EXISTENCE AND COMPARISON RESULTS FOR VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

S. CARL

We consider a prototype of quasilinear elliptic variational-hemivariational inequalities involving the indicator function of some closed convex set and a locally Lipschitz functional. We provide a generalization of the fundamental notion of sub- and supersolutions on the basis of which we then develop the sub-supersolution method for variational-hemivariational inequalities. Furthermore, we give an example to illustrate the abstract theory developed in this paper.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < \infty$, denote the usual Sobolev spaces with their dual spaces V^* and V_0^*, respectively. In this paper, we deal with the following variational-hemivariational inequality:

$$u \in K : \big\langle -\Delta_p u - f, v - u \big\rangle + \int_\Omega j^o(u; v - u) dx \geq 0, \quad \forall v \in K, \tag{1.1}$$

where $j^o(s;r)$ denotes the generalized directional derivative of the locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$ at s in the direction r given by

$$j^o(s;r) = \limsup_{y \to s,\, t \downarrow 0} \frac{j(y + tr) - j(y)}{t}, \tag{1.2}$$

(cf., e.g., [3, Chapter 2]), and $K \subset V_0$ is some closed and convex subset. The operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < \infty$, and $f \in V_0^*$.

The main goal of this paper is to develop the sub-supersolution method for variational-hemivariational inequalities of the form (1.1) which may be considered as a prototype of more general problems of this kind. Problem (1.1) includes various special cases:

(i) for $K = V_0$ and $j : \mathbb{R} \to \mathbb{R}$ smooth, (1.1) is the weak formulation of the Dirichlet problem

$$u \in V_0 : -\Delta_p u + j'(u) = f \text{ in } V_0^*, \tag{1.3}$$

for which the sub-supersolution method is well known,

(ii) if $K = V_0$, and $j : \mathbb{R} \to \mathbb{R}$ not necessarily smooth, then (1.1) is a hemivariational inequality of the form

$$u \in V_0 : \langle -\Delta_p u - f, v - u \rangle + \int_\Omega j^o(u; v - u) dx \geq 0, \quad \forall v \in V_0, \tag{1.4}$$

for which an extension of the sub-supersolution method has been given recently in [2],

(iii) if $j = 0$, then (1.1) becomes a variational inequality for which a sub-supersolution method has been developed in [4, 5], and an extension of this method to systems of variational inequalities has been proved recently in [1].

This paper provides a unified theory on the sub-supersolution method for variational-hemivariational inequalities that includes all the above cited special cases.

2. Notation and hypotheses

For functions $w, z : \Omega \to \mathbb{R}$ and sets W and Z of functions defined on Ω, we use the notations: $w \wedge z = \min\{w, z\}$, $w \vee z = \max\{w, z\}$, $W \wedge Z = \{w \wedge z \mid w \in W, z \in Z\}$, $W \vee Z = \{w \vee z \mid w \in W, z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$. Next we introduce our basic notion of sub-supersolution.

Definition 2.1. A function $\underline{u} \in V$ is called a *subsolution* of (1.1) if the following holds:

(i) $\underline{u} \leq 0$ on $\partial\Omega$,
(ii) $\langle -\Delta_p \underline{u} - f, v - \underline{u} \rangle + \int_\Omega j^o(\underline{u}; v - \underline{u}) dx \geq 0, \forall v \in \underline{u} \wedge K$.

Definition 2.2. $\bar{u} \in V$ is a *supersolution* of (1.1) if the following holds:

(i) $\bar{u} \geq 0$ on $\partial\Omega$,
(ii) $\langle -\Delta_p \bar{u} - f, v - \bar{u} \rangle + \int_\Omega j^o(\bar{u}; v - \bar{u}) dx \geq 0, \forall v \in \bar{u} \vee K$.

Let $\partial j : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\varnothing\}$ denote Clarke's generalized gradient of j defined by

$$\partial j(s) := \{\zeta \in \mathbb{R} \mid j^o(s; r) \geq \zeta r, \ \forall r \in \mathbb{R}\}. \tag{2.1}$$

We assume the following hypothesis for j.

