Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2011, Article ID 686834, 9 pages
doi:10.1155/2011/686834

Research Article

The Optimal Convex Combination Bounds for Seiffert's Mean

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Received 28 November 2010; Accepted 28 February 2011

Academic Editor: P. Y. H. Pang

We derive some optimal convex combination bounds related to Seiffert's mean. We find the greatest values α_1, α_2 and the least values β_1, β_2 such that the double inequalities $\alpha_1 C(a,b) + (1-\alpha_1)G(a,b) < P(a,b) < \beta_1 C(a,b) + (1-\beta_1)G(a,b)$ and $\alpha_2 C(a,b) + (1-\alpha_2)H(a,b) < P(a,b) < \beta_2 C(a,b) + (1-\beta_2)H(a,b)$ hold for all $a, b > 0$ with $a \neq b$. Here, $C(a,b)$, $G(a,b)$, $H(a,b)$, and $P(a,b)$ denote the contraharmonic, geometric, harmonic, and Seiffert's means of two positive numbers a and b, respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Seiffert't mean $P(a,b)$ was introduced by Seiffert [1] as follows:

$$P(a,b) = \frac{a-b}{4\arctan\left(\sqrt{a/b}\right) - \pi}. \tag{1.1}$$

Recently, the inequalities for means have been the subject of intensive research. In particular, many remarkable inequalities for P can be found in the literature [2–6]. Seiffert's mean P can be rewritten as (see [5, equation (2.4)])

$$P(a,b) = \frac{a-b}{2\arcsin((a-b)/(a+b))}. \tag{1.2}$$

Let $C(a,b) = (a^2+b^2)/(a+b)$, $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the contraharmonic, arithmetic, geometric and harmonic means of two positive real numbers a and b with $a \neq b$. Then

$$\min\{a,b\} < H(a,b) < G(a,b) < P(a,b) < A(a,b) < C(a,b) < \max\{a,b\}. \tag{1.3}$$

In [7], Seiffert proved that

$$P(a,b) > \frac{3A(a,b)G(a,b)}{A(a,b)+2G(a,b)}, \qquad P(a,b) > \frac{2}{\pi}A(a,b), \tag{1.4}$$

for all $a,b > 0$ with $a \neq b$.

In [8], the authors found the greatest value α and the least value β such that the double inequality

$$\alpha A(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta A(a,b) + (1-\beta)H(a,b) \tag{1.5}$$

holds for all $a,b > 0$ with $a \neq b$.

For more results, see [9–23].

The purpose of the present paper is to find the greatest values α_1, α_2 and the least values β_1, β_2 such that the double inequalities

$$\begin{gathered} \alpha_1 C(a,b) + (1-\alpha_1)G(a,b) < P(a,b) < \beta_1 C(a,b) + (1-\beta_1)G(a,b), \\ \alpha_2 C(a,b) + (1-\alpha_2)H(a,b) < P(a,b) < \beta_2 C(a,b) + (1-\beta_2)H(a,b) \end{gathered} \tag{1.6}$$

hold for all a, $b > 0$ with $a \neq b$.

2. Main Results

Firstly, we present the optimal convex combination bounds of contraharmonic and geometric means for Seiffert's mean as follows.

Theorem 2.1. *The double inequality $\alpha_1 C(a,b) + (1-\alpha_1)G(a,b) < P(a,b) < \beta_1 C(a,b) + (1-\beta_1)G(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leqslant 2/9$ and $\beta_1 \geqslant 1/\pi$.*

Proof. Firstly, we prove that

$$\begin{gathered} P(a,b) < \frac{1}{\pi}C(a,b) + \left(1-\frac{1}{\pi}\right)G(a,b), \\ P(a,b) > \frac{2}{9}C(a,b) + \frac{7}{9}G(a,b), \end{gathered} \tag{2.1}$$

for all $a,b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = \sqrt{a/b} > 1$ and $p \in \{2/9, 1/\pi\}$. Then (1.1) leads to

$$\begin{aligned}
&\{P(a,b) - [pC(a,b) + (1-p)G(a,b)]\} \\
&\quad = bP\left(t^2,1\right) - b\left[pC\left(t^2,1\right) + (1-p)G\left(t^2,1\right)\right] \\
&\quad = \frac{b\left[pt^4 + (1-p)t^3 + (1-p)t + p\right]}{(t^2+1)(4\arctan t - \pi)} f(t),
\end{aligned} \tag{2.2}$$

where

$$f(t) = \frac{(t^4-1)}{pt^4 + (1-p)t^3 + (1-p)t + p} - 4\arctan t + \pi. \tag{2.3}$$

