THE DIRICHLET PROBLEM FOR A CLASS OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS

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We study the first boundary value problem for a class of nonlinear degenerate parabolic equations $-\partial u/\partial t = \operatorname{div}(\vec{A}(\nabla u))$. We first consider its regularized problem and establish some estimates. Based on these estimates, we prove the existence and uniqueness of the generalized solutions in BV space.

1. Introduction

Let $\Omega \subset R^m$ $(m \geq 1)$ be a bounded set with smooth boundary $\partial\Omega$. We are concerned with the Dirichlet problem

$$\begin{gathered} -\frac{\partial u}{\partial t} = \operatorname{div}\big(\vec{A}(\nabla u)\big) \quad (x,t) \in Q_T = \Omega \times (0,T), \\ u(x,t) = 0 \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \end{gathered} \tag{1.1}$$

where $\vec{A}(p) = (A^1(p),\dots,A^m(p)) \in C^1(R^m,R^m)$, $u_0(x)$ is appropriately smooth on $\overline{\Omega}$ and certain compatibility conditions on the boundary of the lower base of Q_T are fulfilled.

We suppose that

$$0 \leq \frac{\partial A^i(p)}{\partial p_j}\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall \xi \in R^m, \tag{1.2}$$

$$\mu_1|p|^q \leq \vec{A}(p)\cdot p, \quad |\vec{A}(p)| \leq \mu_2\big(|p|^{q-1}+1\big), \quad \forall p \in R^m, \tag{1.3}$$

where $q \geq 2$, Λ, μ_1, μ_2 are positive numbers.

Under some conditions, Gregori [1] considered the elliptic problem

$$\begin{gathered} -\operatorname{div}\big(\vec{A}(\nabla u)\big) = 0 \quad x \in \Omega, \\ u|_{\partial\Omega} = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.4}$$

and proved the existence and the uniqueness of BV solutions. In this note, we generalize the results of [1] to the parabolic case. The Dirichlet problem (1.1) arises from a variety

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of diffusion phenomena which appear widely in nature. The non-Newtonian filtration equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad p \neq 2 \tag{1.5}$$

is a special case of problem (1.1). Problem (1.1) has been widely investigated, for example, see [2, 4, 5, 6] and references therein. For the one-dimensional case, Wu et al. [5] considered the Dirichlet problem $-u_t = (\partial/\partial x)(A((\partial/\partial x)B(u))) + \partial f(u)/\partial x$, and proved the existence and uniqueness of the generalized solutions in BV space under some constrains. Our interest here is to treat the problem for a multi-dimensional case without absorption. Generally speaking, solutions of problem (1.1) are not continuous. The sense of satisfying the boundary value conditions for solutions is also special (see [3]). In present paper, we take some ideas from [6] and investigate the solvability in $BV(Q_T)$, where BV is the class of all integrable functions on Q_T, whose generalized derivatives are measures with bounded variation. The existence of solutions will be proved by means of the method of parabolic regularization.

2. Main results

Definition 2.1. A function $u \in BV(Q_T)\bigcap L^\infty(Q_T)$ is said to be a generalized solution of problem (1.1), if the following conditions are fulfilled:

(1) $u_t \in L^\infty(0,T;L^2(\Omega))$, $u_{x_i} \in L^q(Q_T)$, $i = 1,2,\dots,m$.

(2) For almost all $x \in \Omega$, $\gamma u(x,0) = u_0(x)$, where γu is the trace of u.

(3) For almost all $t \in (0,T)$, $\gamma u(x,t) = 0$ a.e. on $\partial\Omega$.

