CONVEX SETS AND INEQUALITIES

SIN-EI TAKAHASI, YASUJI TAKAHASHI, SHIZUO MIYAJIMA,
AND HIROYUKI TAKAGI

Received 31 May 2002 and in revised form 26 December 2004

We consider a natural correspondence between a family of inequalities and a closed convex set. As an application, we give new types of power mean inequalities and the Hölder-type inequalities.

1. Concept and fundamental result

Given a natural correspondence between a family of inequalities and a closed convex set in a topological linear space, one might expect that an inequality corresponding to a special point (e.g., an extreme point) would be of special interest in view of the convex analysis theory. In this paper, we realize this concept.

Let X be an arbitrary set and $\{\varphi_0,\varphi_1,\varphi\}$ a triple of nonnegative real-valued functions on X. Set

$$m = \inf_{\varphi_0(x)\neq 0} \frac{\varphi(x)}{\varphi_0(x)}, \qquad M = \sup_{\varphi_1(x)\neq 0} \frac{\varphi(x)}{\varphi_1(x)}. \tag{1.1}$$

Suppose that $0 < m$, $M < \infty$. Then we have

$$m\varphi_0(x) \leq \varphi(x) \leq M\varphi_1(x) \quad \forall x \in X. \tag{1.2}$$

For each $x \in X$, put

$$D_\varphi(x) = \{(\alpha,\beta) \in \mathbb{R}^2 : \varphi(x) \leq \alpha\varphi_1(x) + \beta\varphi_0(x)\}. \tag{1.3}$$

We consider the intersection $D_\varphi = \cap_{x\in X} D_\varphi(x)$ of all such sets. Note that D_φ is a nonempty closed convex domain in $\mathbb{R}^2$ and that each point $(\alpha,\beta) \in D_\varphi$ corresponds to the inequality $\varphi \leq \alpha\varphi_1 + \beta\varphi_0$ on X. We want to investigate the closed convex domain D_φ. To do this, we define the constant α_φ by

$$\alpha_\varphi = \sup_{M\varphi_1(x)\neq m\varphi_0(x)} \frac{M\varphi(x) - mM\varphi_0(x)}{M\varphi_1(x) - m\varphi_0(x)}. \tag{1.4}$$

Journal of Inequalities and Applications 2005:2 (2005) 107–117
DOI: 10.1155/JIA.2005.107

Clearly, $0 \le \alpha_\varphi \le M$. Also, we have the following three *fundamental facts*:

(A) if $(\alpha,\beta) \in D_\varphi$ and $\alpha/M + \beta/m = 1$, then $\alpha \ge \alpha_\varphi$,

(B) $\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha/M + \beta/m \ge 1,\ \alpha \ge \alpha_\varphi\} \subset D_\varphi$,

(C) $D_\varphi \subset \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha/M + \beta(m\lambda) \ge 1\}$ for some $1 \le \lambda \le \infty$. In particular, if $\alpha_\varphi < M$, then $D_\varphi \subset \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha/M + \beta/m \ge 1\}$.

These facts will be used in the later sections to realize our concept.

Proof of (A). Suppose $(\alpha,\beta) \in D_\varphi$ and $\alpha/M + \beta/m = 1$. Then

$$\varphi(x) \le \alpha\varphi_1(x) + m\left(1 - \frac{\alpha}{M}\right)\varphi_0(x), \tag{1.5}$$

and hence

$$\frac{M\varphi(x) - mM\varphi_0(x)}{M\varphi_1(x) - m\varphi_0(x)} \le \alpha \tag{1.6}$$

for all $x \in X$ with $M\varphi_1(x) \ne m\varphi_0(x)$. This implies that $\alpha_\varphi \le \alpha$. □

Proof of (B). If $t \ge \alpha_\varphi/M$, then $\varphi(x) - m\varphi_0(x) \le t\ (M\varphi_1(x) - m\varphi_0(x))$ and so $\varphi(x) \le tM\varphi_1(x) + m(1-t)\varphi_0(x)$ for all $x \in X$. Hence, we have

$$\begin{aligned} D_\varphi &\supset \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha \ge tM,\ \beta \ge m(1-t),\ t \ge \frac{\alpha_\varphi}{M} \text{ for some } t \in \mathbb{R}\right\} \\ &= \left\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{\alpha}{M} + \frac{\beta}{m} \ge 1,\ \alpha \ge \alpha_\varphi\right\}. \end{aligned} \tag{1.7}$$

