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Research Article

The Shrinking Projection Method for Common Solutions of Generalized Mixed Equilibrium Problems and Fixed Point Problems for Strictly Pseudocontractive Mappings

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We introduce the shrinking hybrid projection method for finding a common element of the set of fixed points of strictly pseudocontractive mappings, the set of common solutions of the variational inequalities with inverse-strongly monotone mappings, and the set of common solutions of generalized mixed equilibrium problems in Hilbert spaces. Furthermore, we prove strong convergence theorems for a new shrinking hybrid projection method under some mild conditions. Finally, we apply our results to Convex Feasibility Problems (CFP). The results obtained in this paper improve and extend the corresponding results announced by Kim et al. (2010) and the previously known results.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$, and let E be a nonempty closed convex subset of H. Let $T : E \to E$ be a mapping. In the sequel, we will use $F(T)$ to denote the set of *fixed points* of T, that is, $F(T) = \{x \in E : Tx = x\}$. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively.

Let $S : E \to E$ be a mapping. Then S is called

(1) *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in E, \tag{1.1}$$

(2) *strictly pseudocontractive* with the coefficient $k \in [0,1)$ if

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in E, \tag{1.2}$$

(3) *pseudocontractive* if

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in E. \tag{1.3}$$

The class of strictly pseudocontractive mappings falls into the one between classes of nonexpansive mappings and pseudocontractive mappings. Within the past several decades, many authors have been devoted to the studies on the existence and convergence of fixed points for strictly pseudocontractive mappings. In 2008, Zhou [1] considered a convex combination method to study strictly pseudocontractive mappings. More precisely, take $k \in [0,1)$, and define a mapping S_k by

$$S_k x = kx + (1 - k)Sx, \quad \forall x \in E, \tag{1.4}$$

where S is strictly pseudocontractive mappings. Under appropriate restrictions on k, it is proved that the mapping S_k is nonexpansive. Therefore, the techniques of studying nonexpansive mappings can be applied to study more general strictly pseudocontractive mappings.

Recall that letting $A : E \to H$ be a mapping, then A is called

(1) *monotone* if

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall x, y \in E, \tag{1.5}$$

(2) *β-inverse-strongly monotone* if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \beta\|Ax - Ay\|^2, \quad \forall x, y \in E. \tag{1.6}$$

The *domain* of the function $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ is the set $\operatorname{dom}\varphi = \{x \in E : \varphi(x) < +\infty\}$. Let $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and let F be a bifunction of $E \times E$ into $\mathbb{R}$ such that $E \cap \operatorname{dom}\varphi \ne \emptyset$, where $\mathbb{R}$ is the set of real numbers.

There exists the *generalized mixed equilibrium problem* for finding $x \in E$ such that

$$F(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in E. \tag{1.7}$$

The set of solutions of (1.7) is denoted by $\mathrm{GMEP}(F, \varphi, A)$, that is,

$$\mathrm{GMEP}(F, \varphi, A) = \{x \in E : F(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \ge 0,\ \forall y \in E\}. \tag{1.8}$$

We see that x is a solution of a problem (1.7) which implies that $x \in \operatorname{dom}\varphi = \{x \in E : \varphi(x) < +\infty\}$.

In particular, if $A \equiv 0$, then the problem (1.7) is reduced into the *mixed equilibrium problem* [2] for finding $x \in E$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in E. \tag{1.9}$$

The set of solutions of (1.9) is denoted by $\mathrm{MEP}(F, \varphi)$.

If $A \equiv 0$ and $\varphi \equiv 0$, then the problem (1.7) is reduced into the *equilibrium problem* [3] for finding $x \in E$ such that

$$F(x, y) \geq 0, \quad \forall y \in E. \tag{1.10}$$

The set of solutions of (1.10) is denoted by $\mathrm{EP}(F)$. This problem contains fixed point problems and includes as special cases numerous problems in physics, optimization, and economics. Some methods have been proposed to solve the equilibrium problem; please consult [4, 5].

If $F \equiv 0$ and $\varphi \equiv 0$, then the problem (1.7) is reduced into the *Hartmann-Stampacchia variational inequality* [6] for finding $x \in E$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in E. \tag{1.11}$$

The set of solutions of (1.11) is denoted by $\mathrm{VI}(E, A)$. The variational inequality has been extensively studied in the literature. See, for example, [7–10] and the references therein.

Many authors solved the problems $\mathrm{GMEP}(F, \varphi, A)$, $\mathrm{MEP}(F, \varphi)$, and $\mathrm{EP}(F)$ based on iterative methods; see, for instance, [4, 5, 11–25] and reference therein.

In 2007, Tada and Takahashi [26] introduced a hybrid method for finding the common element of the set of fixed point of nonexpansive mapping and the set of solutions of equilibrium problems in Hilbert spaces. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by the following iterative algorithm:

$$\begin{gathered}
x_1 = x \in H, \\
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in E, \\
w_n = (1 - \alpha_n)x_n + \alpha_n S u_n, \\
E_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\
D_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} = P_{E_n \cap D_n} x, \quad \forall n \geq 1.
\end{gathered} \tag{1.12}$$

Then, they proved that, under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.12) converges strongly to $P_{F(S)\cap \mathrm{EP}(F)}x$.

In 2009, Qin and Kang [27] introduced an explicit viscosity approximation method for finding a common element of the set of fixed point of strictly pseudocontractive mappings and the set of solutions of variational inequalities with inverse-strongly monotone mappings in Hilbert spaces:

$$\begin{aligned} x_1 &\in E, \\ z_n &= P_E(x_n - \mu_n C x_n), \\ y_n &= P_E(x_n - \lambda_n B x_n), \\ x_{n+1} &= \epsilon_n f(x_n) + \beta_n x_n + \gamma_n \left[\alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} y_n + \alpha_n^{(3)} z_n\right], \quad \forall n \geq 1. \end{aligned} \tag{1.13}$$

Then, they proved that, under certain appropriate conditions imposed on $\{\epsilon_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, and $\{\alpha_n^{(3)}\}$, the sequence $\{x_n\}$ generated by (1.13) converges strongly to $q \in F(S) \cap \mathrm{VI}(E,B) \cap \mathrm{VI}(E,C)$, where $q = P_{F(S)\cap\mathrm{VI}(E,B)\cap\mathrm{VI}(E,C)} f(q)$.

In 2010, Kumam and Jaiboon [28] introduced a new method for finding a common element of the set of fixed point of strictly pseudocontractive mappings, the set of common solutions of variational inequalities with inverse-strongly monotone mappings, and the set of common solutions of a system of generalized mixed equilibrium problems in Hilbert spaces. Then, they proved that, under certain appropriate conditions imposed on $\{\epsilon_n\}$, $\{\beta_n\}$, and $\{\alpha_n^{(i)}\}$, where $i = 1,2,3,4,5$. The sequence $\{x_n\}$ converges strongly to $q \in \Theta := F(S) \cap \mathrm{VI}(E,B) \cap \mathrm{VI}(E,C) \cap \mathrm{GMEP}(F_1,\varphi,A_1) \cap \mathrm{GMEP}(F_2,\varphi,A_2)$, where $q = P_\Theta(I - A + \gamma f)(q)$.