Simple computations lead to

$$\begin{gathered}
\lim_{t\to 1^+} f(t) = 0, \qquad \lim_{t\to+\infty} f(t) = \frac{1}{p} - \pi, \\
f'(t) = \frac{(t-1)^2}{(t^2+1)\left[pt^4 + (1-p)t^3 + (1-p)t + p\right]^2} g(t),
\end{gathered} \tag{2.4}$$

where

$$\begin{aligned}
g(t) = &- \left(4p^2 + p - 1\right)t^6 - 2(5p-1)t^5 - 3(5p-1)t^4 \\
&+ 4\left(2p^2 - 5p + 1\right)t^3 - 3(5p-1)t^2 \\
&- 2(5p-1)t - 4p^2 - p + 1.
\end{aligned} \tag{2.5}$$

We divide the proof into two cases.

Case 1 ($p = 2/9$). In this case,

$$g(t) = \frac{1}{81}\left(47t^4 + 76t^3 + 78t^2 + 76t + 47\right)(t-1)^2 > 0, \quad \text{for } t > 1. \tag{2.6}$$

Therefore, the second inequality in (2.1) follows from (2.2)–(2.6). Notice that in this case, the second equality in (2.4) becomes

$$\lim_{t\to+\infty} f(t) = \frac{9}{2} - \pi > 0. \tag{2.7}$$

Case 2 ($p = 1/\pi$). From (2.5), we have that

$$g(1) = 8(2-9p) = 8\left(2-\frac{9}{\pi}\right) < 0, \qquad \lim_{t\to+\infty} g(t) = +\infty, \tag{2.8}$$

$$\begin{aligned} g'(t) = &-6\left(4p^2+p-1\right)t^5 - 10(5p-1)t^4 - 12(5p-1)t^3 \\ &+ 12\left(2p^2-5p+1\right)t^2 - 6(5p-1)t - 10p + 2 \end{aligned} \tag{2.9}$$

$$g'(1) = 24(2-9p) = 24\left(2-\frac{9}{\pi}\right) < 0, \qquad \lim_{t\to+\infty} g'(t) = +\infty, \tag{2.10}$$

$$\begin{aligned} g''(t) = &-30\left(4p^2+p-1\right)t^4 - 40(5p-1)t^3 - 36(5p-1)t^2 \\ &+ 24\left(2p^2-5p+1\right)t - 30p + 6, \end{aligned} \tag{2.11}$$

$$g''(1) = 8\left(17-70p-9p^2\right) = 8\left(17-\frac{70}{\pi}-\frac{9}{\pi^2}\right) < 0, \qquad \lim_{t\to+\infty} g''(t) = +\infty, \tag{2.12}$$

$$\begin{aligned} g'''(t) = &-120\left(4p^2+p-1\right)t^3 - 120(5p-1)t^2 - 72(5p-1)t \\ &+ 48p^2 - 120p + 24, \end{aligned} \tag{2.13}$$

$$g'''(1) = 48\left(7-25p-9p^2\right) = 48\left(7-\frac{25}{\pi}-\frac{9}{\pi^2}\right) < 0, \qquad \lim_{t\to+\infty} g'''(t) = +\infty, \tag{2.14}$$

$$g^{(4)}(t) = -360\left(4p^2+p-1\right)t^2 - 240(5p-1)t - 360p + 72, \tag{2.15}$$

$$g^{(4)}(1) = 96\left(7-20p-15p^2\right) = 96\left(7-\frac{20}{\pi}-\frac{15}{\pi^2}\right) < 0, \qquad \lim_{t\to+\infty} g'(t) = +\infty, \tag{2.16}$$

$$g^{(5)}(t) = -720\left(4p^2+p-1\right)t - 1200p + 240, \tag{2.17}$$

$$g^{(5)}(1) = 960\left(1-2p-3p^2\right) = 960\left(1-\frac{2}{\pi}-\frac{3}{\pi^2}\right) > 0. \tag{2.18}$$