□

Proof of (C). By the definition of M, we find a sequence $\{x_n\}$ in X such that

$$\varphi_1(x_n) \ne 0,\ (n = 1,2,\ldots), \quad M = \lim_{n\to\infty} \frac{\varphi(x_n)}{\varphi_1(x_n)}. \tag{1.8}$$

Of course, we can assume that $\varphi(x_n) \ne 0$ for all $n = 1,2,\ldots$. Since $\{\varphi_0(x_n)/\varphi(x_n)\}$ is a bounded sequence with bound $1/m$, we can take a subsequence $\{\varphi_0(x_{n'})/\varphi(x_{n'})\}$ converging to some real number t with $0 \le t \le 1/m$. Set $\lambda = 1/(tm)$ so that $1 \le \lambda \le \infty$. We have

$$\begin{aligned} D_\varphi &\subset \bigcap_{n'} \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha\frac{\varphi_1(x_{n'})}{\varphi(x_{n'})} + \beta\frac{\varphi_0(x_{n'})}{\varphi(x_{n'})} \ge 1\right\} \\ &\subset \left\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{\alpha}{M} + \frac{\beta}{m\lambda} \ge 1\right\}. \end{aligned} \tag{1.9}$$

In particular, if $\alpha_\varphi < M$, then λ must be 1 by an easy geometrical consideration on the $\alpha\beta$-plane $\mathbb{R}^2$. □

2. Application: Djokovic's inequality

Let H be a Hlawka space, that is, a Banach space in which the Hlawka inequality holds. If n and k are natural numbers with $2 \le k \le n-1$, then

$$\sum_{1\le i_1<\cdots<i_k\le n} \|x_{i_1}+\cdots+x_{i_k}\| \le \binom{n-2}{k-1}\sum_{i=1}^{n}\|x_i\| + \binom{n-2}{k-2}\left\|\sum_{i=1}^{n} x_i\right\| \tag{2.1}$$

for all $x_1,\dots,x_n \in H$. This is well known as Djokovic's inequality (cf. [1, 2]).

Let X be the linear space $H\oplus\cdots\oplus H = \{(x_1,\dots,x_n) : x_1,\dots,x_n \in H\}$. For $1 \le k \le n$, set

$$\delta_k(x_1,\dots,x_n) = \sum_{1\le i_1<\cdots<i_k\le n} \|x_{i_1}+\cdots+x_{i_k}\| \tag{2.2}$$

for all $(x_1,\dots,x_n) \in X$. Then $\{\delta_k : 1 \le k \le n\}$ constitutes a system of seminorms on X and satisfies

$$\binom{n-1}{k-1}\delta_n \le \delta_k \le \binom{n-1}{k-1}\delta_1 \quad (1 \le k \le n). \tag{2.3}$$

Fix k and set $\varphi_0 = \delta_n$, $\varphi_1 = \delta_1$, $\varphi = \delta_k$. Then the above Djokovic inequality can be rewritten as

$$\varphi \le \binom{n-2}{k-1}\varphi_1 + \binom{n-2}{k-2}\varphi_0 \quad \text{on } X. \tag{2.4}$$

Also, we can see that $m = M = \binom{n-1}{k-1}$ and $\alpha_\varphi = \binom{n-2}{k-1}$ because $\binom{n-1}{k-1} = \binom{n-2}{k-1} + \binom{n-2}{k-2}$. Then we have

$$\left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha+\beta \ge \binom{n-1}{k-1},\ \alpha \ge \binom{n-2}{k-1}\right\} \subset D_\varphi \subset \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha+\beta \ge \binom{n-1}{k-1}\right\} \tag{2.5}$$

by the fundamental facts (B), (C). However, we have from [3, Theorem 1 (vi)] that

$$D_\varphi \subset \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha+\beta \ge \binom{n-1}{k-1},\ \alpha \ge \binom{n-2}{k-1}\right\}. \tag{2.6}$$

It follows that D_φ coincides with the minimum domain

$$\left\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{\alpha}{M} + \frac{\beta}{m} \ge 1,\ \alpha \ge \alpha_\varphi\right\}. \tag{2.7}$$

Hence $(\alpha_\varphi, M-\alpha_\varphi)$ is the only extreme point of D_φ and the corresponding inequality, that is Djokovic's inequality, is of special interest. The above argument is nearly a restatement of [3, Theorem 1].