In this paper, motivate, by Tada and Takahashi [26], Qin and Kang [27], and Kumam and Jaiboon [28], we introduce a new shrinking projection method for finding a common element of the set of fixed points of strictly pseudocontractive mappings, the set of common solutions of generalized mixed equilibrium problems, and the set of common solutions of the variational inequalities for inverse-strongly monotone mappings in Hilbert spaces. Finally, we apply our results to Convex Feasibility Problems (CFP). The results obtained in this paper improve and extend the corresponding results announced by the previously known results.

2. Preliminaries

Let H be a real Hilbert space, and let E be a nonempty closed convex subset of H. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

For any $x \in H$, there exists a *unique nearest point* in E, denoted by $P_E x$, such that

$$\|x - P_E x\| \leq \|x - y\|, \quad \forall y \in E. \tag{2.2}$$

The mapping P_E is called the *metric projection* of H onto E.

It is well known that P_E is a firmly nonexpansive mapping of H onto E, that is,

$$\langle x - y, P_E x - P_E y\rangle \geq \|P_E x - P_E y\|^2, \quad \forall x, y \in H. \tag{2.3}$$

Moreover, $P_E x$ is characterized by the following properties: $P_E x \in E$ and

$$\langle x - P_E x, y - P_E x\rangle \leq 0,$$
$$\|x - y\|^2 \geq \|x - P_E x\|^2 + \|y - P_E x\|^2 \tag{2.4}$$

for all $x \in H,\ y \in E$.

Lemma 2.1. *Let E be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in E$, then,*

$$z = P_E x \Longleftrightarrow \langle x - z, y - z\rangle \leq 0, \quad \forall y \in E. \tag{2.5}$$

Lemma 2.2. *Let H be a Hilbert space, let E be a nonempty closed convex subset of H, and let B be a mapping of E into H. Let $u \in E$. Then, for $\lambda > 0$,*

$$u \in \mathrm{VI}(E, B) \Longleftrightarrow u = P_E(u - \lambda B u), \tag{2.6}$$

where P_E is the metric projection of H onto E.

Lemma 2.3 (see [1]). *Let E be a nonempty closed convex subset of a real Hilbert space H, and let $S : E \to E$ be a k-strictly pseudocontractive mapping with a fixed point. Then $F(S)$ is closed and convex. Define $S_k : E \to E$ by $S_k = kx + (1-k)Sx$ for each $x \in E$. Then S_k is nonexpansive such that $F(S_k) = F(S)$.*

Lemma 2.4 (see [29]). *Let E be a closed convex subset of a real Hilbert space H, and let $S : E \to E$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero; that is,*

$$x_n \rightharpoonup x, \qquad x_n - Sx_n \longrightarrow 0 \tag{2.7}$$

implies $x = Sx$.

Lemma 2.5 (see [30]). *Each Hilbert space H satisfies the Kadec-Klee property, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together implying $\|x_n - x\| \to 0$.*

Lemma 2.6 (see [31]). *Let E be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that*

(1) *the weak ω-limit set $\omega_w(x_n) \subset E$,*

(2) *for each $z \in E$, $\lim_{n\to\infty}\|x_n - z\|$ exists.*

Then $\{x_n\}$ is weakly convergent to a point in E.

Lemma 2.7 (see [32]). *Let E be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_E u$. If $\{x_n\}$ is $\omega_w(x_n) \subset E$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\| \tag{2.8}$$

for all n, then $x_n \to q$.

Lemma 2.8 (see [33]). *Let E be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on E. Suppose $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let δ_n be a sequence of positive number with $\sum_{n=1}^{\infty} \delta_n = 1$. Then a mapping S on E defined by*

$$Sx = \sum_{n=1}^{\infty} \delta_n T_n x \tag{2.9}$$

for $x \in E$ is well defined, nonexpansive, and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, the function A, and the set E:

(A1) $F(x,x) = 0$ for all $x \in E$

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in E$

(A3) for each $x, y, z \in E$, $\lim_{t \to 0} F(tz + (1-t)x, y) \leq F(x,y)$

(A4) for each $x \in E$, $y \mapsto F(x,y)$ is convex and lower semicontinuous

(A5) for each $y \in E$, $x \mapsto F(x,y)$ is weakly upper semicontinuous

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq E$ and $y_x \in E$ such that, for any $z \in E \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0, \tag{2.10}$$

(B2) E is a bounded set.

By similar argument as in the proof of Lemma 2.9 in [34], we have the following lemma appearing.

Lemma 2.9. *Let E be a nonempty closed convex subset of H. Let $F : E \times E \to \mathbb{R}$ be a bifunction that satisfies (A1)–(A5), and let $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to E$ as follows:*

$$T_r^F(x) = \left\{ z \in E : F(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in E \right\}, \tag{2.11}$$

for all $z \in H$. Then, the following hold:

(1) *for each $x \in H$, $T_r^F(x) \neq \emptyset$,*

(2) *T_r^F is single valued,*

(3) *T_r^F is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| T_r^F x - T_r^F y \right\|^2 \leq \left\langle T_r^F x - T_r^F y, x - y \right\rangle, \tag{2.12}$$

(4) *$F(T_r^F) = MEP(F, \varphi)$,*

(5) *$MEP(F, \varphi)$ is closed and convex.*

Lemma 2.10. *Let H be a Hilbert space, let E be a nonempty closed convex subset of H, and let $A : E \to H$ be ρ-inverse-strongly monotone. If $0 < r \leq 2\rho$, then $I - \rho A$ is a nonexpansive mapping in H.*

Proof. For all $x, y \in E$ and $0 < r \leq 2\rho$, we have

$$\begin{aligned}\|(I-rA)x-(I-rA)y\|^2 &= \|(x-y)-r(Ax-Ay)\|^2 \\ &= \|x-y\|^2 - 2r\langle x-y, Ax-Ay\rangle + r^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 - 2r\rho\|Ax-Ay\| + r^2\|Ax-Ay\|^2 \\ &= \|x-y\|^2 + r(r-2\rho)\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2.\end{aligned} \tag{2.13}$$

So, $I - \rho A$ is a nonexpansive mapping of E into H. □

3. Main Results

In this section, we prove a strong convergence theorem of the new shrinking projection method for finding a common element of the set of fixed points of strictly pseudocontractive mappings, the set of common solutions of generalized mixed equilibrium problems and the set of common solutions of the variational inequalities with inverse-strongly monotone mappings in Hilbert spaces.