(H) The function $j : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and its Clarke's generalized gradient ∂j satisfies the following growth conditions:
(i) there exists a constant $c_1 \geq 0$ such that

$$\xi_1 \leq \xi_2 + c_1 (s_2 - s_1)^{p-1} \tag{2.2}$$

for all $\xi_i \in \partial j(s_i)$, $i = 1, 2$, and for all s_1, s_2 with $s_1 < s_2$,

(ii) there is a constant $c_2 \geq 0$ such that

$$\xi \in \partial j(s) : |\xi| \leq c_2 \left(1 + |s|^{p-1}\right), \quad \forall s \in \mathbb{R}. \tag{2.3}$$

Let $L^p(\Omega)$ be equipped with the natural partial ordering of functions defined by $u \leq w$ if and only if $w - u$ belongs to the positive cone $L^p_+(\Omega)$ of all nonnegative elements of $L^p(\Omega)$. This induces a corresponding partial ordering also in the subspace V of $L^p(\Omega)$, and if $u, w \in V$ with $u \leq w$ then

$$[u, w] = \{z \in V \mid u \leq z \leq w\} \tag{2.4}$$

denotes the order interval formed by u and w.

In the proofs of our main results we make use of the cut-off function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ related with an ordered pair of functions $\underline{u}, \bar{u}$, and given by

$$b(x, s) = \begin{cases} (s - \bar{u}(x))^{p-1} & \text{if } s > \bar{u}(x), \\ 0 & \text{if } \underline{u}(x) \leq s \leq \bar{u}(x), \\ -(\underline{u}(x) - s)^{p-1} & \text{if } s < \underline{u}(x). \end{cases} \tag{2.5}$$

One readily verifies that b is a Carathéodory function satisfying the growth condition

$$|b(x, s)| \leq k(x) + c_3 |s|^{p-1} \tag{2.6}$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, with some function $k \in L^q_+(\Omega)$, and q satisfying $1/p + 1/q = 1$. Moreover, one has the following estimate

$$\int_\Omega b(x, u(x)) u(x) dx \geq c_4 \|u\|_{L^p(\Omega)}^p - c_5, \quad \forall u \in L^p(\Omega), \tag{2.7}$$

where c_4 and c_5 are some positive constants. In view of (2.6), the Nemytskij operator $B : L^p(\Omega) \to L^q(\Omega)$ defined by

$$Bu(x) = b(x, u(x)) \tag{2.8}$$

is continuous and bounded, and thus due to the compact embedding $V \subset L^p(\Omega)$ it follows that $B : V_0 \to V_0^*$ is compact.

3. Preliminaries

In this section, we briefly recall a surjectivity result for multivalued mappings in reflexive Banach spaces (cf., e.g., [6, Theorem 2.12]) which among others will be used in the proof of our main result.

THEOREM 3.1. *Let X be a real reflexive Banach space with dual space X^*, $\Phi : X \to 2^{X^*}$ a maximal monotone operator, and $u_0 \in \mathrm{dom}(\Phi)$. Let $A : X \to 2^{X^*}$ be a pseudomonotone operator, and assume that either A_{u_0} is quasi-bounded or Φ_{u_0} is strongly quasi-bounded. Assume further that $A : X \to 2^{X^*}$ is u_0-coercive, that is, there exists a real-valued function $c : \mathbb{R}_+ \to \mathbb{R}$ with $c(r) \to +\infty$ as $r \to +\infty$ such that for all $(u, u^*) \in \mathrm{graph}(A)$ one has $\langle u^*, u - u_0 \rangle \geq c(\|u\|_X) \|u\|_X$. Then $A + \Phi$ is surjective, that is, $\mathrm{range}(A + \Phi) = X^*$.*

The operators A_{u_0} and Φ_{u_0} that appear in the theorem above are defined by $A_{u_0}(v) := A(u_0 + v)$ and similarly for Φ_{u_0}. As for the notion of *quasi-bounded* and *strongly quasi-bounded*, we refer to [6, page 51]. In particular, one has that any bounded operator is quasi-bounded and strongly quasi-bounded as well. The following proposition provides sufficient conditions for an operator $A : X \to 2^{X^*}$ to be pseudomonotone, which is suitable for our purpose.

PROPOSITION 3.2. *Let X be a real reflexive Banach space, and assume that $A : X \to 2^{X^*}$ satisfies the following conditions:*

(i) *for each $u \in X$ we have that $A(u)$ is a nonempty, closed and convex subset of X^*;*

(ii) *$A : X \to 2^{X^*}$ is bounded;*

(iii) *if $u_n \rightharpoonup u$ in X and $u_n^* \rightharpoonup u^*$ in X^* with $u_n^* \in A(u_n)$ and if $\limsup \langle u_n^*, u_n - u \rangle \leq 0$, then $u^* \in A(u)$ and $\langle u_n^*, u_n \rangle \to \langle u^*, u \rangle$.*

Then the operator $A : X \to 2^{X^*}$ is pseudomonotone.