3. Application: the power mean inequality

Let $X = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_1,\dots,x_n > 0\}$ and take $t \in \mathbb{R}$. We define φ_0, φ_1, φ by

$$\begin{aligned} \varphi_0(x_1,\dots,x_n) &= \min\{x_1,\dots,x_n\}, \\ \varphi_1(x_1,\dots,x_n) &= \max\{x_1,\dots,x_n\}, \\ \varphi(x_1,\dots,x_n) &= \begin{cases} \left(\dfrac{x_1^t + \cdots + x_n{}^t}{n}\right)^{1/t} & \text{if } t \neq 0, \\ \sqrt[n]{x_1 \cdots x_n} & \text{if } t = 0, \end{cases} \end{aligned} \tag{3.1}$$

for all $(x_1,\dots,x_n) \in X$. Then $m = M = 1$. We determine the domain D_φ. For $(\alpha,\beta) \in \mathbb{R}^2$, $(\alpha,\beta) \in D_\varphi$ if and only if

$$\varphi(x_1,\dots,x_n) \leq \alpha \max\{x_1,\dots,x_n\} + \beta \min\{x_1,\dots,x_n\} \quad \forall (x_1,\dots,x_n) \in X. \tag{3.2}$$

Dividing (3.2) by $\max\{x_1,\dots,x_n\}$, we see that (3.2) is equivalent to the following condition:

$$\alpha + \beta u \geq \sup\left\{\varphi(x_1,\dots,x_n) : \begin{matrix} \min\{x_1,\dots,x_n\} = u \\ \max\{x_1,\dots,x_n\} = 1 \end{matrix}\right\} \quad \text{for } 0 < u \leq 1. \tag{3.3}$$

Denote by $f(u)$ the right side of (3.3). Then (3.3) becomes

$$\alpha + \beta u \geq f(u) \quad \text{for } 0 < u \leq 1. \tag{3.4}$$

Also, we can easily see that

$$f(u) = \begin{cases} \left(\dfrac{n-1}{n} + \dfrac{u^t}{n}\right)^{1/t} & \text{if } t \neq 0 \\ \sqrt[n]{u} & \text{if } t = 0 \end{cases} \quad (0 < u \leq 1). \tag{3.5}$$

If $t \neq 0$, then we have

$$\begin{aligned} f'(u) &= \frac{1}{n} u^{t-1} \left(\frac{n-1}{n} + \frac{u^t}{n}\right)^{1/t-1}, \\ f''(u) &= \frac{n-1}{n^2}(t-1)u^{t-2}\left(\frac{n-1}{n} + \frac{u^t}{n}\right)^{1/t-2}. \end{aligned} \tag{3.6}$$

(i) *The case of $t < 1$ and $t \neq 0$.* In this case, (3.6) implies that f is a concave function on $(0,1]$. Hence (3.4) is equivalent to the following condition:

$$\begin{aligned} \beta \geq f'(u(\alpha)) \quad &\text{for } \lim_{u \downarrow 0} f(u) \leq \alpha \leq f(1) - f'(1), \\ \alpha + \beta \geq f(1) \quad &\text{for } \alpha > f(1) - f'(1), \end{aligned} \tag{3.7}$$

where $u(\alpha)$ is the unique solution of the equation $\alpha + f'(u)u = f(u)$. Note that $f(1) = 1$, $f(1) - f'(1) = (n-1)/n$, and

$$\lim_{u\downarrow 0} f(u) = \begin{cases} \left(\dfrac{n-1}{n}\right)^{1/t} & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases} \tag{3.8}$$

To investigate $f'(u(\alpha))$, set $v = u(\alpha)$ and $\gamma = f'(v)$. Then

$$\left(\frac{n-1}{n} + \frac{v^t}{n}\right)^{1/t-1} = n\gamma v^{1-t}, \qquad \alpha + \gamma v = f(v). \tag{3.9}$$