Theorem 3.1. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $E \times E$ to $\mathbb{R}$ satisfying (A1)–(A5), and let $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let A_1, A_2, B, C be four ρ, ω, β, ξ-inverse-strongly monotone mappings of E into H, respectively. Let $S : E \to E$ be a k-strictly pseudocontractive mapping with a fixed point. Define a mapping $S_k : E \to E$ by $S_k x = kx+(1-k)Sx$, for all $x \in E$. Suppose that*

$$\Theta := F(S) \cap \mathrm{GMEP}(F_1, \varphi, A_1) \cap \mathrm{GMEP}(F_2, \varphi, A_2) \cap \mathrm{VI}(E, B) \cap \mathrm{VI}(E, C) \neq \emptyset. \tag{3.1}$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{gathered} x_0 \in H, \quad E_1 = E, \quad x_1 = P_{E_1}x_0, \quad u_n \in E, \quad v_n \in E, \\ F_1(u_n, u) + \varphi(u) - \varphi(u_n) + \langle A_1 x_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in E, \\ F_2(v_n, v) + \varphi(v) - \varphi(v_n) + \langle A_2 x_n, v - v_n\rangle + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in E, \\ y_n = P_E(x_n - \lambda_n B x_n), \qquad z_n = P_E(x_n - \mu_n C x_n), \end{gathered}$$

$$t_n = \alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} y_n + \alpha_n^{(3)} z_n + \alpha_n^{(4)} u_n + \alpha_n^{(5)} v_n,$$
$$E_{n+1} = \{w \in E_n : \|t_n - w\| \le \|x_n - w\|\},$$
$$x_{n+1} = P_{E_{n+1}} x_0, \quad \forall n \ge 0, \tag{3.2}$$

where $\{\alpha_n^{(i)}\}$ *are sequences in* $(0,1)$, *where* $i = 1,2,3,4,5$, $r_n \in (0,2\rho)$, $s_n \in (0,2\omega)$, *and* $\{\lambda_n\}$, $\{\mu_n\}$ *are positive sequences. Assume that the control sequences satisfy the following restrictions:*

(C1) $\sum_{i=1}^{5} \alpha_n^{(i)} = 1$,

(C2) $\lim_{n\to\infty}\alpha_n^{(i)} = \alpha^{(i)} \in (0,1)$, *where* $i = 1,2,3,4,5$,

(C3) $a \le r_n \le 2\rho$ *and* $b \le s_n \le 2\omega$, *where* a, b *are two positive constants,*

(C4) $c \le \lambda_n \le 2\beta$ *and* $d \le \mu_n \le 2\xi$, *where* c, d *are two positive constants,*

(C5) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty}|\mu_{n+1} - \mu_n| = 0$.

Then, $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

Proof. Letting $p \in \Theta$ and by Lemma 2.9, we obtain

$$p = P_E(p - \lambda_n Bp) = P_E(p - \mu_n Cp) = T_{r_n}^{F_1}(I - r_n A_1)p = T_{s_n}^{F_2}(I - s_n A_2)p. \tag{3.3}$$

Note that $u_n = T_{r_n}^{F_1}(I - r_n A_1)x_n \in \operatorname{dom}\varphi$ and $v_n = T_{s_n}^{F_2}(I - s_n A_2)x_n \in \operatorname{dom}\varphi$, then we have

$$\|u_n - p\| = \|T_{r_n}^{F_1}(I - r_n A_1)x_n - T_{r_n}^{F_1}(I - r_n A_1)p\| \le \|x_n - p\|,$$
$$\|v_n - p\| = \left\|T_{s_n}^{F_2}(I - s_n A_2)x_n - T_{s_n}^{F_2}(I - s_n A_2)p\right\| \le \|x_n - p\|. \tag{3.4}$$

Next, we will divide the proof into six steps.

Step 1. We show that $\{x_n\}$ is well defined and E_n is closed and convex for any $n \ge 1$.

From the assumption, we see that $E_1 = E$ is closed and convex. Suppose that E_k is closed and convex for some $k \ge 1$. Next, we show that E_{k+1} is closed and convex for some k. For any $p \in E_k$, we obtain

$$\|t_k - p\| \le \|x_k - p\| \tag{3.5}$$

is equivalent to

$$\|t_k - p\|^2 + 2\langle t_k - x_k, x_k - p\rangle \le 0. \tag{3.6}$$

Thus, E_{k+1} is closed and convex. Then, E_n is closed and convex for any $n \ge 1$. This implies that $\{x_n\}$ is well defined.

Step 2. We show that $\Theta \subset E_n$ for each $n \geq 1$. From the assumption, we see that $\Theta \subset E = E_1$. Suppose $\Theta \subset E_k$ for some $k \geq 1$. For any $p \in \Theta \subset E_k$, since $y_n = P_E(x_n - \lambda_n B x_n)$ and $z_n = P_E(x_n - \mu_n C x_n)$, for each $\lambda_n \leq 2\beta$ and $\mu_n \leq 2\xi$ by Lemma 2.10, we have $I - \lambda_n B$ and $I - \mu_n C$ are nonexpansive. Thus, we obtain

$$\begin{aligned}
\|y_n - p\| &= \|P_E(x_n - \lambda_n B x_n) - P_E(p - \lambda_n B p)\| \\
&\leq \|(x_n - \lambda_n B x_n) - (p - \lambda_n B p)\| \\
&= \|(I - \lambda_n B)x_n - (I - \lambda_n B)p\| \\
&\leq \|x_n - p\|, \\
\|z_n - p\| &= \|P_E(x_n - \mu_n C x_n) - P_E(p - \mu_n C p)\| \\
&\leq \|(x_n - \mu_n C x_n) - (p - \mu_n C p)\| \\
&= \|(I - \mu_n C)x_n - (I - \mu_n C)p\| \\
&\leq \|x_n - p\|.
\end{aligned} \tag{3.7}$$

From Lemma 2.3, we have S_k is nonexpansive with $F(S_k) = F(S)$. It follows that

$$\begin{aligned}
\|t_n - p\| &= \left\|\alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} y_n + \alpha_n^{(3)} z_n + \alpha_n^{(4)} u_n + \alpha_n^{(5)} v_n - p\right\| \\
&\leq \alpha_n^{(1)} \|S_k x_n - p\| + \alpha_n^{(2)} \|y_n - p\| + \alpha_n^{(3)} \|z_n - p\| + \alpha_n^{(4)} \|u_n - p\| + \alpha_n^{(5)} \|v_n - p\| \\
&\leq \alpha_n^{(1)} \|x_n - p\| + \alpha_n^{(2)} \|x_n - p\| + \alpha_n^{(3)} \|x_n - p\| + \alpha_n^{(4)} \|x_n - p\| + \alpha_n^{(5)} \|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned} \tag{3.8}$$

It follows that $p \in E_{k+1}$. This implies that $\Theta \subset E_n$ for each $n \geq 1$.

Step 3. We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - t_n\| = 0$.

From $x_n = P_{E_n} x_0$, we get

$$\langle x_0 - x_n, x_n - y \rangle \geq 0 \tag{3.9}$$

for each $y \in E_n$. Using $\Theta \subset E_n$, we have

$$\langle x_0 - x_n, x_n - p \rangle \geq 0 \quad \text{for each } p \in \Theta,\ n \in \mathbb{N}. \tag{3.10}$$

Hence, for $p \in \Theta$, we obtain

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - p \rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - p \rangle \\
&= -\langle x_0 - x_n, x_0 - x_n \rangle + \langle x_0 - x_n, x_0 - p \rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - p\|.
\end{aligned} \tag{3.11}$$

It follows that

$$\|x_0 - x_n\| \leq \|x_0 - p\|, \quad \forall p \in \Theta, \ n \in \mathbb{N}. \tag{3.12}$$

From $x_n = P_{E_n} x_0$ and $x_{n+1} = P_{E_{n+1}} x_0 \in E_{n+1} \subset E_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0. \tag{3.13}$$

For $n \in \mathbb{N}$, we compute

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\
&= -\langle x_0 - x_n, x_0 - x_n \rangle + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\
&\leq -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|,
\end{aligned} \tag{3.14}$$

and then

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|, \quad \forall n \in \mathbb{N}. \tag{3.15}$$

Thus, the sequence $\{\|x_n - x_0\|\}$ is a bounded and nondecreasing sequence, so $\lim_{n\to\infty}\|x_n - x_0\|$ exists; that is, there exists m such that

$$m = \lim_{n\to\infty} \|x_n - x_0\|. \tag{3.16}$$

From (3.13), we get

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n\rangle + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\le -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.\end{aligned} \tag{3.17}$$

By (3.16), we obtain

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{3.18}$$

Since $x_{n+1} = P_{E_{n+1}}x_0 \in E_{n+1} \subset E_n$, we have

$$\|x_n - t_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - t_n\| \le 2\|x_n - x_{n+1}\|. \tag{3.19}$$

By (3.18), we obtain

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{3.20}$$

Step 4. We claim that the following statements hold:

(S1) $\lim_{n\to\infty}\|x_n - u_n\| = 0$,

(S2) $\lim_{n\to\infty}\|x_n - y_n\| = 0$,

(S3) $\lim_{n\to\infty}\|x_n - z_n\| = 0$,

(S4) $\lim_{n\to\infty}\|x_n - v_n\| = 0$.