From (2.17) and (2.18), we clearly see that $g^{(5)}(t) > 0$ for $t \geq 1$; hence $g^{(4)}(t)$ is strictly increasing in $[1,+\infty)$, which together with (2.16) implies that there exists $\lambda_1 > 1$ such that $g^{(4)}(t) < 0$ for $t \in [1,\lambda_1)$ and $g^{(4)}(t) > 0$ for $t \in (\lambda_1,+\infty)$; and hence $g'''(t)$ is strictly decreasing in $[1,\lambda_1]$ and strictly increasing for $[\lambda_1,+\infty)$. From (2.14) and the monotonicity of $g'''(t)$, there exists $\lambda_2 > 1$ such that $g'''(t) < 0$ for $t \in [1,\lambda_2)$ and $g'''(t) > 0$ for $t \in (\lambda_2,+\infty)$; hence $g''(t)$ is strictly decreasing in $[1,\lambda_2]$ and strictly increasing for $[\lambda_2,+\infty)$. As this goes on, there exists $\lambda_3 > 1$ such that $f(t)$ is strictly decreasing in $[1,\lambda_3]$ and strictly increasing in $[\lambda_3,+\infty)$. Note that if $p = 1/\pi$, then the second equality in (2.4) becomes

$$\lim_{t\to+\infty} f(t) = 0. \tag{2.19}$$

Thus $f(t) < 0$ for all $t > 1$. Therefore, the first inequality in (2.1) follows from (2.2) and (2.3).

Secondly, we prove that $2/9C(a,b) + 7/9G(a,b)$ is the best possible lower convex combination bound of the contraharmonic and geometric means for Seiffert's mean.

If $\alpha_1 > 2/9$, then (2.5) (with α_1 in place of p) leads to

$$g(1) = 8(2-9\alpha_1) < 0. \tag{2.20}$$

From this result and the continuity of $g(t)$ we clearly see that there exists $\delta = \delta(\alpha_1) > 0$ such that $g(t) < 0$ for $t \in (1, 1+\delta)$. Then the last equality in (2.4) implies that $f'(t) < 0$ for $t \in (1, 1+\delta)$. Thus $f(t)$ is decreasing for $t \in (1, 1+\delta)$. Due to (2.4), $f(t) < 0$ for $t \in (1, 1+\delta)$, which is equivalent to, by (2.2),

$$P\left(t^2, 1\right) < \alpha_1 C\left(t^2, 1\right) + (1-\alpha_1) G\left(t^2, 1\right), \tag{2.21}$$

for $t \in (1, 1+\delta)$.

Finally, we prove that $1/\pi C(a,b) + (1-1/\pi)G(a,b)$ is the best possible upper convex combination bound of the contraharmonic and geometric means for Seiffert's mean.

If $\beta_1 < 1/\pi$, then from (1.1) one has

$$\begin{aligned} &\lim_{t\to+\infty} \frac{\beta_1 C(t^2,1) + (1-\beta_1)G(t^2,1)}{P(t^2,1)} \\ &\quad = \lim_{t\to+\infty} \frac{\left[\beta_1 t^4 + (1-\beta_1)t^3 + (1-\beta_1)t + \beta_1\right](4\arctan t - \pi)}{t^4 - 1} = \beta_1 \pi < 1. \end{aligned} \tag{2.22}$$

Inequality (2.22) implies that for any $\beta_1 < 1/\pi$ there exists $X = X(\beta_1) > 1$ such that

$$\beta_1 C\left(t^2, 1\right) + (1-\beta_1) G\left(t^2, 1\right) < P\left(t^2, 1\right) \tag{2.23}$$

for $t \in (X, +\infty)$. □

Secondly, we present the optimal convex combination bounds of the contraharmonic and harmonic means for Seiffert's mean as follows.

Theorem 2.2. *The double inequality* $\alpha_2 C(a,b) + (1-\alpha_2)H(a,b) < P(a,b) < \beta_2 C(a,b) + (1-\beta_2)H(a,b)$ *holds for all* $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_2 \leqslant 1/\pi$ *and* $\beta_2 \geqslant 5/12$.