Hence

$$f(v) = n\gamma v^{1-t}\left(\frac{n-1}{n} + \frac{v^t}{n}\right) = (n-1)\gamma v^{1-t} + \gamma v, \tag{3.10}$$

so that $\alpha = (n-1)\gamma v^{1-t}$. Therefore by a simple computation, we obtain the equation $(n-1)^{1/(1-t)}\alpha^{t/(t-1)} + \gamma^{t/(t-1)} = n^{1/(1-t)}$. Consequently, if $t > 0$, then

$$D_\varphi = \left\{ (\alpha,\beta) \in \mathbb{R}^2 : \begin{array}{l} \beta \geq \left(n^{1/(1-t)} - (n-1)^{1/(1-t)}\alpha^{t/(t-1)}\right)^{(t-1)/t} \text{ for } \left(\dfrac{n-1}{n}\right)^{1/t} \alpha \leq \dfrac{n-1}{n} \\ \alpha + \beta \geq 1 \text{ for } \alpha > \dfrac{n-1}{n} \end{array} \right\}, \tag{3.11}$$

and if $t < 0$, then

$$D_\varphi = \left\{ (\alpha,\beta) \in \mathbb{R}^2 : \begin{array}{l} \beta \geq \left(n^{1/(1-t)} - (n-1)^{1/(1-t)}\alpha^{t/(t-1)}\right)^{(t-1)/t} \text{ for } 0 \leq \alpha \leq \dfrac{n-1}{n} \\ \alpha + \beta \geq 1 \text{ for } \alpha > \dfrac{n-1}{n} \end{array} \right\}. \tag{3.12}$$

Also, $\alpha_\varphi = (n-1)/n$ from the fundamental facts (A), (B).

(ii) *The case of* $t = 0$. Note that f is a concave function on $(0,1]$ and $\lim_{u\downarrow 0} f(u) = 0$, $f(1) = 1$, and $f(1) - f'(1) = (n-1)/n$. By executing the argument of (i), we obtain that

$$D_\varphi = \left\{ (\alpha,\beta) \in \mathbb{R}^2 : \begin{array}{l} \beta \geq \dfrac{(n-1)^{n-1}}{n^n}\alpha^{1-n} \text{ for } 0 < \alpha \leq \dfrac{n-1}{n} \\ \alpha + \beta \geq 1 \text{ for } \alpha > \dfrac{n-1}{n} \end{array} \right\}, \tag{3.13}$$

and $\alpha_\varphi = (n-1)/n$.

(iii) *The case of* $t \geq 1$. By (3.6), f is a convex function on $(0,1]$. Therefore, (3.4) holds precisely when

$$\lim_{u\downarrow 0} f(u) \leq \alpha, \qquad f(1) \leq \alpha + \beta. \tag{3.14}$$

Since $\lim_{u\downarrow 0} f(u) = ((n-1)/n)^{1/t}$ and $f(1) = 1$, it follows that

$$D_\varphi = \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha \geq \left(\frac{n-1}{n}\right)^{1/t},\ \alpha+\beta \geq 1\right\}, \tag{3.15}$$

and $\alpha_\varphi = ((n-1)/n)^{1/t}$.

We are now in a position to give the inequalities of special interest. We describe the corresponding inequality in each case.

(i) Let $0 < t < 1$. Then

$$\left(\frac{x_1^t + \cdots + x_n^t}{n}\right)^{1/t} \leq \alpha x_n + \left(n^{1/(1-t)} - (n-1)^{1/(1-t)}\alpha^{t/(t-1)}\right)^{(t-1)/t} x_1 \tag{3.16}$$

for $0 < x_1 \leq \cdots \leq x_n$ and $((n-1)/n)^{1/t} < \alpha \leq (n-1)/n$. In particular,

$$\left(\frac{x_1^t + \cdots + x_n^t}{n}\right)^{1/t} \leq \frac{(n-1)x_n + x_1}{n} \tag{3.17}$$

for $0 < x_1 \leq \cdots \leq x_n$.

Let $t < 0$. Then

$$\left(\frac{x_1^t + \cdots + x_n^t}{n}\right)^{1/t} \leq \alpha x_n + \left(n^{1/(1-t)} - (n-1)^{1/(1-t)}\alpha^{t/(t-1)}\right)^{(t-1)/t} x_1 \tag{3.18}$$

for $0 < x_1 \leq \cdots \leq x_n$ and $0 < \alpha \leq (n-1)/n$. In particular,

$$\left(\frac{x_1^t + \cdots + x_n^t}{n}\right)^{1/t} \leq \frac{(n-1)x_n + x_1}{n} \tag{3.19}$$

for $0 < x_1 \leq \cdots \leq x_n$.