For $p \in \Theta$, we note that

$$\begin{aligned}\|z_n - p\|^2 &= \|P_E(x_n - \mu_n Cx_n) - P_E(p - \mu_n Cp)\|^2 \\ &\le \|(x_n - \mu_n Cx_n) - (p - \mu_n Cp)\|^2 \\ &= \|(x_n - p) - \mu_n(Cx_n - Cp)\|^2 \\ &\le \|x_n - p\|^2 - 2\mu_n\langle x_n - p, Cx_n - Cp\rangle + \mu_n^2\|Cx_n - Cp\|^2 \\ &\le \|x_n - p\|^2 + \mu_n(\mu_n - 2\xi)\|Cx_n - Cp\|^2 \\ &= \|x_n - p\|^2 - \mu_n(2\xi - \mu_n)\|Cx_n - Cp\|^2.\end{aligned} \tag{3.21}$$

Similarly, we also have

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \lambda_n(2\beta - \lambda_n)\|Bx_n - B\mathrm{p}\|^2. \tag{3.22}$$

We note that

$$\begin{aligned}\|u_n - p\|^2 &= \left\|T_{r_n}^{F_1}(I - r_nA_1)x_n - T_{r_n}^{F_1}(I - r_nA_1)p\right\|^2 \\ &\le \|(I - r_nA_1)x_n - (I - r_nA_1)p\|^2 \\ &= \|(x_n - p) - r_n(A_1x_n - A_1p)\|^2 \\ &= \|x_n - p\|^2 - 2r_n\langle x_n - p, A_1x_n - A_1p\rangle + r_n^2\|A_1x_n - A_1p\|^2 \\ &\le \|x_n - p\|^2 - 2r_n\rho\|A_1x_n - A_1p\|^2 + r_n^2\|A_1x_n - A_1p\|^2 \\ &= \|x_n - p\|^2 + r_n(r_n - 2\rho)\|A_1x_n - A_1p\|^2 \\ &= \|x_n - p\|^2 - r_n(2\rho - r_n)\|A_1x_n - A_1p\|^2.\end{aligned} \tag{3.23}$$

Similarly, we also have

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - s_n(2\omega - s_n)\|A_2x_n - A_2p\|^2. \tag{3.24}$$

Observing that

$$\begin{aligned}\|t_n - p\|^2 &\le \alpha_n^{(1)}\|S_kx_n - p\|^2 + \alpha_n^{(2)}\|y_n - p\|^2 + \alpha_n^{(3)}\|z_n - p\|^2 + \alpha_n^{(4)}\|u_n - p\|^2 + \alpha_n^{(5)}\|v_n - p\|^2 \\ &\le \alpha_n^{(1)}\|x_n - p\|^2 + \alpha_n^{(2)}\|y_n - p\|^2 + \alpha_n^{(3)}\|z_n - p\|^2 + \alpha_n^{(4)}\|u_n - p\|^2 + \alpha_n^{(5)}\|v_n - p\|^2.\end{aligned} \tag{3.25}$$

Substituting (3.21), (3.22), (3.23), and (3.24) into (3.25), we obtain

$$\begin{aligned}\|t_n - p\|^2 &\le \alpha_n^{(1)}\|x_n - p\|^2 + \alpha_n^{(2)}\left\{\|x_n - p\|^2 - \lambda_n(2\beta - \lambda_n)\|Bx_n - Bp\|^2\right\} \\ &\quad + \alpha_n^{(3)}\left\{\|x_n - p\|^2 - \mu_n(2\xi - \mu_n)\|Cx_n - Cp\|^2\right\} \\ &\quad + \alpha_n^{(4)}\left\{\|x_n - p\|^2 - r_n(2\rho - r_n)\|A_1x_n - A_1p\|^2\right\} \\ &\quad + \alpha_n^{(5)}\left\{\|x_n - p\|^2 - s_n(2\omega - s_n)\|A_2x_n - A_2p\|^2\right\} \\ &= \|x_n - p\|^2 - \alpha_n^{(2)}\lambda_n(2\beta - \lambda_n)\|Bx_n - Bp\|^2 - \alpha_n^{(3)}\mu_n(2\xi - \mu_n)\|Cx_n - Cp\|^2 \\ &\quad - \alpha_n^{(4)}r_n(2\rho - r_n)\|A_1x_n - A_1p\|^2 - \alpha_n^{(5)}s_n(2\omega - s_n)\|A_2x_n - A_2p\|^2.\end{aligned} \tag{3.26}$$

It follows that

$$\begin{aligned}
&\alpha_n^{(3)}\mu_n(2\xi-\mu_n)\left\|Cx_n-Cp\right\|^2\\
&\quad\le \left\|x_n-p\right\|^2-\left\|t_n-p\right\|^2-\alpha_n^{(2)}\lambda_n(2\beta-\lambda_n)\left\|Bx_n-Bp\right\|^2\\
&\qquad-\alpha_n^{(4)}r_n(2\rho-r_n)\left\|A_1x_n-A_1p\right\|^2-\alpha_n^{(5)}s_n(2\omega-s_n)\left\|A_2x_n-A_2p\right\|^2\\
&\quad\le \left(\left\|x_n-p\right\|+\left\|t_n-p\right\|\right)\|x_n-t_n\|.
\end{aligned}\tag{3.27}$$

From (C2), (C4), and (3.20), we have

$$\lim_{n\to\infty}\left\|Cx_n-Cp\right\|=0.\tag{3.28}$$

Since $s_n\in(0,2\omega)$, we also have

$$\begin{aligned}
&\alpha_n^{(5)}s_n(2\omega-s_n)\left\|A_2x_n-A_2p\right\|^2\\
&\quad\le \left\|x_n-p\right\|^2-\left\|t_n-p\right\|^2-\alpha_n^{(2)}\lambda_n(2\beta-\lambda_n)\left\|Bx_n-Bp\right\|^2\\
&\qquad-\alpha_n^{(3)}\mu_n(2\xi-\mu_n)\left\|Cx_n-Cp\right\|^2-\alpha_n^{(4)}r_n(2\rho-r_n)\left\|A_1x_n-A_1p\right\|^2\\
&\quad\le \left(\left\|x_n-p\right\|+\left\|t_n-p\right\|\right)\|x_n-t_n\|.
\end{aligned}\tag{3.29}$$