(4) u satisfies

$$\begin{aligned}&\iint_{Q_T} \operatorname{sgn}(u-k)\left\{(u-k)\frac{\partial \varphi_1}{\partial t} - \vec{A}(\nabla u)\cdot\nabla\varphi_1\right\}dx\,dt\\ &\quad + \iint_{Q_T} \operatorname{sgn} k\left\{u\frac{\partial \varphi_2}{\partial t} - \vec{A}(\nabla u)\cdot\nabla\varphi_2\right\}dx\,dt \geq 0,\end{aligned} \tag{2.1}$$

where $\varphi_1,\varphi_2 \in C^1(\overline{Q_T})$, $\varphi_1,\varphi_2 \geq 0$, $\varphi_1 = \varphi_2$ on $\partial\Omega\times(0,T)$, $\operatorname{supp}\varphi_1,\operatorname{supp}\varphi_2 \subset \overline{\Omega}\times(0,T)$ and $k \in R$.

Remark 2.2. If $u \in BV(Q_T)\bigcap L^\infty(Q_T)$ satisfies conditions (1) and (4) in Definition 2.1, then

(4′) u satisfies

$$\iint_{Q_T} \operatorname{sgn}(u-k)\left\{(u-k)\frac{\partial \varphi_1}{\partial t} - \vec{A}(\nabla u)\cdot\nabla\varphi_1\right\}dx\,dt \geq 0, \tag{2.2}$$

for $\forall\varphi \in C^1(\overline{Q_T})$, $\varphi \geq 0$ and $k \in R$.

Our main results are the following.

Theorem 2.3. *Assume that (1.2) and (1.3) hold. Then problem (1.1) admits at least one solution $u \in BV(Q_T)\bigcap L^\infty(Q_T)$.*

THEOREM 2.4. *Suppose that u_1, u_2 are solutions of problem (1.1) satisfying*

$$u_1 = u_2 \quad on\ \partial\Omega \times (0,T), \qquad u_{01} \le u_{02} \quad on\ \overline{\Omega}, \qquad \lim_{t\to 0} \|u_i - u_{0i}\|_{L^1(\Omega)} = 0, \quad i = 1,2. \tag{2.3}$$

Then $u_1 \le u_2$ in Q_T.

Remark 2.5. Theorem 2.4 implies the uniqueness of solutions of problem (1.1).

3. Proof of Theorems 2.3 and 2.4

To prove the existence of solutions of problem (1.1), we consider the following regularized problem:

$$\begin{gathered} -\frac{\partial u}{\partial t} = \operatorname{div}(\vec{A}(\nabla u)) + \epsilon\Delta u \quad (x,t) \in Q_T = \Omega \times (0,T), \\ u(x,t) = 0 \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x). \end{gathered} \tag{3.1}$$

Under the assumptions of Theorem 2.3, by the classical parabolic theory, problem (3.1) has a unique solution $u_\epsilon \in C^3(Q_T) \bigcap C^2(\overline{Q_T})$ and

$$\sup_{t\in(0,T)} |u_\epsilon(x,t)| \le M, \tag{3.2}$$

where M is a positive constant independent of ϵ.

LEMMA 3.1. *Under the assumptions of Theorem 2.3, the following estimates for the solution u_ϵ hold.*

$$\sup_{t\in(0,T)} \int_\Omega \left|\frac{\partial u_\epsilon}{\partial t}\right|^2 dx \le C, \tag{3.3}$$

$$\iint_{Q_T} |\nabla u_\epsilon|^q \,dx\,dt \le C, \tag{3.4}$$

$$\iint_{Q_T} \vec{A}(\nabla u_\epsilon)\cdot\nabla u_\epsilon \,dx\,dt + \epsilon\iint_{Q_T} |\nabla u_\epsilon|^2 \,dx\,dt \le C. \tag{3.5}$$

Proof. Differentiate (3.1) with respect to t, multiply the resulting relation by $\partial u_\epsilon/\partial t$ and integrate over $Q_t = \Omega \times (0,t)$, we derive that

$$\begin{aligned} \frac{1}{2}\int_\Omega \left|\frac{\partial u_\epsilon}{\partial t}\right|^2 dx = &-\iint_{Q_T} \frac{\partial A^i(\nabla u_\epsilon)}{\partial p_j}\frac{\partial^2 u_\epsilon}{\partial t\partial x_i}\frac{\partial^2 u_\epsilon}{\partial t\partial x_j}\,dx\,dt \\ &-\epsilon\sum_{i=1}^m \iint_{Q_t} \left|\frac{\partial^2 u_\epsilon}{\partial t\partial x_i}\right|^2 dx\,dt + \frac{1}{2}\int_\Omega \left|\frac{\partial u_\epsilon}{\partial t}\right|^2_{t=0} dx, \end{aligned} \tag{3.6}$$

which, together with (1.2) yields the desired estimate (3.3).

