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Research Article

A New Class of Sequences Related to the l_p Spaces Defined by Sequences of Orlicz Functions

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We introduce new sequence space $m(M, \phi, q, \Lambda)$ defined by combining an Orlicz function, seminorms, and λ-sequences. We study its different properties and obtain some inclusion relation involving the space $m(M, \phi, q, \lambda)$. Inclusion relation between statistical convergent sequence spaces and Cesaro statistical convergent sequence spaces is also given.

1. Introduction

By w, we denote the space of all real or complex valued sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=1}^{\infty}$. Also, we will use the conventions that $e = (1, 1, \ldots)$. Any vector subspace of w is called a sequence space. We will write l_∞, c, and c_0 for the sequence spaces of all bounded, convergent, and null sequences, respectively. Further, by l_p $(1 \leq p < \infty)$, we denote the sequence space of all p-absolutely convergent series, that is, $l_p = \{x = (x_k) \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \leq p < \infty$. Throughout the article, $w(X)$, $l_\infty(X)$, and $\mathrm{l}_p(X)$ denote, respectively, the spaces of all, bounded, and p-absolutely summable sequences with the elements in X, where (X, q) is a seminormed space. By $\theta = (0, 0, \ldots)$, we denote the zero element in X. P_s denotes the set of all subsets of $\mathbb{N}$, that do not contain more than s elements. With (ϕ_s), we will denote a nondecreasing sequence of positive real numbers such that $(s-1)\phi_{s-1} \leq (s-1)\phi_s$ and $\phi_s \to \infty$, as $s \to \infty$. The class of all the sequences (ϕ_s) satisfying this property is denoted by Φ.

In paper [1], the notion of λ-convergent and bounded sequences is introduced as follows: let $\lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive reals tending to infinity, that is

$$0 < \lambda_0 < \lambda_1 < \cdots, \qquad \lambda_k \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \tag{1.1}$$

We say that a sequence $x = (x_k) \in w$ is λ-convergent to the number $l \in \mathbb{C}$, called as the λ-limit of x, if $\Lambda_n(x) \rightarrow l$ as $n \rightarrow \infty$, where

$$\Lambda_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})x_k, \quad n \in \mathbb{N}. \tag{1.2}$$

In particular, we say that x is a λ-null sequence if $\Lambda_n(x) \rightarrow 0$ as $n \rightarrow \infty$. Further, we say that x is λ-bounded if $\sup|\Lambda_n(x)| < \infty$. Here and in the sequel, we will use the convention that any term with a negative subscript is equal to naught, for example, $\lambda_{-1} = 0$ and $x_{-1} = 0$. Now, it is well known [1] that if $\lim_n x_n = a$ in the ordinary sense of convergence, then

$$\lim_n\left(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})|x_k - a|\right) = 0. \tag{1.3}$$

This implies that

$$\lim_n|\Lambda_n(x) - a| = \lim_n\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(x_k - a)\right| = 0, \tag{1.4}$$

which yields that $\lim_n\Lambda_n(x) = a$ and hence x is λ-convergent to a. We therefore deduce that the ordinary convergence implies the λ-convergence to the same limit.

2. Definitions and Background

The space $m(\phi)$ introduced and studied by Sargent [2] is defined as follows:

$$m(\phi) = \left\{x = (x_i) \in s : \|x\|_{m(\phi)} = \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}|x_i| < \infty\right\}. \tag{2.1}$$

Sargent [2] studied some of its properties and obtained its relationship with the space l_p. Later on it was investigated from sequence space point of view by Rath [3], Rath and Tripathy [4], Tripathy and Sen [5], Tripathy and Mahanta [6], and others. Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to define the following sequence spaces:

$$l_M = \left\{x = (x_i) \in s : \sum_{i=1}^{\infty}M\left(\frac{|x_i|}{\varrho}\right) < \infty, \ \varrho > 0\right\}, \tag{2.2}$$

which is called an Orlicz sequence space. The space l_M is a Banach space with the norm

$$\|x\| = \inf\left\{\varrho > 0 : \sum_{i=1}^{\infty}M\left(\frac{|x_i|}{\varrho}\right) \leq 1\right\}. \tag{2.3}$$