From (C2), (C3), and (3.20), we obtain

$$\lim_{n\to\infty}\left\|A_2x_n-A_2p\right\|=0.\tag{3.30}$$

Similarly, by (3.28) and (3.30), we can prove that

$$\lim_{n\to\infty}\left\|Bx_n-Bp\right\|=\lim_{n\to\infty}\left\|A_1x_n-A_1p\right\|=0.\tag{3.31}$$

On the other hand, letting $p \in \Theta$ for each $n \geq 1$, we get $p = T_{r_n}^{F_1}(I - r_n A_1)p$. Since $T_{r_n}^{F_1}$ is firmly nonexpansive, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \left\|T_{r_n}^{F_1}(I - r_n A_1)x_n - T_{r_n}^{F_1}(I - r_n A_1)p\right\|^2 \\
&\leq \langle (I - r_n A_1)x_n - (I - r_n A_1)p, u_n - p\rangle \\
&= \frac{1}{2}\Big\{ \|(I - r_n A_1)x_n - (I - r_n A_1)p\|^2 + \|u_n - p\|^2 \\
&\quad - \|(I - r_n A_1)x_n - (I - r_n A_1)p - (u_n - p)\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(A_1 x_n - A_1 p)\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2r_n\|x_n - u_n\|\|A_1 x_n - A_1 p\| - r_n^2\|A_1 x_n - A_1 p\|^2 \Big\}.
\end{aligned} \tag{3.32}$$

So, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|A_1 x_n - A_1 p\|. \tag{3.33}$$

Observe that

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_E(x_n - \lambda_n B x_n) - P_E(p - \lambda_n B p)\|^2 \\
&\leq \langle (I - \lambda_n B)x_n - (I - \lambda_n B)p, y_n - p\rangle \\
&= \frac{1}{2}\Big\{ \|(I - \lambda_n B)x_n - (I - \lambda_n B)p\|^2 + \|y_n - p\|^2 \\
&\quad - \|(I - \lambda_n B)x_n - (I - \lambda_n B)p - (y_n - p)\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|y_n - p\|^2 - \|(x_n - y_n) - \lambda_n(B x_n - B p)\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 - \lambda_n^2\|B x_n - B p\|^2 \\
&\quad + 2\lambda_n\langle x_n - y_n, B x_n - B p\rangle \Big\},
\end{aligned} \tag{3.34}$$

and hence

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\|x_n - y_n\|\|B x_n - B p\|. \tag{3.35}$$

By using the same argument in (3.33) and (3.35), we can get

$$\begin{aligned} \|v_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - v_n\|^2 + 2s_n\|x_n - v_n\|\|A_2x_n - A_2p\|, \\ \|z_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\mu_n\|x_n - z_n\|\|Cx_n - Cp\|. \end{aligned} \tag{3.36}$$

Substituting (3.33), (3.35), and (3.36) into (3.25), we obtain

$$\begin{aligned} \|t_n - p\|^2 &\le \alpha_n^{(1)}\|x_n - p\|^2 + \alpha_n^{(2)}\|y_n - p\|^2 + \alpha_n^{(3)}\|z_n - p\|^2 \\ &\quad + \alpha_n^{(4)}\|u_n - p\|^2 + \alpha_n^{(5)}\|v_n - p\|^2 \\ &\le \alpha_n^{(1)}\|x_n - p\|^2 + \alpha_n^{(2)}\left\{\|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\|x_n - y_n\|\|Bx_n - Bp\|\right\} \\ &\quad + \alpha_n^{(3)}\left\{\|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\mu_n\|x_n - z_n\|\|Cx_n - Cp\|\right\} \\ &\quad + \alpha_n^{(4)}\left\{\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|A_1x_n - A_1p\|\right\} \\ &\quad + \alpha_n^{(5)}\left\{\|x_n - p\|^2 - \|x_n - v_n\|^2 + 2s_n\|x_n - v_n\|\|A_2x_n - A_2p\|\right\} \\ &= \|x_n - p\|^2 - \alpha_n^{(2)}\|x_n - y_n\|^2 + 2\lambda_n\alpha_n^{(2)}\|x_n - y_n\|\|Bx_n - Bp\| \\ &\quad - \alpha_n^{(3)}\|x_n - z_n\|^2 + 2\mu_n\alpha_n^{(3)}\|x_n - z_n\|\|Cx_n - Cp\| \\ &\quad - \alpha_n^{(4)}\|x_n - u_n\|^2 + 2r_n\alpha_n^{(4)}\|x_n - u_n\|\|A_1x_n - A_1p\| \\ &\quad - \alpha_n^{(5)}\|x_n - v_n\|^2 + 2s_n\alpha_n^{(5)}\|x_n - v_n\|\|A_2x_n - A_2p\|. \end{aligned} \tag{3.37}$$

It follows that

$$\begin{aligned} \alpha_n^{(4)}\|x_n - u_n\|^2 &\le \|x_n - p\|^2 - \|t_n - p\|^2 - \alpha_n^{(2)}\|x_n - y_n\|^2 + 2\lambda_n\alpha_n^{(2)}\|x_n - y_n\|\|Bx_n - Bp\| \\ &\quad - \alpha_n^{(3)}\|x_n - z_n\|^2 + 2\mu_n\alpha_n^{(3)}\|x_n - z_n\|\|Cx_n - Cp\| \\ &\quad + 2r_n\alpha_n^{(4)}\|x_n - u_n\|\|A_1x_n - A_1p\| - \alpha_n^{(5)}\|x_n - v_n\|^2 \\ &\quad + 2s_n\alpha_n^{(5)}\|x_n - v_n\|\|A_2x_n - A_2p\| \\ &\le (\|x_n - p\| + \|t_n - p\|)\|x_n - t_n\| + 2\lambda_n\alpha_n^{(2)}\|x_n - y_n\|\|Bx_n - Bp\| \\ &\quad + 2\mu_n\alpha_n^{(3)}\|x_n - z_n\|\|Cx_n - Cp\| + 2r_n\alpha_n^{(4)}\|x_n - u_n\|\|A_1x_n - A_1p\| \\ &\quad + 2s_n\alpha_n^{(5)}\|x_n - v_n\|\|A_2x_n - A_2p\|. \end{aligned} \tag{3.38}$$

From (C2), (3.20), (3.28), (3.30), and (3.31), we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.39}$$

By using the same argument, we can prove that

$$\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|x_n - z_n\| = \lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{3.40}$$

Applying (3.20), (3.39), and (3.40), we can obtain

$$\lim_{n\to\infty}\|t_n - u_n\| = \lim_{n\to\infty}\|t_n - y_n\| = \lim_{n\to\infty}\|t_n - z_n\| = \lim_{n\to\infty}\|t_n - v_n\| = 0. \tag{3.41}$$

Step 5. We show that

$$z \in F(S) \cap \mathrm{GMEP}(F_1, \varphi, A_1) \cap \mathrm{GMEP}(F_2, \varphi, A_2) \cap \mathrm{VI}(E, B) \cap \mathrm{VI}(E, C). \tag{3.42}$$