Multiplying (3.1) by u_ϵ and integrating over Q_T, we get

$$\iint_{Q_T} \vec{A}(\nabla u_\epsilon)\cdot\nabla u_\epsilon\,dx\,dt+\epsilon\iint_{Q_T}|\nabla u_\epsilon|^2\,dx\,dt=-\frac{1}{2}\int_\Omega {u_\epsilon}^2(x,T)\,dx\,dt+\frac{1}{2}\int_\Omega u_0^2(x)\,dx. \tag{3.7}$$

This, together with (1.3) implies estimates (3.4) and (3.5). The proof of Lemma 3.1 is complete. □

Proof of Theorem 2.3. By (3.2) and Lemma 3.1, there exists a subsequence of $\{u_\epsilon\}$, still denoted by u_ϵ and a function $u\in BV(Q_T)\bigcap L^\infty(Q_T)$ with $u_t\in L^\infty(0,T;L^2(\Omega))$, $|\nabla u|\in L^q(Q_T)$ such that

$$\begin{gathered} u_\epsilon \longrightarrow u \quad \text{a.e. on } Q_T,\\ |\nabla u_\epsilon| \rightharpoonup |\nabla u| \quad \text{weakly in } L^q(Q_T),\\ \vec{A}(\nabla u_\epsilon)\rightharpoonup w=(w_1,w_2,\dots,w_m) \quad \text{weakly in } L^{q/q-1}(Q_T,R^m),\\ \frac{\partial u_\epsilon}{\partial t}\rightharpoonup\frac{\partial u}{\partial t} \quad \text{weakly in } L^\infty(0,T;L^2(\Omega)). \end{gathered} \tag{3.8}$$

We now prove $w=\vec{A}(\nabla u)$. Multiplying (3.1) by $u_\epsilon-u$ and integrating by parts over Q_T, we get

$$\begin{aligned} \iint_{Q_T}\vec{A}(\nabla u_\epsilon)(\nabla u_\epsilon-\nabla u)\,dx\,dt=&-\iint_{Q_T}u_\epsilon\left(\frac{\partial u_\epsilon}{\partial t}-\frac{\partial u}{\partial t}\right)dx\,dt\\ &-\epsilon\iint_{Q_T}\nabla u_\epsilon(\nabla u_\epsilon-\nabla u)\,dx\,dt. \end{aligned} \tag{3.9}$$

On the other hand,

$$\begin{aligned} &\lim_{\epsilon\to0}\iint_{Q_T}u_\epsilon\left(\frac{\partial u_\epsilon}{\partial t}-\frac{\partial u}{\partial t}\right)dx\,dt=0,\\ &\lim_{\epsilon\to0}\epsilon\left|\iint_{Q_T}\nabla u_\epsilon(\nabla u_\epsilon-\nabla u)\,dx\,dt\right|\le\lim_{\epsilon\to0}\epsilon\iint_{Q_T}|\nabla u_\epsilon|^2\,dx\,dt\\ &\qquad+\lim_{\epsilon\to0}\epsilon\left(\iint_{Q_T}|\nabla u_\epsilon|^2\,dx\,dt\right)^{1/2}\left(\iint_{Q_T}|\nabla u|^2\,dx\,dt\right)^{1/2}=0. \end{aligned} \tag{3.10}$$

Thus

$$\iint_{Q_T}\vec{A}(\nabla u_\epsilon)(\nabla u_\epsilon-\nabla u)\,dx\,dt=0. \tag{3.11}$$

Note that

$$\lim_{\epsilon\to0}\iint_{Q_T}\vec{A}(\nabla u)(\nabla u_\epsilon-\nabla u)\,dx\,dt=0. \tag{3.12}$$