The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$, $1 \leq p < \infty$. An Orlicz function is a function $M : (0,\infty] \rightarrow (0,\infty]$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda \cdot M(x)$ for all λ with $0 < \lambda \leq 1$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $L > 0$ such that $M(2u) \leq LM(u)$, $u \geq 0$ (see, Krasnoselskii and Rutitsky [8]). In the later stage, different Orlicz sequence spaces were introduced and studied by Bhardwaj and Singh [9], Güngör et al. [10], Tripathy and Mahanta [6], Esi [11], Esi and Et [12], Parashar and Choudhary [13], and many others.

The following inequality will be used throughout the paper,

$$|a_i + b_i|^{p_i} \leq \max\left(1, 2^{H-1}\right)\left(|a_i|^{p_i} + |b_i|^{p_i}\right), \tag{2.4}$$

where a_i and b_i are complex numbers, and $H = \sup p_i < \infty$, $h = \inf p_i$. Tripathy and Mahanta [6] defined and studied the following sequence space. Let M be an Orlicz function, then

$$m(M, \phi) = \left\{ x = (x_i) \in s : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{i \in \sigma} M\left(\frac{|x_i|}{\varrho}\right) < \infty, \text{ for some } \varrho > 0 \right\}. \tag{2.5}$$

Recently, Altun and Bilgin [14] defined and studied the following sequence spaces:

$$m(M, A, \phi, p) = \left\{ x = (x_i) \in s : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{i \in \sigma} M\left(\frac{|A_i(x)|}{\varrho}\right)^{p_i} < \infty, \text{ for some } \varrho > 0 \right\}, \tag{2.6}$$

where $A_i(x) = \sum_{k=1}^{\infty} a_{ik} x_k$ and converges for each i. In this paper, we will define the following sequence spaces:

$$m(M, \phi, q, \Lambda) = \left\{ x = (x_i) \in w : \lim_n \frac{1}{\phi_s} \sum_{n \in \sigma, \sigma \in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} = 0, \text{ for some } \varrho > 0 \right\}. \tag{2.7}$$

3. Results

Since the proofs of the following theorems are not hard we omit them.

Theorem 3.1. *The sequence spaces $m(M, \phi, q, \Lambda)$ are linear spaces over the complex field $\mathbb{C}$.*

Theorem 3.2. *The space $m(M, \phi, q, \Lambda)$ is a linear topological space paranormed by*

$$g(x) = \left\{ \varrho^{p_r/H} : \left[\sup_s \frac{1}{\phi_s} \sum_{n \in \sigma, \sigma \in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \right]^{1/H} \leq 1, \; r = 1, 2, \ldots \right\}. \tag{3.1}$$

In what follows, we will show inclusion theorems between spaces$m(M, \phi, q, \Lambda)$.

Theorem 3.3. *$m(M,\phi^1,q,\Lambda)\subset m(M,\phi^2,q,\Lambda)$ if and only if*

$$\sup_{s\geq 1}\frac{\phi_s^1}{\phi_s^2}<\infty. \tag{3.2}$$

Proof. Let $x\in m(M,\phi^1,q,\Lambda)$ and $K=\sup_{s\geq 1}(\phi_s^1/\phi_s^2)<\infty$. Then we get

$$\begin{aligned}\frac{1}{\phi_s^2}\sum_{n\in\sigma,\sigma\in P_s}M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n}&\leq\sup_{s\geq 1}\frac{\phi_s^1}{\phi_s^2}\frac{1}{\phi_s^1}\sum_{n\in\sigma,\sigma\in P_s}M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n}\\&=K\cdot\frac{1}{\phi_s^1}\sum_{n\in\sigma,\sigma\in P_s}M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n},\end{aligned} \tag{3.3}$$

hence $x\in m(M,\phi^2,q,\Lambda)$. Conversely, let us suppose that $m(M,\phi^1,q,\Lambda)\subset m(M,\phi^2,q,\Lambda)$ and $x\in m(M,\phi^1,q,\Lambda)$. Then there exists a $\varrho>0$ such that