Assume that $\lambda_n \to \lambda \in [c, 2\beta]$ and $\mu_n \to \mu \in [d, 2\xi]$.
Define a mapping $\wp : E \to E$ by

$$\begin{aligned}\wp x &= \alpha^{(1)} S_k x + \alpha^{(2)} P_E(1 - \lambda B)x + \alpha^{(3)} P_E(1 - \mu C)x + \alpha^{(4)} T_r^{F_1}(I - rA_1)x \\ &\quad + \alpha^{(5)} T_s^{F_2}(I - sA_2)x, \quad \forall x \in E,\end{aligned} \tag{3.43}$$

where $\lim_{n\to\infty}\alpha_n^{(i)} = \alpha^{(i)} \in (0,1)$, when $i = 1,2,3,4,5$. By (C1), then we have $\sum_{i=1}^{5}\alpha_n^{(i)} = 1$. From Lemma 2.8, we have $\wp$ is nonexpansive and

$$\begin{aligned}F(\wp) &= F(S_k) \cap F(P_E(1 - \lambda B)) \cap F(P_E(1 - \mu C)) \cap F\left(T_r^{F_1}(I - rA_1)\right) \cap F\left(T_s^{F_2}(I - sA_2)\right) \\ &= F(S_k) \cap \mathrm{GMEP}(F_1, \varphi, A_1) \cap \mathrm{GMEP}(F_2, \varphi, A_2) \cap \mathrm{VI}(E, B) \cap \mathrm{VI}(E, C).\end{aligned} \tag{3.44}$$

We note that

$$\begin{aligned}
\|\wp x_n - x_n\| &\le \|\wp x_n - t_n\| + \|t_n - x_n\| \\
&= \Big\| \Big[\alpha^{(1)} S_k x_n + \alpha^{(2)} P_E (1 - \lambda B) x_n + \alpha^{(3)} P_E (1 - \mu C) x_n \\
&\qquad + \alpha^{(4)} T_r^{F_1} (I - r A_1) x_n + \alpha^{(5)} T_s^{F_2} (I - s A_2) x_n \Big] \\
&\qquad - \Big[\alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} P_E (1 - \lambda_n B) x_n + \alpha_n^{(3)} P_E (1 - \mu_n C) x_n \\
&\qquad + \alpha_n^{(4)} T_r^{F_1} (I - r A_1) x_n + \alpha_n^{(5)} T_s^{F_2} (I - s A_2) x_n \Big] \Big\| + \|t_n - x_n\| \\
&\le \left| \alpha^{(1)} - \alpha_n^{(1)} \right| \|S_k x_n\| + \alpha^{(2)} \|P_E (I - \lambda B) x_n - P_E (I - \lambda_n B) x_n\| \\
&\quad + \left| \alpha^{(2)} - \alpha_n^{(2)} \right| \|P_E (I - \lambda_n B) x_n\| \\
&\quad + \alpha^{(3)} \|P_E (I - \mu C) x_n - P_E (I - \mu_n C) x_n\| + \left| \alpha^{(3)} - \alpha_n^{(3)} \right| \|P_E (I - \mu_n C) x_n\| \\
&\quad + \left| \alpha^{(4)} - \alpha_n^{(4)} \right| \left\| T_r^{F_1} (I - r A_1) x_n \right\| + \left| \alpha^{(5)} - \alpha_n^{(5)} \right| \left\| T_s^{F_2} (I - s A_2) x_n \right\| + \|t_n - x_n\| \\
&\le \left| \alpha^{(1)} - \alpha_n^{(1)} \right| \|S_k x_n\| + \alpha^{(2)} |\lambda_n - \lambda| \|B x_n\| + \left| \alpha^{(2)} - \alpha_n^{(2)} \right| \|P_E (I - \lambda_n B) x_n\| \\
&\quad + \alpha^{(3)} |\mu_n - \mu| \|C x_n\| + \left| \alpha^{(3)} - \alpha_n^{(3)} \right| \|P_E (I - \mu_n C) x_n\| \\
&\quad + \left| \alpha^{(4)} - \alpha_n^{(4)} \right| \left\| T_r^{F_1} (I - r A_1) x_n \right\| + \left| \alpha^{(5)} - \alpha_n^{(5)} \right| \left\| T_s^{F_2} (I - s A_2) x_n \right\| + \|t_n - x_n\| \\
&\le K_1 \left(\sum_{i=1}^{5} \left| \alpha^{(i)} - \alpha_n^{(i)} \right| + |\lambda_n - \lambda| + |\mu_n - \mu| \right) + \|t_n - x_n\|,
\end{aligned} \tag{3.45}$$

where K_1 is an appropriate constant such that

$$\begin{aligned}
K_1 = \max \Big\{ &\sup_{n \ge 1} \left\| T_r^{F_1} (I - r A_1) x_n \right\|, \ \sup_{n \ge 1} \left\| T_s^{F_2} (I - s A_2) x_n \right\|, \ \sup_{n \ge 1} \|P_E (I - \lambda_n B) x_n\|, \\
&\sup_{n \ge 1} \|P_E (I - \mu_n C) x_n\|, \ \sup_{n \ge 1} \|B x_n\|, \ \sup_{n \ge 1} \|C x_n\|, \ \sup_{n \ge 1} \|S_k x_n\| \Big\}.
\end{aligned} \tag{3.46}$$

From (C2), (C5), and (3.20), we obtain

$$\lim_{n \to \infty} \|x_n - \wp x_n\| = 0. \tag{3.47}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup z$. It follows from (3.47), that

$$\lim_{n\to\infty} \|x_{n_i} - \wp x_{n_i}\| = 0. \tag{3.48}$$

It follows from Lemma 2.4 that $z \in F(\wp)$. By (3.44), we have $z \in \Theta$.

Step 6. Finally, we show that $x_n \to z$, where $z = P_\Theta x_0$.

Since Θ is nonempty closed convex subset of H, there exists a unique $z' \in \Theta$ such that $z' = P_\Theta x_0$. Since $z' \in \Theta \subset E_n$ and $x_n = P_{E_n} x_0$, we have

$$\|x_0 - x_n\| = \|x_0 - P_{E_n} x_0\| \le \|x_0 - z'\| \tag{3.49}$$

for all $n \ge 1$. From (3.49), $\{x_n\}$ is bounded, so $\omega_w(x_n) \neq \emptyset$. By the weak lower semicontinuity of the norm, we have

$$\|x_0 - z\| \le \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \le \|x_0 - z'\|. \tag{3.50}$$

Since $z \in \omega_w(x_n) \subset \Theta$, we obtain

$$\|x_0 - z'\| = \|x_0 - P_\Theta x_0\| \le \|x_0 - z\|. \tag{3.51}$$

Using (3.49) and (3.50), we obtain $z' = z$. Thus, $\omega_w(x_n) = \{z\}$ and $x_n \rightharpoonup z$. So we have

$$\|x_0 - z'\| \le \|x_0 - z\| \le \liminf_{i\to\infty} \|x_0 - x_n\| \le \limsup_{i\to\infty} \|x_0 - x_n\| \le \|x_0 - z'\|. \tag{3.52}$$

Thus,

$$\|x_0 - z\| = \lim_{i\to\infty} \|x_0 - x_n\| = \|x_0 - z'\|. \tag{3.53}$$

From $x_n \rightharpoonup z$, we obtain $(x_0 - x_n) \rightharpoonup (x_0 - z)$. Using Lemma 2.5, we obtain that

$$\|x_n - z\| = \|(x_n - x_0) - (z - x_0)\| \longrightarrow 0 \tag{3.54}$$

as $n \to \infty$ and hence $x_n \to z$ in norm. This completes the proof. □

If the mapping S is nonexpansive, then $S_k = S_0 = S$. We can obtain the following result from Theorem 3.1 immediately.

