - f(b)|| \leq \delta(\|a\|^p + \|b\|^p) \quad (a,b \in A), \\ ||f(a^2) - f(a)^2|| \leq \delta\|a\|^{2p} \quad (a \in A) \end{gathered} \tag{1.9}$$

for some $\delta \geq 0$ and $p \in \mathbb{R}$. If $p \geq 0$ and $p \neq 1$, or $p < 0$ and $f(0) = 0$, then there is a unique ring homomorphism $\tau : A \to B$ such that

$$||f(a) - \tau(a)|| \leq \frac{2\delta}{|2-2^p|}\|a\|^p \quad (a \in A). \tag{1.10}$$

2. Proof of results

Before we turn to the proof of Theorem 1.1, we need the following lemma. It should be mentioned that the following proof is just a slight modification of [13, Proof of Theorem 1] by Żelazko.

Lemma 2.1. *Suppose A is a Banach algebra, which need not be commutative. Then each Jordan homomorphism $\phi : A \to \mathbb{C}$ is a ring homomorphism.*

Proof. Recall that ϕ is an additive mapping such that $\phi(a^2) = \phi(a)^2$ for all $a \in A$. Replacement of a by $x + y$ results in

$$\phi(xy + yx) = 2\phi(x)\phi(y) \quad (x \in A,\ y \in A). \tag{2.1}$$

Then (2.1), with $x = x^2$, implies

$$\phi(x^2y + yx^2) = 2\phi(x)^2\phi(y). \tag{2.2}$$

Taking $y = xy + yx$ in (2.1), we see that

$$\phi\big(x(xy + yx) + (xy + yx)x\big) = 2\phi(x)\phi(xy + yx), \tag{2.3}$$

and hence, by (2.1)

$$\phi(x^2y + 2xyx + yx^2) = 4\phi(x)^2\phi(y) \quad (x \in A,\ y \in A). \tag{2.4}$$

Subtraction (2.4) from (2.2) gives

$$\phi(xyx) = \phi(x)^2\phi(y) \quad \text{if } x \in A,\ y \in A. \tag{2.5}$$

Fix $a \in A$ and $b \in A$ arbitrarily, and put

$$2t = \phi(ab - ba). \tag{2.6}$$

It follows from (2.1) and (2.6) that

$$\phi(ab) = \phi(a)\phi(b) + t, \qquad \phi(ba) = \phi(a)\phi(b) - t. \tag{2.7}$$

By (2.5), (2.6), (2.7),

$$\begin{aligned} 4t^2 &= \phi\big((ab - ba)^2\big) \\ &= \phi(ab)^2 - \phi(ab^2a) - \phi(ba^2b) + \phi(ba)^2 \\ &= \{\phi(a)\phi(b) + t\}^2 - 2\phi(a)^2\phi(b)^2 + \{\phi(a)\phi(b) - t\}^2 \\ &= 2t^2; \end{aligned} \tag{2.8}$$

hence $t = 0$, which proves $\phi(ab) = \phi(ba)$. It follows from (2.1) that $\phi(ab) = \phi(a)\phi(b)$, and the proof is complete. □

Proof of Theorem 1.1. We show that τ is multiplicative. Let M_B be the maximal ideal space of B. We associate to each $\varphi \in M_B$ a function $\tau_\varphi : A \to \mathbb{C}$ defined by

$$\tau_\varphi(a) = \varphi\big(\tau(a)\big) \quad (a \in A). \tag{2.9}$$

Pick $\varphi \in M_B$ arbitrarily. We see that $\tau_\varphi(a^2) = \tau_\varphi(a)^2$ for all $a \in A$, and so Lemma 2.1, applied to τ_φ, implies that τ_φ is multiplicative. By the definition of τ_φ, we get $\varphi(\tau(ab)) = \varphi(\tau(a)\tau(b))$ for all $a, b \in A$. Since $\varphi \in M_B$ was arbitrary and since B is assumed to be semisimple, we obtain $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in A$. We thus conclude that τ is a ring homomorphism, and the proof is complete. □

Proof of Theorem 1.3. It follows from [8] and [4] (cf. [5]) that there is an additive mapping $\tau : A \to B$ such that

$$\|f(a)-\tau(a)\| \le \frac{2\delta}{|2-2^p|}\|a\|^p \quad (a\in A). \tag{2.10}$$

We first show that $\tau(a^2)=\tau(a)^2$ for all $a\in A$. Pick $a\in A$ arbitrarily, and put $s=|1-p|/(1-p)$. Note that $s=1$ if $0\le p<1$ and that $s=-1$ if $p>1$. Since τ is additive, it follows from (2.10) that

$$\begin{aligned}\|n^{-2s}f(n^{2s}a^2)-\tau(a^2)\| &= \|n^{-2s}f(n^{2s}a^2)-n^{-2s}\tau(n^{2s}a^2)\|\\ &\le n^{-2s}\frac{2\delta}{|2-2^p|}\|n^{2s}a^2\|^p\end{aligned} \tag{2.11}$$

for all $n\in\mathbb{N}$, and hence

$$\|n^{-2s}f(n^{2s}a^2)-\tau(a^2)\| \le n^{2s(p-1)}\frac{2\delta}{|2-2^p|}\|a^2\|^p \tag{2.12}$$

for all $n\in\mathbb{N}$. A similar argument to the above shows for each $n\in\mathbb{N}$ that

$$\|n^{-s}f(n^{s}a)-\tau(a)\| \le n^{s(p-1)}\frac{2\delta}{|2-2^p|}\|a\|^p. \tag{2.13}$$