$$\frac{1}{\phi_s^1}\sum_{n\in\sigma,\sigma\in P_s}M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n}<\epsilon, \tag{3.4}$$

for every $\epsilon>0$. Suppose that $\sup_{s\geq 1}(\phi_s^1/\phi_s^2)=\infty$, then there exists a sequence of natural numbers (s_i) such that $\lim_{j\to\infty}(\phi_{s_j}^1/\phi_{s_j}^2)=\infty$. Hence we can write

$$\frac{1}{\phi_s^2}\sum_{n\in\sigma,\sigma\in P_s}M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n}\geq\sup_{j\geq 1}\frac{\phi_{s_j}^1}{\phi_s^2}\cdot\frac{1}{\phi_{s_j}^1}\sum_{n\in\sigma,\sigma\in P_s}M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n}=\infty. \tag{3.5}$$

Therefore, $x\notin m(M,\phi^2,q,\Lambda)$, which is contradiction. □

Corollary 3.4. *Let M be an Orlicz function. Then $m(M,\phi^1,q,\Lambda)=m(M,\phi^2,q,\Lambda)$ if and only if*

$$\sup_{s\geq 1}\frac{\phi_s^1}{\phi_s^2}<\infty,\qquad \sup_{s\geq 1}\frac{\phi_s^2}{\phi_s^1}<\infty. \tag{3.6}$$

Theorem 3.5. *Let M, M_1, M_2 be Orlicz functions which satisfy the Δ_2-condition and q, q_1, and q_2 seminorms. Then*

(1) $m(M_1,\phi,q,\Lambda)\subset m(M\circ M_1,\phi,q,\Lambda)$,

(2) $m(M_1,\phi,q,\Lambda)\cap m(M_2,\phi,q,\Lambda)\subset m(M_1+M_2,\phi,q,\Lambda)$,

(3) $m(M,\phi,q_1,\Lambda)\cap m(M,\phi,q_2,\Lambda)\subset m(M,\phi,q_1+q_2,\Lambda)$,

(4) *If q_1 is stronger than q_2, then $m(M,\phi,q_1,\Lambda)\subset m(M,\phi,q_2,\Lambda)$, and*

(5) *If q_1 is equivalent to q_2, then $m(M,\phi,q_1,\Lambda)=m(M,\phi,q_2,\Lambda)$.*

Proof. Proof is similar to [14, Theorem 2.5]. □

Corollary 3.6. *Let M be an Orlicz function which satisfy the Δ_2-condition. Then $m(\phi, q, \Lambda) \subset m(M, \phi, q, \Lambda)$.*

Theorem 3.7. *Let $\Omega = (M_i)$ be a sequence of Orlicz functions. Then the sequence space $m(M, \phi, q, \Lambda)$ is solid and monotone.*

Proof. Let $x \in m(M, \phi, q, \Lambda)$, then there exists $\varrho > 0$ such that

$$\frac{1}{\phi_s} \sum_{n \in \sigma, \sigma \in P_s} M\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} < \epsilon, \tag{3.7}$$

for every $\epsilon > 0$. Let (λ_n) be a sequence of scalars with $|\lambda_n| \leq 1$ for all $n \in \mathbb{N}$. Then from properties of Orlicz functions and seminorm, we get

$$\begin{aligned} \frac{1}{\phi_s} \sum_{n \in \sigma, \sigma \in P_s} M_n\left(q\left(\frac{|\Lambda_n(\lambda_n x)|}{\varrho}\right)\right)^{p_n} &= \frac{1}{\phi_s} \sum_{n \in \sigma, \sigma \in P_s} M_n\left(q\left(\frac{|\lambda_n||\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \\ &\leq \frac{1}{\phi_s} \sum_{n \in \sigma, \sigma \in P_s} |\lambda_n| M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \\ &\leq \frac{1}{\phi_s} \sum_{n \in \sigma, \sigma \in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n}, \end{aligned} \tag{3.8}$$