Corollary 3.2. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $E \times E$ to $\mathbb{R}$ satisfying (A1)–(A5), and let $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let A_1, A_2, B, C be four ρ, ω, β,*

ξ-inverse-strongly monotone mappings of E into H, respectively. Let $S : E \to E$ be a nonexpansive mapping with a fixed point. Suppose that

$$\Theta := F(S) \cap \mathrm{GMEP}(F_1, \varphi, A_1) \cap \mathrm{GMEP}(F_2, \varphi, A_2) \cap \mathrm{VI}(E, B) \cap \mathrm{VI}(E, C) \neq \emptyset. \tag{3.55}$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm (3.1)*, where $\{\alpha_n^{(i)}\}$ are sequences in $(0,1)$, where $i = 1,2,3,4,5$, $r_n \in (0, 2\rho)$, $s_n \in (0, 2\omega)$, and $\{\lambda_n\}$, $\{\mu_n\}$ are positive sequences. Assume that the control sequences satisfy (C1)–(C5) in Theorem 3.1. Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.*

If $\varphi = 0$ and $A_1 = A_2 = 0$ in Theorem 3.1, then we can obtain the following result immediately.

Corollary 3.3. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $E \times E$ to $\mathbb{R}$ satisfying (A1)–(A5), and let $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let B, C be two β, ξ-inverse-strongly monotone mappings of E into H, respectively. Let $S : E \to E$ be a nonexpansive mapping with a fixed point. Suppose that*

$$\Theta := F(S) \cap \mathrm{EP}(F_1) \cap \mathrm{EP}(F_2) \cap \mathrm{VI}(E, B) \cap \mathrm{VI}(E, C) \neq \emptyset. \tag{3.56}$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{gathered}
x_0 \in H, \quad E_1 = E, \quad x_1 = P_{E_1} x_0, \quad u_n \in E, \quad v_n \in E, \\
F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in E, \\
F_2(v_n, v) + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in E, \\
z_n = P_E(x_n - \mu_n C x_n), \\
y_n = P_E(x_n - \lambda_n B x_n), \\
t_n = \alpha_n^{(1)} S x_n + \alpha_n^{(2)} y_n + \alpha_n^{(3)} z_n + \alpha_n^{(4)} u_n + \alpha_n^{(5)} v_n, \\
E_{n+1} = \{w \in E_n : \|t_n - w\| \leq \|x_n - w\|\}, \\
x_{n+1} = P_{E_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered} \tag{3.57}$$

where $\{\alpha_n^{(i)}\}$ are sequences in (0,1), where $i = 1,2,3,4,5$, $r_n \in (0, \infty)$, $s_n \in (0, \infty)$ and $\{\lambda_n\}$, $\{\mu_n\}$ are positive sequences. Assume that the control sequences satisfy the condition (C1)–(C5) in Theorem 3.1. Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

If $B = 0$, $C = 0$, and $F_1(u_n, u) = F_1(v_n, v) = 0$ in Corollary 3.3, then $P_E = I$ and we get $u_n = y_n = x_n$ and $v_n = z_n = x_n$; hence, we can obtain the following result immediately.

Corollary 3.4. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $S : E \to E$ be a k-strictly pseudocontractive mapping with a fixed point. Define a mapping $S_k : E \to E$ by $S_k x = kx + (1-k)Sx$, for all $x \in E$. Suppose that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following* *iterative algorithm:*

$$\begin{aligned} &x_0 \in H, \quad E_1 = E, \quad x_1 = P_{E_1} x_0, \\ &t_n = \alpha_n S_k x_n + (1 - \alpha_n) x_n, \\ &E_{n+1} = \{ w \in E_n : \| t_n - w \| \le \| x_n - w \| \}, \\ &x_{n+1} = P_{E_{n+1}} x_0, \quad \forall n \ge 1, \end{aligned} \tag{3.58}$$

where $\{\alpha_n\}$ *are sequences in* $(0,1)$*. Assume that the control sequences satisfy the condition* $\lim_{n\to\infty}\alpha_n = \alpha \in (0,1)$ *in Theorem 3.1. Then,* $\{x_n\}$ *converges strongly to a point* $P_{F(S)}x_0$*.*

4. Convex Feasibility Problem

Finally, we consider the following *Convex Feasibility Problem* (CFP): finding an $x \in \bigcap_{j=1}^{M} C_j$, where $M \ge 1$ is an integer and each C_i is assumed to be the solutions of equilibrium problem with the bifunction F_j, $j = 1,2,3,\ldots,M$ and the solution set of the variational inequality problem. There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [35, 36], computer tomography [37], and radiation therapy treatment planning [38].

The following result can be obtained from Theorem 3.1. We, therefore, omit the proof.

Theorem 4.1. *Let* E *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $\{F_j\}_{j=1}^{M}$ *be a family of bifunction from* $E \times E$ *to* $\mathbb{R}$ *satisfying (A1)–(A5), and let* $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function with either (B1) or (B2). Let* $A_j : E \to H$ *be* ρ_j*-inverse-strongly monotone mapping for each* $j \in \{1,2,3,\ldots,M\}$*. Let* $B_i : E \to H$ *be* β_i*-inverse-strongly monotone mapping for each* $i \in \{1,2,3,\ldots,N\}$*. Let* $S : E \to E$ *be a* k*-strictly pseudocontractive mapping with a fixed point. Define a mapping* $S_k : E \to E$ *by* $S_k x = kx + (1-k)Sx$*, for all* $x \in E$*. Suppose that*

$$\Theta := F(S) \cap \left(\bigcap_{j=1}^{M} \mathrm{GMEP}(F_j, \varphi, A_j) \right) \cap \left(\bigcap_{i=1}^{N} \mathrm{VI}(E, B_i) \right) \ne \emptyset. \tag{4.1}$$

Let $\{x_n\}$ *be a sequence generated by the following iterative algorithm:*

$$\begin{aligned} &x_0 \in H, \quad E_1 = E, \quad x_1 = P_{E_1} x_0, \quad v_1, v_2, \ldots, v_M \in E, \\ &F_1(v_{n,1}, v_1) + \varphi(v_1) - \varphi(v_{n,1}) + \langle A_1 x_n, v_1 - v_{n,1} \rangle + \frac{1}{r_1} \langle v_1 - v_{n,1}, v_{n,1} - x_n \rangle \ge 0, \quad \forall v_1 \in E, \end{aligned}$$

$$F_2(v_{n,2}, v_2) + \varphi(v_2) - \varphi(v_{n,2}) + \langle A_2 x_n, v_2 - v_{n,2}\rangle + \frac{1}{r_2}\langle v_2 - v_{n,2}, v_{n,2} - x_n\rangle \geq 0, \quad \forall v_2 \in E,$$

$$\vdots$$

$$F_M(v_{n,M}, v_M) + \varphi(v_M) - \varphi(v_{n,M}) + \langle A_M x_n, v_M - v_{n,M}\rangle$$
$$+\frac{1}{r_M}\langle v_M - v_{n,M}, v_{n,M} - x_n\rangle \geq 0, \quad \forall v_M \in E,$$
$$y_{n,1} = P_E(x_n - \lambda_{n,1} B_1 x_n),$$
$$y_{n,2} = P_E(x_n - \lambda_{n,2} B_2 x_n),$$
$$\vdots$$
$$y_{n,N} = P_E(x_n - \lambda_{n,N} B_N x_n),$$
$$t_n = \alpha_{n,0} S_k x_n + \sum_{i=1}^{N} \alpha_{n,i} y_{n,i} + \sum_{j=1}^{M} \alpha'_{n,j} v_{n,j},$$
$$E_{n+1} = \{w \in E_n : \|t_n - w\| \leq \|x_n - w\|\},$$
$$x_{n+1} = P_{E_{n+1}} x_0, \quad \forall n \geq 1, \tag{4.2}$$

where $\alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,N}$ *and* $\alpha'_{n,1}, \alpha'_{n,2}, \dots, \alpha'_{n,M} \in (0,1)$ *such that* $\sum_{i=0}^{N} \alpha_{n,i} + \sum_{j=1}^{M} \alpha'_{n,j} = 1$, $\{\lambda_{n,i}\}$ *are positive sequences in* $(0,1)$. *Assume that the control sequences satisfy the following restrictions:*