By (3.11), we infer that

$$\lim_{\epsilon\to 0}\iint_{Q_T}(\vec{A}(\nabla u_\epsilon)-\vec{A}(\nabla u))(\nabla u_\epsilon-\nabla u)\,dx\,dt=0. \tag{3.13}$$

Set $a^{ij}=\int_0^1(A^i(p)/p_j)d\lambda$, $p=\lambda\nabla u_\epsilon+(1-\lambda)\nabla u$, then (3.13) can be rewritten as

$$\lim_{\epsilon\to 0}\iint_{Q_T}a^{ij}\frac{\partial}{\partial x_i}(u_\epsilon-u)\frac{\partial}{\partial x_j}(u_\epsilon-u)\,dx\,dt=0. \tag{3.14}$$

By Hölder inequality and (3.14), for $\forall\vec{\varphi}=(\varphi_1,\varphi_2,\dots,\varphi_m)\in C_0^1(Q_T,R^m)$, we obtain

$$\begin{aligned}
&\left|\iint_{Q_T}(\vec{A}(\nabla u_\epsilon)-\vec{A}(\nabla u))\cdot\vec{\varphi}\,dx\,dt\right|\\
&\quad=\left|\iint_{Q_T}a^{ij}\frac{\partial}{\partial x_j}(u_\epsilon-u)\varphi_i\,dx\,dt\right|\\
&\quad\le\left(\iint_{Q_T}a^{ij}\frac{\partial}{\partial x_i}(u_\epsilon-u)\frac{\partial}{\partial x_j}(u_\epsilon-u)\,dx\,dt\right)^{1/2}\\
&\qquad\times\left(\iint_{Q_T}a^{ij}\varphi_i\varphi_j\,dx\,dt\right)^{1/2}\longrightarrow 0\quad\text{as }\epsilon\longrightarrow 0.
\end{aligned} \tag{3.15}$$

Thus

$$\iint_{Q_T}(w-\vec{A}(\nabla u))\cdot\vec{\varphi}\,dx\,dt=0, \tag{3.16}$$

which implies that $w=\vec{A}(\nabla u)$ a.e. on Q_T.

Now let $\varphi_1\in C^1(\overline{Q_T})$, $\varphi_1\ge 0$, $\operatorname{supp}\varphi_1\subset\overline{\Omega}\times(0,T)$. Multiplying (3.1) by $\varphi_1\operatorname{sgn}_\eta(u_\epsilon-k)$, $k\in R$ and integrating by parts over Q_T, we obtain

$$\begin{aligned}
&\iint_{Q_T}I_\eta(u_\epsilon-k)\frac{\partial\varphi_1}{\partial t}\,dx\,dt-\iint_{Q_T}\vec{A}(\nabla u_\epsilon)\cdot\nabla\varphi_1\cdot\operatorname{sgn}_\eta(u_\epsilon-k)\,dx\,dt\\
&\quad-\iint_{Q_T}\vec{A}(\nabla u_\epsilon)\cdot\nabla u_\epsilon\cdot\operatorname{sgn}'_\eta(u_\epsilon-k)\varphi_1\,dx\,dt\\
&\quad-\epsilon\iint_{Q_T}|\nabla u_\epsilon|^2\operatorname{sgn}'_\eta(u_\epsilon-k)\varphi_1\,dx\,dt\\
&\quad-\epsilon\iint_{Q_T}\nabla u_\epsilon\cdot\nabla\varphi_1\cdot\operatorname{sgn}_\eta(u_\epsilon-k)\,dx\,dt\\
&\quad+\sum_{i=1}^m\int_0^T\int_{\partial\Omega}\left(A^i(\nabla u_\epsilon)+\frac{\partial u_\epsilon}{\partial x_i}\right)\operatorname{sgn}_\eta(u_\epsilon-k)\varphi_1 n_i\,d\sigma\,dt=0,
\end{aligned} \tag{3.17}$$

where

$$\operatorname{sgn}_\eta\tau=\begin{cases}1 & \text{if }\tau\ge\eta,\\ \dfrac{\tau}{\eta} & \text{if }|\tau|<\eta,\\ -1 & \text{if }\tau\le-\eta,\end{cases}\qquad I_\eta(s)=\int_0^s\operatorname{sgn}_\eta\tau\,d\tau,\quad \eta>0. \tag{3.18}$$