which proves that $m(M, \phi, q, \Lambda)$ is solid space and monotone. □

4. Statistical Convergence

In [15], Fast introduced the idea of statistical convergence. This ideas was later studied by Connor [16], Freedman and Sember [17], and many others. A sequence of positive integers $\theta = (k_r)$ is called lacunary if $k_0 = 0$, $0 < k_r < k_{r+1}$, and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. A sequence $x = (x_i)$ is said to be $S_\theta(\phi, \Lambda)$ statistically convergent to s if for any $\epsilon > 0$,

$$\lim_i \frac{1}{h_r} k\left(\left\{i \in \sigma,\ \sigma \in P_r,\ r \geq 1 : \left|\frac{\Lambda_i(x)}{\varrho} - s\right| \geq \epsilon\right\}\right) = 0, \tag{4.1}$$

for some $\varrho > 0$, where $k(A)$ denotes the cardinality of A. A sequence $x = (x_i)$ is said to be $S_\theta^0(\phi, \Lambda)$ statistically convergent to s if for any $\epsilon > 0$,

$$\lim_i \frac{1}{h_r} k\left(\left\{i \in \sigma,\ \sigma \in P_r,\ r \geq 1 : \left|\frac{\Lambda_i(x)}{\varrho}\right| \geq \epsilon\right\}\right) = 0, \tag{4.2}$$

for some $\varrho > 0$.

Theorem 4.1. *If M is any Orlicz function, (ϕ_n) strictly increasing sequence, then $m(M, \phi, q, \Lambda) \subset S_\theta^0(\phi, \Lambda)$.*

Proof. Let $x \in m(M, \phi, q, \Lambda)$. Then

$$\frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} < \epsilon_1, \tag{4.3}$$

for every $\epsilon_1 > 0$. Let $k_s = s\phi_s$ be a sequence of positive numbers. Then it follows that k_s is lacunary sequence. Then we get the following relation:

$$\begin{aligned}
&\frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \\
&\geq \frac{1}{s\phi_s - (s-1)\phi_{s-1}} \sum_{n\in\sigma,\sigma\in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \\
&= \frac{1}{h_s} \sum_{n\in\sigma,\sigma\in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \\
&\geq \frac{1}{h_s} \sum_{1} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \\
&\geq \frac{1}{h_s} \sum_{1} M_n(q(\epsilon))^{p_n} \\
&\geq \frac{1}{h_s} \sum_{1} \min\left\{M_n(q(\epsilon))^{h}, M_n(q(\epsilon))^{H}\right\} \quad \left(\text{where the summation } \sum_1 \text{ is over } \left(\frac{|\Lambda_n(x)|}{\varrho}\right) \geq \epsilon\right) \\
&\geq \frac{1}{h_s} k\left\{n \in \sigma,\ \sigma \in P_s,\ s \geq 1 : \left(\frac{|\Lambda_n(x)|}{\varrho}\right) \geq \epsilon\right\} \cdot \min\left\{M_n(q(\epsilon))^{h}, M_n(q(\epsilon))^{H}\right\}.
\end{aligned} \tag{4.4}$$

Taking the limit as $n \to \infty$, it follows that $x \in S^0_\theta(M, \phi, q, \Lambda)$. □

Theorem 4.2. *If M is any Orlicz bounded function, (ϕ_s) strictly increasing sequence, then $m(M, \phi_s, q, \Lambda(\cdot/s)) = S^0_\theta(\phi_s, \Lambda(\cdot/s))$, for every $s \geq 1$.*

Proof. Inclusion $m(M, \phi_s, q, \Lambda(\cdot/s)) \subset S^0_\theta(\phi_s, \Lambda(\cdot/s))$, is valid (from Theorem 4.1). In what follows, we will show converse inclusion. Let $x \in S^0_\theta(\phi_s, \Lambda(\cdot/s))$, since M_n is bounded, there exists a constant K such that $M_n(q(|\Lambda_n(x/s)|/\varrho)) < K$. Then for every given $\epsilon > 0$, we have

$$\begin{aligned}
\frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} M_n\left(q\left(\frac{|\Lambda_n(x/s)|}{\varrho}\right)\right)^{p_n} &= \frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} M_n\left(q\left(\frac{1}{s} \cdot \frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \\
&\leq \frac{1}{s} \cdot \frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n}.
\end{aligned} \tag{4.5}$$