As for the proof of **Proposition 3.2** we refer, for example, to [6, Chapter 2].

4. Main result

The main result of this paper is given by the following theorem which provides an existence and comparison result for the variational-hemivariational inequality (1.1).

THEOREM 4.1. *Let $\bar{u}$ and $\underline{u}$ be super- and subsolutions of (1.1), respectively, satisfying $\underline{u} \leq \bar{u}$, and assume $\bar{u} \wedge K \subset K$ and $\underline{u} \vee K \subset K$. Then under hypothesis (H) there exist solutions of (1.1) within the order interval $[\underline{u}, \bar{u}]$.*

Proof. Let $I_K : V_0 \to \mathbb{R} \cup \{+\infty\}$ denote the indicator function related with the given closed convex set $K \neq \varnothing$ and defined by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K, \end{cases} \tag{4.1}$$

which is proper, convex, and lower semicontinuous. By means of the indicator function the variational-hemivariational inequality (1.1) can be rewritten in the following form. Find $u \in V_0$ such that

$$\langle -\Delta_p u - f, v - u \rangle + I_K(v) - I_K(u) + \int_\Omega j^o(u; v - u)dx \geq 0, \quad \forall v \in V_0. \tag{4.2}$$

Since we are looking for solutions of (4.2) within $[\underline{u}, \bar{u}]$, we consider the following auxiliary problem. Find $u \in V_0$ such that

$$\langle -\Delta_p u - f + \lambda B(u), v - u \rangle + I_K(v) - I_K(u) + \int_\Omega j^o(u; v - u)dx \geq 0, \quad \forall v \in V_0, \tag{4.3}$$

where B is the cut-off operator introduced in **Section 2**, and $\lambda \geq 0$ is some parameter to be specified later. As will be seen in the course of the proof, the role of λB is twofold. First it provides a coercivity generating term, and second, it allows for comparison. The proof of the theorem will be done in two steps. In **Step 1**, we prove the existence of solutions of

the auxiliary problem (4.3), and in **Step 2**, we are going to show that any solution of (4.3) belongs to the interval $[\underline{u}, \bar{u}]$, which completes the proof, since then $B(u) = 0$ and (4.2) holds.

Step 1 (existence for (4.3)). We introduce the functional $J : L^p(\Omega) \to \mathbb{R}$ defined by

$$J(v) = \int_\Omega j(v(x)) dx, \quad \forall v \in L^p(\Omega), \tag{4.4}$$

which by hypothesis (H) is locally Lipschitz, and moreover, by Aubin-Clarke theorem (see [3, page 83]) for each $u \in L^p(\Omega)$ we have

$$\xi \in \partial J(u) \Longrightarrow \xi \in L^q(\Omega) \quad \text{with} \quad \xi(x) \in \partial j(u(x)) \text{ for a.e. } x \in \Omega. \tag{4.5}$$

Consider now the multivalued operator

$$-\Delta_p + \lambda B + \partial(J|_{V_0}) + \partial I_K : V_0 \longrightarrow 2^{V_0^*}, \tag{4.6}$$

where $J|_{V_0}$ denotes the restriction of J to V_0 and ∂I_K is the subdifferential of I_K in the sense of convex analysis. It is well known that $\Phi := \partial I_K : V_0 \to 2^{V_0^*}$ is a maximal monotone operator, (cf., e.g., [7]). Since $-\Delta_p : V_0 \to V_0^*$ is strictly monotone, bounded, and continuous, and $\lambda B : V_0 \to V_0^*$ is bounded, continuous and compact, it follows that $-\Delta_p + \lambda B : V_0 \to V_0^*$ is a (singlevalued) pseudomonotone, continuous, and bounded operator. In [2], it has been shown that $\partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is a (multivalued) pseudomonotone operator, which, due to (H), is bounded. Thus $A := -\Delta_p + \lambda B + \partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is a pseudomonotone and bounded operator. Hence, it follows by Theorem 3.1 that range$(A + \Phi) = V_0^*$ provided A is u_0-coercive for some $u_0 \in K$, which can readily be seen as follows: for any $v \in V_0$ and any $w \in \partial(J|_{V_0})(v)$, we obtain by applying (H)(ii) and (2.7) the estimate