Since $s(p-1)<0$, it follows from (2.12) and (2.13) that

$$\tau(a^2)=\lim_{n\to\infty}n^{-2s}f(n^{2s}a^2), \qquad \tau(a)=\lim_{n\to\infty}n^{-s}f(n^{s}a). \tag{2.14}$$

By (1.6), we get $\|f(n^{2s}a^2)-f(n^sa)^2\|\le\delta\|n^sa\|^{2p}$ for all $n\in\mathbb{N}$. So,

$$\lim_{n\to\infty}n^{-2s}\left(f(n^{2s}a^2)-f(n^sa)^2\right)\le\lim_{n\to\infty}n^{2s(p-1)}\delta\|a\|^{2p}=0, \tag{2.15}$$

since $s(p-1)<0$. Now it follows from (2.14) and (2.15) that

$$\begin{aligned}\tau(a^2) &= \lim_{n\to\infty}n^{-2s}f(n^{2s}a^2)\\ &= \lim_{n\to\infty}\left\{n^{-2s}f(n^{2s}a^2)-n^{-2s}\left(f(n^{2s}a^2)-f(n^sa)^2\right)\right\}\\ &= \left\{\lim_{n\to\infty}n^{-s}f(n^sa)\right\}^2=\tau(a)^2.\end{aligned} \tag{2.16}$$

Since $a\in A$ was arbitrary, we obtain $\tau(a^2)=\tau(a)^2$ for all $a\in A$, and hence τ is a Jordan homomorphism.

Finally, suppose that $\tau^*: A\to B$ is another Jordan homomorphism such that $\|f(a)-\tau^*(a)\|\le 2\delta\|a\|^p/|2-2^p|$ for all $a\in A$. Then (2.13), with $\tau=\tau^*$, is also valid. We thus obtain

$$\begin{aligned}\|\tau(a)-\tau^*(a)\| &\le \|\tau(a)-n^{-s}f(n^sa)\|+\|n^{-s}f(n^sa)-\tau^*(a)\|\\ &\le n^{s(p-1)}\frac{4\delta}{|2-2^p|}\|a\|^p\end{aligned} \tag{2.17}$$

for all $a \in A$ and $n \in \mathbb{N}$. Since $s(p-1) < 0$, it follows that $\tau = \tau^*$, and hence the uniqueness have been proved. □

Proof of Theorem 1.4. It follows from [8] that there exists an additive mapping $\tau : A \to B$ such that

$$\|f(a) - \tau(a)\| \leq \frac{2\delta}{|2-2^p|}\|a\|^p \quad (a \in A), \tag{2.18}$$

where we assume $\|0\|^p = \infty$. It suffices to show that $\tau(a^2) = \tau(a)^2$ for all $a \in A$. Since τ is additive, we obtain $\tau(0) = 0$, and so the case $a = 0$ is omitted. Pick $a \in A \setminus \{0\}$ arbitrarily. There are now two possibilities. Either $a^2 = 0$ or $a^2 \neq 0$, in which case the proof of Theorem 1.3 works well, and so $\tau(a^2) = \tau(a)^2$. Thus we need consider only the case $a^2 = 0$ (In this case, we cannot apply the proof of Theorem 1.3. In fact, if $a^2 = 0$, then $\|a^2\|^p = \infty$ and hence (2.13), with $a = a^2$, is meaningless). We will show that $\tau(a)^2 = 0$ whenever $a^2 = 0$.

Pick $a \in A \setminus \{0\}$ such that $a^2 = 0$. It follows from (1.6), with the hypothesis $f(0) = 0$, that

$$\|n^{-2}f(na)^2\| \leq n^{-2}\delta\|na\|^{2p} = n^{2(p-1)}\delta\|a\|^{2p}. \tag{2.19}$$

Since $a \neq 0$ and since $p - 1 < 0$, we obtain

$$\lim_{n\to\infty} n^{-2}f(na)^2 = 0. \tag{2.20}$$

Note also that

$$\|n^{-1}f(na) - \tau(a)\| \leq n^{-1}\frac{2\delta}{|2-2^p|}\|na\|^p = n^{p-1}\frac{2\delta}{|2-2^p|}\|a\|^p \tag{2.21}$$

for all $n \in \mathbb{N}$, and hence

$$\tau(a) = \lim_{n\to\infty} n^{-1}f(na). \tag{2.22}$$

It follows from (2.20) and (2.22) that

$$\tau(a)^2 = \lim_{n\to\infty} n^{-2}f(na)^2 = 0, \tag{2.23}$$

which proves $\tau(a^2) = 0 = \tau(a)^2$ whenever $a^2 = 0$. This completes the proof. □

In this paper, we have proved the Hyers-Ulam-Rassias stability of Jordan homomorphisms for $p \in \mathbb{R} \setminus \{1\}$. On the other hand, Šemrl [10] gave an example that the stability result fails for $p = 1$: In fact, to each $\delta > 0$ there corresponds a multiplicative continuous function $f : \mathbb{C} \to \mathbb{C}$ satisfying $f(ia) = if(a)$ for all $a \in \mathbb{C}$ such that

$$|f(a+b) - f(a) - f(b)| \leq \delta(|a| + |b|) \quad (a, b \in \mathbb{C}) \tag{2.24}$$

and that

$$\sup_{a\in\mathbb{C}\setminus\{0\}} \frac{|f(a)-\tau(a)|}{|a|} \geq 1 \tag{2.25}$$

for all ring homomorphism $\tau : \mathbb{C} \to \mathbb{C}$.

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