(ii) The inequality

$$\sqrt[n]{x_1 \cdots x_n} \leq \alpha x_n + \frac{(n-1)^{n-1}}{n^n}\alpha^{1-n} x_1 \tag{3.20}$$

holds for $0 < x_1 \leq \cdots \leq x_n$ and $0 < \alpha \leq (n-1)/n$. In particular,

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_n + (n-1)^{n-1} x_1}{n} \tag{3.21}$$

holds for $0 < x_1 \leq \cdots \leq x_n$.

(iii) Let $t \geq 1$. Then

$$\left(\frac{x_1^t + \cdots + x_n^t}{n}\right)^{1/t} \leq \alpha x_n + (1-\alpha)x_1 \tag{3.22}$$

for $0 < x_1 \leq \cdots \leq x_n$ and $((n-1)/n)^{1/t} \leq \alpha$. In particular,

$$\left(\frac{x_1^t + \cdots + x_n^t}{n}\right)^{1/t} \leq \left(\frac{n-1}{n}\right)^{1/t} x_n + \left(1 - \left(\frac{n-1}{n}\right)^{1/t}\right) x_1 \tag{3.23}$$

for $0 < x_1 \leq \cdots \leq x_n$.

4. Application: the Hölder inequality

Let (Ω,μ) be a finite measure space and $0 < p < q < r \le \infty$. Let $X = L^r(\Omega,\mu)$ and set

$$\varphi_0(f) = \|f\|_p, \qquad \varphi_1(f) = \|f\|_r, \qquad \varphi(f) = \|f\|_q \tag{4.1}$$

for all $f \in X$. Then

$$m = \mu(\Omega)^{1/q-1/p}, \qquad M = \mu(\Omega)^{1/q-1/r}, \tag{4.2}$$

because the map $t \mapsto \mu(\Omega)^{-1/t}\|f\|_t$ is a monotone increasing function. If $\dim X = 1$, then we have

$$\begin{aligned} D_\varphi &= \{(\alpha,\beta) \in \mathbb{R}^2 : \mu(\Omega)^{1/q} \le \alpha\mu(\Omega)^{1/r} + \beta\mu(\Omega)^{1/p}\} \\ &= \left\{(\alpha,\beta) \in \mathbb{R}^2 : 1 \le \frac{\alpha}{M} + \frac{\beta}{m}\right\}. \end{aligned} \tag{4.3}$$

In general, it is hard to determine the domain D_φ. We consider the following two special cases:

(I) $\Omega = \{1,2\}$, $\mu(\{1\}) = a > 0$, $\mu(\{2\}) = b > 0$, $p = 1$, and $r = \infty$,

(II) $1 \le p < q < r$ and μ is nonatomic.

(I) We first consider the case (I). In this case, $m = (a+b)^{-1+1/q}$ and $M = (a+b)^{1/q}$. Let $(\alpha,\beta) \in \mathbb{R}^2$. Then $(\alpha,\beta) \in D_\varphi$ if and only if

$$\left(ax^q + by^q\right)^{1/q} \le \alpha \max\{x,y\} + \beta(ax+by) \quad \forall x,y \ge 0. \tag{4.4}$$

This is equivalent to the following condition:

$$\alpha + \beta t \ge \sup\left\{\left(ax^q + by^q\right)^{1/q} : \begin{array}{l} ax+by = t,\ 0 \le x,\ y \le 1 \\ \max\{x,y\} = 1 \end{array}\right\} \quad \text{for } \min\{a,b\} \le t \le a+b, \tag{4.5}$$

namely,

$$\begin{aligned} \alpha + \beta t &\ge \left(a^{1-q}(t-b)^q + b\right)^{1/q} \quad \text{for } b \le t \le a+b, \\ \alpha + \beta t &\ge \left(b^{1-q}(t-a)^q + a\right)^{1/q} \quad \text{for } a \le t \le a+b. \end{aligned} \tag{4.6}$$