(C1) $\lim_{n\to\infty} \alpha_n^{(i)} = \alpha^{(i)} \in (0,1)$, *for each* $0 \leq i \leq N$,

(C2) $\lim_{n\to\infty} \alpha_n'^{(j)} = \alpha'^{(j)} \in (0,1)$, *for each* $1 \leq j \leq M$,

(C3) $a_j \leq r_j \leq 2\rho_j$, *where* a_j *is some positive constants for each* $1 \leq j \leq M$,

(C4) $c_i \leq \lambda_{n,i} \leq 2\beta_i$, *where* c_i *is some positive constants for each* $1 \leq i \leq N$,

(C5) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$, *for each* $1 \leq i \leq N$.

Then, $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

If $A_j = 0$, for each $1 \leq j \leq M$ and $F_i(v_{n,i}, v_i) = 0$, for each $1 \leq i \leq N$ in Theorem 4.1, then $v_{n,i} = x_n$; hence, we can obtain the following result immediately.

Theorem 4.2. *Let* E *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function with either (B1) or (B2). Let* $B_i : E \to H$ *be* β_i*-inverse-strongly monotone mapping for each* $i \in \{1, 2, 3, \dots, N\}$. *Let* $S : E \to E$ *be a* k*-strictly pseudocontractive mapping with a fixed point. Define a mapping* $S_k : E \to E$ *by* $S_k x = kx + (1-k)Sx$, *for all* $x \in E$. *Suppose that*

$$\Theta := F(S) \cap \left(\bigcap_{i=1}^{N} \mathrm{VI}(E, B_i)\right) \neq \emptyset. \tag{4.3}$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{aligned}
x_0 &\in H, \quad E_1 = E, \quad x_1 = P_{E_1}x_0,\\
y_{n,1} &= P_E(x_n - \lambda_{n,1}B_1x_n),\\
y_{n,2} &= P_E(x_n - \lambda_{n,2}B_2x_n),\\
&\vdots\\
y_{n,N} &= P_E(x_n - \lambda_{n,N}B_Nx_n),\\
t_n &= \alpha_{n,0}S_kx_n + \sum_{i=1}^{N}\alpha_{n,i}y_{n,i},\\
E_{n+1} &= \{w \in E_n : \|t_n - w\| \le \|x_n - w\|\},\\
x_{n+1} &= P_{E_{n+1}}x_0, \quad \forall n \ge 1,
\end{aligned} \tag{4.4}$$

where $\alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,N} \in (0,1)$ such that $\sum_{i=0}^{N} \alpha_{n,i} = 1$, $\{\lambda_{n,i}\}$ are positive sequences in $(0,1)$. Assume that the control sequences satisfy the following restrictions:

(C1) $\lim_{n\to\infty}\alpha_n^{(i)} = \alpha^{(i)} \in (0,1)$, *for each* $0 \le i \le N$,

(C2) $c_i \le \lambda_{n,i} \le 2\beta_i$, *where c_i is some positive constants for each* $1 \le i \le N$,

(C3) $\lim_{n\to\infty}|\lambda_{n+1,i} - \lambda_{n,i}| = 0$, *for each* $1 \le i \le N$.

Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

If $B_i = 0$, for each $1 \le i \le N$ in Theorem 4.1, then we get $y_{n,i} = x_n$. Hence, we can obtain the following result immediately.

Theorem 4.3. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let be a $\{F_j\}_{j=1}^{M}$ be a family of bifunction from $E\times E$ to $\mathbb{R}$ satisfying (A1)–(A5), and let $\varphi : E \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let $A_j : E \to H$ be ρ_j-inverse-strongly monotone mapping for each $j \in \{1,2,3,\ldots,M\}$. Let $S : E \to E$ be a k-strictly pseudocontractive mapping with a fixed point. Define a mapping $S_k : E \to E$ by $S_kx = kx + (1-k)Sx$, for all $x \in E$. Suppose that*

$$\Theta := F(S) \cap \left(\bigcap_{j=1}^{M} \mathrm{GMEP}(F_j, \varphi, A_j)\right) \ne \emptyset. \tag{4.5}$$

Let $\{x_n\}$ *be a sequence generated by the following iterative algorithm:*

$$\begin{gathered}
x_0 \in H, \quad E_1 = E, \quad x_1 = P_{E_1}x_0, \quad v_1, v_2, \ldots, v_M \in E, \\
F_1(v_{n,1}, v_1) + \varphi(v_1) - \varphi(v_{n,1}) + \langle A_1 x_n, v_1 - v_{n,1} \rangle + \frac{1}{r_1}\langle v_1 - v_{n,1}, v_{n,1} - x_n \rangle \geq 0, \quad \forall v_1 \in E, \\
F_2(v_{n,2}, v_2) + \varphi(v_2) - \varphi(v_{n,2}) + \langle A_2 x_n, v_2 - v_{n,2} \rangle + \frac{1}{r_2}\langle v_2 - v_{n,2}, v_{n,2} - x_n \rangle \geq 0, \quad \forall v_2 \in E, \\
\vdots \\
F_M(v_{n,M}, v_M) + \varphi(v_M) - \varphi(v_{n,M}) + \langle A_M x_n, v_M - v_{n,M} \rangle \\
+ \frac{1}{r_M}\langle v_M - v_{n,M}, v_{n,M} - x_n \rangle \geq 0, \quad \forall v_M \in E, \\
t_n = \alpha_{n,0} S_k x_n + \sum_{j=1}^{M} \alpha'_{n,j} v_{n,j}, \\
E_{n+1} = \{ w \in E_n : \|t_n - w\| \leq \|x_n - w\| \}, \\
x_{n+1} = P_{E_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered} \tag{4.6}$$

where $\alpha_{n,0}$ *and* $\alpha'_{n,1}, \alpha'_{n,2}, \ldots, \alpha'_{n,M} \in (0,1)$ *such that* $\alpha_{n,0} + \sum_{j=1}^{M} \alpha'_{n,j} = 1$. *Assume that the control sequences satisfy the following restrictions:*

(C1) $\lim_{n\to\infty} \alpha_n^{(0)} = \alpha^{(0)} \in (0,1)$,

(C2) $\lim_{n\to\infty} \alpha_n^{\prime(j)} = \alpha^{\prime(j)} \in (0,1)$, *for each* $1 \leq j \leq M$,

(C3) $a_j \leq r_j \leq 2\rho_j$, *where* a_j *is some positive constants for each* $1 \leq j \leq M$.

Then, $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

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