Proof. Firstly, we prove that

$$\begin{aligned} P(a,b) &< \frac{5}{12}C(a,b) + \frac{7}{12}H(a,b), \\ P(a,b) &> \frac{1}{\pi}C(a,b) + \left(1 - \frac{1}{\pi}\right)H(a,b), \end{aligned} \tag{2.24}$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = \sqrt{a/b} > 1$ and $p \in \{1/\pi, 5/12\}$. Then (1.1) leads to

$$\begin{aligned}
&\{P(a,b) - [pC(a,b) + (1-p)H(a,b)]\} \\
&\quad = bP\left(t^2,1\right) - b\left[pC\left(t^2,1\right) + (1-p)H\left(t^2,1\right)\right] \\
&\quad = \frac{b\left[pt^4 + 2(1-p)t^2 + p\right]}{(t^2+1)(4\arctan t - \pi)} f(t),
\end{aligned} \tag{2.25}$$

where

$$f(t) = \frac{(t^4-1)}{pt^4 + 2(1-p)t^2 + p} - 4\arctan t + \pi. \tag{2.26}$$

Simple computations lead to

$$\begin{aligned}
&\lim_{t\to 1^+} f(t) = 0, \qquad \lim_{t\to+\infty} f(t) = \frac{1}{p} - \pi, \\
&f'(t) = \frac{4(t-1)^2}{(t^2+1)\left[pt^4 + 2(1-p)t^2 + p\right]^2} g(t),
\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}
g(t) = {}& -p^2t^6 + \left(-2p^2 - p + 1\right)t^5 + \left(p^2 - 6p + 2\right)t^4 \\
& + 2\left(2p^2 - 5p + 2\right)t^3 + \left(p^2 - 6p + 2\right)t^2 + \left(-2p^2 - p + 1\right)t - p^2.
\end{aligned} \tag{2.28}$$

We divide the proof into two cases.

Case 1 ($p = 5/12$). In this case,

$$g(t) = -\frac{1}{144}\left(25t^4 + 16t^3 + 54t^2 + 16t + 25\right)(t-1)^2 < 0, \quad \text{for } t > 1. \tag{2.29}$$

Therefore, the first inequality in (2.24) follows from (2.25)–(2.29). Notice that in this case, the second equality in (2.27) becomes

$$\lim_{t\to+\infty} f(t) = \frac{12}{5} - \pi < 0. \tag{2.30}$$

Case 2 ($p = 1/\pi$). From (2.28) we have that

$$g(1) = 2(5 - 12p) = 2\left(5 - \frac{12}{\pi}\right) > 0, \qquad \lim_{t \to +\infty} g(t) = -\infty, \tag{2.31}$$

$$\begin{aligned} g'(t) = &-6p^2t^5 + 5\left(-2p^2 - p + 1\right)t^4 + 4\left(p^2 - 6p + 2\right)t^3 \\ &+ 6\left(2p^2 - 5p + 2\right)t^2 + 2\left(p^2 - 6p + 2\right)t - 2p^2 - p + 1, \end{aligned} \tag{2.32}$$

$$g'(t) = 6(5 - 12p) = 6\left(5 - \frac{12}{\pi}\right) > 0, \qquad \lim_{t \to +\infty} g'(t) = -\infty, \tag{2.33}$$

$$\begin{aligned} g''(t) = &-30p^2t^4 + 20\left(-2p^2 - p + 1\right)t^3 + 12\left(p^2 - 6p + 2\right)t^2 \\ &+ 12\left(2p^2 - 5p + 2\right)t + 2p^2 - 12p + 4, \end{aligned} \tag{2.34}$$

$$g''(t) = 4\left(18 - 41p - 8p^2\right) = 4\left(18 - \frac{41}{\pi} - \frac{8}{\pi^2}\right) > 0, \qquad \lim_{t \to +\infty} g''(t) = -\infty, \tag{2.35}$$

$$\begin{aligned} g'''(t) = &-120p^2t^3 + 60\left(-2p^2 - p + 1\right)t^2 + 24\left(p^2 - 6p + 2\right)t^2 \\ &+ 24p^2 - 60p + 24, \end{aligned} \tag{2.36}$$

$$g'''(1) = 12\left(11 - 22p - 16p^2\right) = 12\left(11 - \frac{22}{\pi} - \frac{16}{\pi^2}\right) > 0, \qquad \lim_{t \to +\infty} g'''(\mathrm{t}) = -\infty, \tag{2.37}$$

$$g^{(4)}(t) = -360p^2t^2 + 120\left(-2p^2 - p + 1\right)t + 24p^2 - 144p + 48. \tag{2.38}$$

$$g^{(4)}(1) = 24\left(7 - 11p - 24p^2\right) = 24\left(7 - \frac{11}{\pi} - \frac{24}{\pi^2}\right) > 0, \qquad \lim_{t \to +\infty} g'(t) = -\infty, \tag{2.39}$$

$$g^{(5)}(t) = -720p^2t - 240p^2 - 120p + 120, \tag{2.40}$$

$$g^{(5)}(1) = 120\left(1 - p - 8p^2\right) = 120\left(1 - \frac{1}{\pi} - \frac{8}{\pi^2}\right) < 0. \tag{2.41}$$