Set $f(t) = (a^{1-q}(t-b)^q + b)^{1/q}$ for $b \le t \le a+b$. Since $1 < q < \infty$, f is a convex function on $[b,a+b]$. Hence $\alpha + \beta t \ge f(t)$ for $b \le t \le a+b$ if and only if $\alpha + \beta b \ge f(b) = b^{1/q}$ and $\alpha + \beta(a+b) \ge f(a+b) = (a+b)^{1/q}$. Also, set $g(t) = (b^{1-q}(t-a)^q + a)^{1/q}$ for $a \le t \le a+b$. The same argument shows that $\alpha + \beta t \ge g(t)$ for $a \le t \le a+b$ if and only if $\alpha + \beta a \ge a^{1/q}$ and $\alpha + \beta(a+b) \ge (a+b)^{1/q}$. Therefore, in view of the condition (4.6), we have

$$D_\varphi = \left\{(\alpha,\beta) \in \mathbb{R}^2 : \begin{array}{l} \alpha + \beta b \ge b^{1/q},\ \alpha + \beta a \ge a^{1/q} \\ \alpha + \beta(a+b) \ge (a+b)^{1/q} \end{array}\right\}. \tag{4.7}$$

Moreover, since $\alpha+\beta(a+b)\geq(a+b)^{1/q}$ means $\alpha/M+\beta/m\geq 1$, it follows from the fundamental facts (A), (B) that

$$\alpha_\varphi=\frac{\max\{a,b\}^{1/q}(a+b)-(a+b)^{1/q}\max\{a,b\}}{\min\{a,b\}}. \tag{4.8}$$

Also, D_φ has two extreme points:

$$\left(\frac{\max\{a,b\}^{1/q}(a+b)-(a+b)^{1/q}\max\{a,b\}}{\min\{a,b\}},\frac{(a+b)^{1/q}-\max\{a,b\}^{1/q}}{\min\{a,b\}}\right),\quad \left(\frac{a^{1/q}b-ab^{1/q}}{b-a},\frac{b^{1/q}-a^{1/q}}{b-a}\right). \tag{4.9}$$

The first extreme point corresponds to the following inequality:

$$\begin{aligned}(ax^q+by^q)^{1/q}\leq{}&\frac{\max\{x,y\}}{\min\{a,b\}}\left(\max\{a,b\}^{1/q}(a+b)-(a+b)^{1/q}\max\{a,b\}\right)\\&+\frac{ax+by}{\min\{a,b\}}\left((a+b)^{1/q}-\max\{a,b\}^{1/q}\right)\end{aligned} \tag{4.10}$$

for all $a,b,x,y>0$ and $q>1$. In particular, if $a=b$, then we have

$$(x^q+y^q)^{1/q}\leq\max\{x,y\}(2-2^{1/q})+(x+y)(2^{1/q}-1) \tag{4.11}$$

for all $x,y>0$ and $q>1$. Since $x+y=\max\{x,y\}+\min\{x,y\}$, it follows that

$$(x^q+y^q)^{1/q}\leq\max\{x,y\}+(2^{1/q}-1)\min\{x,y\} \tag{4.12}$$

for all $x,y>0$ and $q>1$. This is just equal to (3.23) in case of $n=2$. The second extreme point corresponds to the following inequality:

$$(ax^q+by^q)^{1/q}\leq\frac{a^{1/q}b-ab^{1/q}}{b-a}\max\{x,y\}+\frac{b^{1/q}-a^{1/q}}{b-a}(ax+by) \tag{4.13}$$

for all $a,b,x,y>0$ and $q>1$.

(II) We consider the case (II). Take $f\in X$. Set $t=(r-p)/(q-p)$ and $s=(r-p)/(r-q)$. Then $r/t+p/s=q$ and $1/t+1/s=1$. Also, we have

$$\frac{p}{sq}=\frac{rp-pq}{rq-pq}=\frac{1/q-1/r}{1/p-1/r},\qquad \frac{r}{tq}=1-\frac{p}{sq}. \tag{4.14}$$

Now put $\gamma=(1/q-1/r)/(1/p-1/r)$. Then $0<\gamma<1$, $p/(sq)=\gamma$, and $r/(tq)=1-\gamma$. We use the Hölder inequality to see that

$$\begin{aligned}\|f\|_q&=\left(\int|f|^q dx\right)^{1/q}=\left(\int|f|^{r/t}|f|^{p/s}dx\right)^{1/q}\\&\leq\left(\int|f|^r dx\right)^{1/tq}\left(\int|f|^p dx\right)^{1/sq}=\|f\|_r^{r/tq}\|f\|_p^{p/sq}=\|f\|_r^{1-\gamma}\|f\|_p^{\gamma}.\end{aligned} \tag{4.15}$$