Note that the third term and the fourth term are nonnegative, let $\eta \to 0$ in (3.17), we get

$$\begin{aligned} &\iint_{Q_T} \operatorname{sgn}_\eta (u_\epsilon - k)\left\{(u_\epsilon - k)\frac{\partial \varphi_1}{\partial t} - \vec{A}(\nabla u_\epsilon)\cdot\nabla\varphi_1 - \epsilon\nabla u_\epsilon\cdot\nabla\varphi_1\right\}dx\,dt \\ &\qquad - \operatorname{sgn} k\sum_{i=1}^m \int_0^T\int_{\partial\Omega}\left(A^i(\nabla u_\epsilon) + \frac{\partial u_\epsilon}{\partial x_i}\right)\varphi_1 n_i\,d\sigma dt = 0. \end{aligned} \tag{3.19}$$

Take $\varphi_2 \in C^1(\overline{Q_T})$, $\varphi_2 \geq 0$, $\operatorname{supp}\varphi_2 \subset \overline{\Omega}\times(0,T)$, $\varphi_1 = \varphi_2$ on $\partial\Omega\times(0,T)$, and from (3.1), we get

$$\begin{aligned} &\iint_{Q_T} \left\{(u_\epsilon - k)\frac{\partial \varphi_2}{\partial t} - \vec{A}(\nabla u_\epsilon)\cdot\nabla\varphi_2 - \epsilon\nabla u_\epsilon\cdot\nabla\varphi_2\right\}dx\,dt \\ &\qquad + \sum_{i=1}^m \int_0^T\int_{\partial\Omega}\left(A^i(\nabla u_\epsilon) + \frac{\partial u_\epsilon}{\partial x_i}\right)\varphi_1 n_i\,d\sigma dt = 0. \end{aligned} \tag{3.20}$$

Combining (3.19) with (3.20), we get

$$\begin{aligned} J(u_\epsilon,k,\varphi_1,\varphi_2) = &\iint_{Q_T} \operatorname{sgn}_\eta (u_\epsilon - k)\left\{(u_\epsilon - k)\frac{\partial \varphi_1}{\partial t} - \vec{A}(\nabla u_\epsilon)\cdot\nabla\varphi_1\right\}dx\,dt \\ &+ \iint_{Q_T} \operatorname{sgn} k\left\{(u_\epsilon - k)\frac{\partial \varphi_2}{\partial t} - \vec{A}(\nabla u_\epsilon)\cdot\nabla\varphi_2\right\}dx\,dt \\ &- \epsilon\iint_{Q_T} \operatorname{sgn}_\eta (u_\epsilon - k)\cdot\nabla u_\epsilon\cdot\nabla\varphi_1\,dx\,dt \\ &- \epsilon\iint_{Q_T} \operatorname{sgn} k\cdot\nabla u_\epsilon\cdot\nabla\varphi_2\,dx\,dt \geq 0. \end{aligned} \tag{3.21}$$

By (3.4), the last two terms in (3.21) tend to zero as $\epsilon \to 0$. Let $\epsilon \to 0$ in (3.21), we easily get (2.1). By (3.2) and (3.3), we derive that

$$\gamma u(x,0) = u_0(x) \quad \text{a.e. on } \Omega. \tag{3.22}$$

We now prove $u(x,t)|_{\partial\Omega} = 0$ a.e. on $(0,T)$.