From (2.40) and (2.41) we clearly see that $g^{(5)}(t) < 0$ for $t \geq 1$; hence $g^{(4)}(t)$ is strictly decreasing in $[1, +\infty)$, which together with (2.39) implies that there exists $\lambda_4 > 1$ such that $g^{(4)}(t) > 0$ for $t \in [1, \lambda_4)$ and $g^{(4)}(t) < 0$ for $t \in (\lambda_4, +\infty)$, and hence $g'''(t)$ is strictly increasing in $[1, \lambda_4]$ and strictly decreasing for $[\lambda_1, +\infty)$. From (2.37) and the monotonicity of $g'''(t)$, there exists $\lambda_5 > 1$ such that $g'''(t) > 0$ for $t \in [1, \lambda_5)$ and $g'''(t) < 0$ for $t \in (\lambda_5, +\infty)$; hence $g''(t)$ is strictly increasing in $[1, \lambda_5]$ and strictly decreasing for $[\lambda_5, +\infty)$. As this goes on, there exists $\lambda_6 > 1$ such that $f(t)$ is strictly increasing in $[1, \lambda_6]$ and strictly decreasing in $[\lambda_6, +\infty)$. Notice that if $p = 1/\pi$, then the second equality in (2.27) becomes

$$\lim_{t \to +\infty} f(t) = 0. \tag{2.42}$$

Thus $f(t) > 0$ for all $t > 1$. Therefore, the second inequality in (2.24) follows from (2.25) and (2.26).

Secondly, we prove that $5/12C(a,b)+7/12H(a,b)$ is the best possible upper convex combination bound of the contraharmonic and harmonic means for Seiffert's mean.

If $\beta_2 < 5/12$, then (2.28) (with β_2 in place of p) leads to

$$g(1) = 2(5-12\beta_2) > 0. \tag{2.43}$$

From this result and the continuity of $g(t)$ we clearly see that there exists $\delta = \delta(\beta_2) > 0$ such that $g(t) > 0$ for $t \in (1, 1+\delta)$. Then the last equality in (2.27) implies that $f'(t) > 0$ for $t \in (1, 1+\delta)$. Thus $f(t)$ is increasing for $t \in (1, 1+\delta)$. Due to (2.27), $f(t) > 0$ for $t \in (1, 1+\delta)$, which is equivalent to, by (2.25),

$$P\left(t^2,1\right) > \beta_2 C\left(t^2,1\right) + (1-\beta_2)H\left(t^2,1\right), \tag{2.44}$$

for $t \in (1, 1+\delta)$.

Finally, we prove that $1/\pi C(a,b) + (1-1/\pi)H(a,b)$ is the best possible lower convex combination bound of the contraharmonic and harmonic means for Seiffert's mean.

If $\alpha_2 > 1/\pi$, then from (1.1) one has

$$\begin{aligned} \lim_{t\to+\infty} & \frac{\alpha_2 C(t^2,1) + (1-\alpha_2)H(t^2,1)}{P(t^2,1)} \\ &= \lim_{t\to+\infty} \frac{\left[\alpha_2 t^4 - 2(1-\alpha_2)t^2 + \alpha_2\right](4\arctan t - \pi)}{(t^2+1)(t^2-1)} = \alpha_2\pi > 1. \end{aligned} \tag{2.45}$$

Inequality (2.45) implies that for any $\alpha_2 > 1/\pi$ there exists $X = X(\alpha_2) > 1$ such that

$$\alpha_2 C\left(t^2,1\right) + (1-\alpha_2)H\left(t^2,1\right) > P\left(t^2,1\right) \tag{2.46}$$

for $t \in (X, +\infty)$. □

Acknowledgments

The authors wish to thank the anonymous referees for their very careful reading of the paper and fruitful comments and suggestions. This research is partly supported by N S Foundation of Hebei Province (Grant A2011201011), and the Youth Foundation of Hebei University (Grant 2010Q24).

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