$$\langle -\Delta_p v + \lambda B(v) + w, v - u_0 \rangle$$
$$= \int_\Omega |\nabla v|^p \, dx + \lambda \langle B(v), v \rangle + \int_\Omega wv \, dx - \langle -\Delta_p v + \lambda B(v) + w, u_0 \rangle$$
$$\geq \int_\Omega |\nabla v|^p \, dx + c_4 \lambda \|v\|_{L^p(\Omega)}^p - c_5 \lambda - c_2 \int_\Omega \left(1 + |v|^{p-1}\right) |v| dx \tag{4.7}$$
$$- |\langle -\Delta_p v + \lambda B(v) + w, u_0 \rangle| \geq \|v\|_{V_0}^p - C\left(1 + \|v\|_{V_0}^{p-1}\right),$$

for some constant $C > 0$, by choosing the constant λ in such a way that $c_4 \lambda > c_2$. Since $p > 1$, the coercivity of A follows from (4.7). In view of the surjectivity of the operator $A + \Phi$ there exists a $u \in \text{dom}(A) \cap \text{dom}(\Phi)$ such that $f \in A(u) + \Phi(u)$, that is, there is an $\xi \in \partial(J|_{V_0})(u)$ with $\xi \in L^q(\Omega)$ and $\xi(x) \in \partial j(u(x))$ for a.e. $x \in \Omega$, and an $\eta \in \Phi(u)$ such that

$$-\Delta_p u - f + \lambda B(u) + \xi + \eta = 0 \quad \text{in } V_0^*, \tag{4.8}$$

where

$$\langle \xi, \varphi \rangle = \int_\Omega \xi(x)\varphi(x)dx, \quad \forall \varphi \in V_0, \tag{4.9}$$

$$I_K(v) \geq I_K(u) + \langle \eta, v - u \rangle, \quad \forall v \in V_0. \tag{4.10}$$

By the definition of Clarke's generalized gradient ∂j from (4.9) we get

$$\langle \xi, \varphi \rangle = \int_\Omega \xi(x)\varphi(x)dx \le \int_\Omega j^o(u(x); \varphi(x))dx, \quad \forall \varphi \in V_0. \tag{4.11}$$

Thus from (4.8), (4.9), (4.10), and (4.11) with φ replaced by $v - u$, we obtain (4.3), which proves the existence of solutions of problem (4.3).

Step 2 ($\underline{u} \le u \le \bar{u}$ for any solution u of (4.3)). We first show $u \le \bar{u}$. By definition, the supersolution $\bar{u}$ satisfies: $\bar{u} \ge 0$ on $\partial\Omega$, and

$$\langle -\Delta_p \bar{u} - f, v - \bar{u} \rangle + \int_\Omega j^o(\bar{u}; v - \bar{u})dx \ge 0, \quad \forall v \in \bar{u} \vee K. \tag{4.12}$$

Let u be any solution of (4.3) which is equivalent with the following variational-hemivariational inequality:

$$u \in K : \langle -\Delta_p u - f, v - u \rangle + \langle \lambda B(u), v - u \rangle + \int_\Omega j^o(u; v - u)dx \ge 0, \quad \forall v \in K. \tag{4.13}$$

We apply the special test function $v = \bar{u} \vee u = \bar{u} + (u - \bar{u})^+$ in (4.12) and $v = \bar{u} \wedge u = u - (u - \bar{u})^+$ in (4.13), and get by adding the resulting inequalities the following one:

$$\begin{aligned}
\langle -\Delta_p \bar{u} - (-\Delta_p u), (u - \bar{u})^+ \rangle + \lambda \langle B(u), -(u - \bar{u})^+ \rangle \\
+ \int_\Omega (j^o(\bar{u}; (u - \bar{u})^+) + j^o(u; -(u - \bar{u})^+))dx \ge 0,
\end{aligned} \tag{4.14}$$

which yields, due to

$$\langle -\Delta_p u - (-\Delta_p \bar{u}), (u - \bar{u})^+ \rangle \ge 0, \tag{4.15}$$

the inequality

$$\lambda \langle B(u), (u - \bar{u})^+ \rangle \le \int_\Omega (j^o(\bar{u}; (u - \bar{u})^+) + j^o(u; -(u - \bar{u})^+))dx. \tag{4.16}$$

By using (H) and the properties on j^o and ∂j, we get for certain $\bar{\xi}(x) \in \partial j(\bar{u}(x))$ and $\xi(x) \in \partial j(u(x))$ the following estimate of the right-hand side of (4.16):

$$\begin{aligned}
\int_\Omega (j^o(\bar{u}; (u - \bar{u})^+) &+ j^o(u; -(u - \bar{u})^+))dx \\
&= \int_{\{u > \bar{u}\}} (j^o(\bar{u}; u - \bar{u}) + j^o(u; -(u - \bar{u})))dx \\
&= \int_{\{u > \bar{u}\}} (\bar{\xi}(x)(u(x) - \bar{u}(x)) + \xi(x)(-(u(x) - \bar{u}(x))))dx \\
&= \int_{\{u > \bar{u}\}} (\bar{\xi}(x) - \xi(x))(u(x) - \bar{u}(x))dx \\
&\le \int_{\{u > \bar{u}\}} c_1(u(x) - \bar{u}(x))^P dx.
\end{aligned} \tag{4.17}$$