Take $\varepsilon > 0$ arbitrarily and put $\alpha = (1-\gamma)\varepsilon$. If $u = 1/(1-\gamma)$ and $v = 1/\gamma$, then the Young inequality yields

$$\begin{aligned}
\|f\|_r^{1-\gamma}\|f\|_p^{\gamma} &= \left(\varepsilon\|f\|_r\right)^{1-\gamma}\left(\varepsilon^{(\gamma-1)/\gamma}\|f\|_p\right)^{\gamma} \\
&\le \frac{\left(\varepsilon\|f\|_r\right)^{(1-\gamma)u}}{u} + \frac{\left(\varepsilon^{(\gamma-1)/\gamma}\|f\|_p\right)^{\gamma v}}{v} \\
&= (1-\gamma)\varepsilon\|f\|_r + \gamma\varepsilon^{(\gamma-1)/\gamma}\|f\|_p \\
&= \alpha\|f\|_r + \gamma\left(\frac{\alpha}{1-\gamma}\right)^{(\gamma-1)/\gamma}\|f\|_p.
\end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16), we obtain

$$\|f\|_q \le \alpha\|f\|_r + \gamma\left(\frac{\alpha}{1-\gamma}\right)^{(\gamma-1)/\gamma}\|f\|_p. \tag{4.17}$$

Since $\varepsilon > 0$ is arbitrary, so is $\alpha > 0$. Hence we have

$$\left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0,\ \beta \ge h(\alpha)\right\} \subset D_{\varphi}, \tag{4.18}$$

where $h(\alpha) = \gamma(\alpha/(1-\gamma))^{(\gamma-1)/\gamma}$. Now, set

$$\alpha_0 = (1-\gamma)\mu(\Omega)^{\gamma/p-\gamma/r}. \tag{4.19}$$

We observe that

$$\left\{(\alpha,\beta) \in \mathbb{R}^2 : 0 < \alpha \le \alpha_0,\ \beta \ge h(\alpha)\right\} = D_{\varphi} \cap \left\{(\alpha,\beta) \in \mathbb{R}^2 : 0 < \alpha \le \alpha_0\right\}. \tag{4.20}$$

Actually, the equality holds in (4.17) if and only if the equalities hold in both (4.15) and (4.16). Hence the equality condition in (4.17) is that

$$\left\{|f(\omega)| : \omega \in \Omega\right\} \subset \{0,c\} \quad \text{for some } c \in \mathbb{R}, \qquad \left(\frac{\alpha}{1-\gamma}\right)^{1/\gamma}\|f\|_r = \|f\|_p. \tag{4.21}$$

Define

$$a(\alpha) = \left(\frac{\alpha}{1-\gamma}\right)^{pr/(r-p)\gamma} \tag{4.22}$$

for all $\alpha > 0$. Let $0 < \alpha \le \alpha_0$. Then $0 < a(\alpha) \le a(\alpha_0) = \mu(\Omega)$, and hence we can take a measurable set A such that $\mu(A) = a(\alpha)$, because μ is nonatomic. Since the characteristic function χ_A on A satisfies the condition (4.21), the equality in (4.17) holds for $f = \chi_A$. Consequently, we easily see that (4.20) is valid. Notice that

$$\begin{aligned}
h'(\alpha) &= \frac{\gamma}{1-\gamma}\frac{\gamma-1}{\gamma}\left(\frac{\alpha}{1-\gamma}\right)^{(\gamma-1)/\gamma-1} = -\left(\frac{\alpha}{1-\gamma}\right)^{-1/\gamma} < 0 \quad (\alpha > 0), \\
h''(\alpha) &= \frac{1}{1-\gamma}\frac{1}{\gamma}\left(\frac{\alpha}{1-\gamma}\right)^{-1/\gamma-1} = \frac{1}{\gamma(1-\gamma)}\left(\frac{\alpha}{1-\gamma}\right)^{-(1+\gamma)/\gamma} > 0 \quad (\alpha > 0).
\end{aligned} \tag{4.23}$$