Since $u_{x_i} \in L^q(Q_T)$, $i = 1,2,\dots,m$, we have for $\forall\varphi \in C^1(\overline{Q_T})$

$$\begin{aligned} \iint_{Q_T}\varphi\cdot u_{x_i}\,dx\,dt &= \lim_{\epsilon\to 0}\iint_{Q_T}\varphi\cdot(u_\epsilon)_{x_i}\,dx\,dt = \lim_{\epsilon\to 0}\iint_{Q_T}\varphi_{x_i}\cdot u_\epsilon\,dx\,dt \\ &= -\iint_{Q_T}\varphi_{x_i}\cdot u\,dx\,dt = -\int_0^T\int_{\partial\Omega}\varphi\cdot\gamma u\cdot n_i\,d\sigma dt + \iint_{Q_T}\varphi\cdot u_{x_i}\,dx\,dt, \quad i = 1,2,\dots,m. \end{aligned} \tag{3.23}$$

Thus

$$\int_0^T\int_{\partial\Omega}\varphi\cdot\gamma u\cdot n_i\,d\sigma dt = 0 \quad \forall\varphi \in C^1(\overline{Q_T}),\ i = 1,2,\dots,m, \tag{3.24}$$

which implies $\gamma u = 0$ a.e. on $\partial\Omega\times(0,T)$. The proof of Theorem 2.3 is complete. □

Proof of Theorem 2.4. Take $k > |u|_{L^\infty}$ and $k < -|u|_{L^\infty}$ in (2.1)′ respectively, we get

$$\iint_{Q_T} \left(u\frac{\partial\varphi}{\partial t} - \vec{A}(\nabla u)\cdot\nabla\varphi\right)dx\,dt, \quad \forall\varphi\in C_0^1(\overline{Q_T}). \tag{3.25}$$

By approximating, we may take $\varphi = ((u_1-u_2)_+)/((u_1-u_2)_+ + \epsilon)$ in (3.25) to get

$$\begin{aligned}&\int_{s_1}^{s_2}\int_\Omega \frac{(u_1-u_2)_+}{(u_1-u_2)_+ + \epsilon}(u_1-u_2)_t\,dx\,dt\\ &\qquad + \epsilon\int_{s_1}^{s_2}\int_\Omega (\vec{A}(\nabla u_1) - \vec{A}(\nabla u_2))\cdot\frac{\nabla(u_1-u_2)}{((u_1-u_2)_+ + \epsilon)^2}\,dx\,dt = 0,\end{aligned} \tag{3.26}$$

where $(u_1-u_2)_t$ means measure, $0 < s_1 < s_2 \le T$.

Since

$$\begin{aligned}&\int_{s_1}^{s_2}\int_\Omega \frac{(u_1-u_2)_+}{(u_1-u_2)_+ + \epsilon}(u_1-u_2)_t\,dx\,dt\\ &\quad = \int_\Omega (u_1(x,s_2) - u_2(x,s_2))_+\,dx - \int_\Omega (u_1(x,s_1) - u_2(x,s_1))_+\,dx\\ &\qquad - \epsilon\int_{s_1}^{s_2}\int_\Omega \frac{(u_1-u_2)_+}{((u_1-u_2)_+ + \epsilon)^2}(u_1-u_2)_t\,dx\,dt,\\ &\lim_{\epsilon\to 0}\epsilon\int_{s_1}^{s_2}\int_\Omega \frac{(u_1-u_2)_+}{((u_1-u_2)_+ + \epsilon)^2}(u_1-u_2)_t\,dx\,dt = 0.\end{aligned} \tag{3.27}$$

Note that the second term of the left side of (3.26) is nonnegative. Thus, let $\epsilon \to 0$ in (3.26), we obtain

$$\int_\Omega (u_1(x,s_2) - u_2(x,s_2))_+\,dx \le \int_\Omega (u_1(x,s_1) - u_2(x,s_1))_+\,dx. \tag{3.28}$$

Hence, let $s_1 \to 0$, we get

$$\int_\Omega (u_1(x,s_2) - u_2(x,s_2))_+\,dx \le \int_\Omega (u_{01}(x) - u_{02}(x))_+\,dx. \tag{3.29}$$

The proof of Theorem 2.4 is complete. □

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