Since

$$\langle B(u), (u - \bar{u})^+ \rangle = \int_{\{u > \bar{u}\}} (u - \bar{u})^p \, dx, \tag{4.18}$$

we get from (4.16) and (4.17) the estimate

$$(\lambda - c_1) \int_{\{u > \bar{u}\}} (u - \bar{u})^p \, dx \le 0. \tag{4.19}$$

Selecting the parameter λ, in addition, such that $\lambda - c_1 > 0$ then (4.19) yields

$$\int_\Omega ((u - \bar{u})^+)^p \, dx \le 0, \tag{4.20}$$

which implies $(u - \bar{u})^+ = 0$ and thus $u \le \bar{u}$. The proof for the inequality $\underline{u} \le u$ can be carried out in a similar way which completes the proof of the theorem. $\qquad\square$

4.1. Example. We consider (1.1) with $f \in L^\infty(\Omega)$ and K representing the following obstacle problem

$$K = \{v \in V_0 \mid v(x) \le \psi(x) \text{ for a.e. } x \in \Omega\}, \tag{4.21}$$

and assume $\psi \in L^\infty(\Omega)$. We are going to provide sufficient conditions for the existence of an ordered pair of constant sub- and supersolutions α and β, respectively.

PROPOSITION 4.2. *Let K be given by (4.21) and assume $f, \psi \in L^\infty(\Omega)$. Then the following hold.*

(a) *the constant function $\underline{u}(x) \equiv \alpha \le 0$ is a subsolution of (1.1) if*

$$f(x) \ge -j^o(\alpha; -1) \quad \text{for a.e. } x \in \Omega; \tag{4.22}$$

(b) *the constant function $\bar{u}(x) \equiv \beta \ge 0$ is a supersolution of (1.1) if*

$$f(x) \le j^o(\beta; 1) \quad \text{for a.e. } x \in \Omega. \tag{4.23}$$

Proof. Let $\alpha \le 0$ satisfy (4.22). According to Definition 2.1, we only need to verify that α satisfies (ii) in Definition 2.1. To this end let $v \in \alpha \wedge K$ be given. Then $v - \alpha \le 0$ in Ω and in view of (4.22) we get

$$\langle -\Delta_p \alpha - f, v - \alpha \rangle + \int_\Omega j^o(\alpha; v(x) - \alpha) \, dx$$

$$= \int_\Omega (j^o(\alpha; v(x) - \alpha) - f(x)(v(x) - \alpha)) \, dx \tag{4.24}$$

$$= \int_\Omega (j^o(\alpha; -1) + f(x))(\alpha - v(x)) \, dx \ge 0 \quad \forall v \in \alpha \wedge K,$$

which proves that α is a subsolution. In a similar way one can show that under (4.23) the constant $\beta \ge 0$ is a supersolution. $\qquad\square$

In order to apply Theorem 4.1 to our example, we only need to make sure that, in addition, $\beta \wedge K \subset K$ and $\alpha \vee K \subset K$ is satisfied. For the obstacle problem $\beta \wedge K \subset K$ is trivially satisfied and $\alpha \vee K \subset K$ holds provided $\alpha \leq \psi(x)$ for a.e. $x \in \Omega$.

4.2. Remarks. (i) As already mentioned problem (1.1) can be treated also with the p-Laplacian replaced by a more general quasilinear elliptic operator of Leray-Lions type involving lower order terms.

(ii) It can be shown that the set $\mathcal{S}$ of all solutions of (1.1) lying within the interval $[\underline{u}, \bar{u}]$ is a compact set in V_0.

(iii) Under additional lattice conditions such as $K \wedge K \subset K$ and $K \vee K \subset K$ one can prove that the solution set $\mathcal{S}$ possesses extremal elements, that is, there exist the greatest and smallest solution with respect to the underlying partial ordering.

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S. Carl: Fachbereich Mathematik und Informatik, Institut für Analysis, Martin-Luther-Universität Halle-Wittenberg, 06099 Halle, Germany

E-mail address: carl@mathematik.uni-halle.de