Hence $h(\alpha)$ is a strictly monotone decreasing concave function on $(0,\infty)$. Note also that $h'(\alpha_0) = -m/M$, since $m/M = \mu(\Omega)^{-1/p+1/r}$. Next we assert that the point $(\alpha_0, h(\alpha_0))$ is on the line $\alpha/M + \beta/m = 1$. Indeed,

$$\frac{\gamma-1}{p} + \frac{1-\gamma}{r} = (\gamma-1)\left(\frac{1}{p} - \frac{1}{r}\right) = -\frac{r}{q}\frac{q-p}{r-p}\frac{r-p}{pr} = \frac{p-q}{pq} = \frac{1}{q} - \frac{1}{p}, \tag{4.24}$$

and so

$$\begin{aligned} m\left(1 - \frac{\alpha_0}{M}\right) &= \mu(\Omega)^{1/q-1/p} - \mu(\Omega)^{1/r-1/p}(1-\gamma)\mu(\Omega)^{\gamma/p-\gamma/r} \\ &= \mu(\Omega)^{1/q-1/p} - \mu(\Omega)^{(\gamma-1)/p+(1-\gamma)/r} + \gamma\mu(\Omega)^{(\gamma-1)/p+(1-\gamma)/r} \\ &= \gamma\mu(\Omega)^{(\gamma-1)/p+(1-\gamma)/r} = h(\alpha_0). \end{aligned} \tag{4.25}$$

This implies the assertion. Therefore α_φ is just equal to α_0 by the fundamental facts (A), (B). Hence the above observations imply that

$$\begin{gathered} \alpha_\varphi = (1-\gamma)\mu(\Omega)^{\gamma/p-\gamma/r}, \\ D_\varphi = \{(\alpha,\beta) \in \mathbb{R}^2 : 0 < \alpha \le \alpha_\varphi,\ \beta \ge h(\alpha)\} \cup \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha \ge \alpha_\varphi,\ \frac{\alpha}{M} + \frac{\beta}{m} \ge 1\right\}. \end{gathered} \tag{4.26}$$

Thus the corresponding inequality is

$$\|f\|_q \le \alpha\|f\|_r + \gamma\left(\frac{\alpha}{1-\gamma}\right)^{(\gamma-1)/\gamma}\|f\|_p \tag{4.27}$$

for all $f \in L^r(\Omega,\mu)$, $1 \le p < q < r$, $0 < \alpha \le (1-\gamma)\mu(\Omega)^{\gamma/p-\gamma/r}$, and $\gamma = (1/q - 1/r)/(1/p - 1/r)$. In particular, we have

$$\|f\|_q \le (1-\gamma)\mu(\Omega)^{\gamma/p-\gamma/r}\|f\|_r + \gamma\mu(\Omega)^{(\gamma-1)/p-(1-\gamma)/r}\|f\|_p \tag{4.28}$$

for all $f \in L^r(\Omega,\mu)$, $1 \le p < q < r$, and $\gamma = (1/q - 1/r)/(1/p - 1/r)$. Moreover, as $r \to \infty$, we have

$$\mu(\Omega)^{-1/q}\|f\|_q \le \left(1 - \frac{p}{q}\right)\|f\|_\infty + \frac{p}{q}\mu(\Omega)^{-1/p}\|f\|_p \tag{4.29}$$

for all $f \in L^\infty(\Omega,\mu)$ and $1 \le p < q < \infty$.

Acknowledgment

The first and second authors are partially supported by the Grant-in-Aid for Scientific Research via the Japan Society for the Promotion of Science.

References

[1] D. Ž. Doković, *Generalizations of Hlawka's inequality*, Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 169–175.

[2] D. M. Smiley and M. F. Smiley, *The polygonal inequalities*, Amer. Math. Monthly **71** (1964), 755–760.

[3] S.-E. Takahasi, Y. Takahashi, and A. Honda, *A new interpretation of Djoković's inequality*, J. Nonlinear Convex Anal. **1** (2000), no. 3, 343–350.

Sin-Ei Takahasi: Department of Basic Technology, Yamagata University, Yonezawa 992-8510, Japan
E-mail address: sin-ei@emperor.yz.yamagata-u.ac.jp

Yasuji Takahashi: Department of Systems Engineering, Okayama Prefectural University, Soja, Okayama 719-1197, Japan
E-mail address: takahasi@cse.oka-pu.ac.jp

Shizuo Miyajima: Department of Mathematics, Tokyo University of Science, Shinjuku-ku, Wakamiya 26, Tokyo 162-8601, Japan
E-mail address: miyajima@rs.kagu.sut.ac.jp

Hiroyuki Takagi: Department of Mathematical Sciences, Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan
E-mail address: takagi@math.shinshu-u.ac.jp