Let us denote by $k_s = s \cdot \phi_s$, as we know this sequence is lacunary and finally we get the following relation:

$$\begin{aligned}
&\frac{1}{k_s}\sum_{n\in\sigma,\sigma\in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \\
&\quad\le \frac{1}{k_s-k_{s-1}}\sum_{n\in\sigma,\sigma\in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \\
&\quad= \frac{1}{h_s}\sum_1 M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} + \frac{1}{h_s}\sum_2 M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \\
&\quad\le K^H\cdot\frac{1}{h_s}k\left\{n\in\sigma,\ \sigma\in P_s,\ s\ge 1:\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\ge\epsilon\right\} + \max\left\{M_n(q(\epsilon))^h, M_n(q(\epsilon))^H\right\},
\end{aligned}\tag{4.6}$$

where the summation $\sum_1$ is over $(|\Lambda_n(x)|/\varrho)\ge\epsilon$ and the summation $\sum_2$ is over $(|\Lambda_n(x)|/\varrho)\le\epsilon$. Taking the limit as $\epsilon\to 0$ and $n\to\infty$, it follows that $x\in m(M,\phi_s,q,\Lambda(\cdot/s))$. □

5. Cesaro Convergence

In this paragraph, we will consider that (ϕ_s) is a nondecreasing sequence of positive real numbers such that $\phi_s\le s$, $\phi_s\to\infty$, as $s\to\infty$. Let us denote by

$$m_\theta^c(M,\phi,q,\Lambda) = \left\{x=(x_i):\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} = 0,\ \text{for some } \varrho>0\right\}. \tag{5.1}$$

Theorem 5.1. *If M is an Orlicz function. Then $m_\theta^c(M,\phi,q,\Lambda)\subset m(M,\phi,q,\Lambda)$.*

Proof. From the definition of the sequences ϕ_n, it follows that $\inf_n((n+1)/(n+1-\phi_n))\ge 1$. Then there exist a $\delta>0$, such that

$$\frac{n+1}{\phi_n}\le\frac{1+\delta}{\delta}. \tag{5.2}$$

Then we get the following relation:

$$\begin{aligned}
&\frac{1}{\phi_s}\sum_{n\in\sigma,\sigma\in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \\
&\quad= \frac{n+1}{\phi_s}\frac{1}{n+1}\sum_{k=1}^{n+1} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} - \frac{1}{\phi_s}\sum_{k\in\{1,2,\dots,n+1\}\setminus\sigma,\sigma\in P_s} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} \\
&\quad\le \frac{s+1}{\phi_s}\frac{1}{n+1}\sum_{k=1}^{n+1} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} - \frac{1}{\phi_s}M_{n_0}\left(q\left(\frac{|\Lambda_{n_0}(x)|}{\varrho}\right)\right)^{p_{n_0}} \\
&\quad\le \frac{1+\delta}{\delta}\frac{1}{n+1}\sum_{k=1}^{n+1} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} - \frac{1}{\phi_s}M_{n_0}\left(q\left(\frac{|\Lambda_{n_0}(x)|}{\varrho}\right)\right)^{p_{n_0}},
\end{aligned}\tag{5.3}$$

where $n_0 \in \{1,2,\ldots,n+1\} \setminus \sigma$, $\sigma \in P_s$. Knowing that $x \in m_\theta^c(M,\phi,q,\Lambda)$ and M_i are continuous, letting $n \to \infty$ on last relation, we obtain

$$\frac{1}{\phi_s}\sum_{n\in\sigma,\sigma\in P_s} M_n\left(q\left(\frac{|\Lambda_n(x)|}{\varrho}\right)\right)^{p_n} \longrightarrow 0. \tag{5.4}$$

Hence $x \in m(M,\phi,q,\Lambda)$. □

Theorem 5.2. *Let* $\sup_s(\phi_s/\phi_{s-1}) < \infty$. *Then for any Orlicz function,* M, $m(M,\phi,q,\Lambda) \subset m_\theta^c(M,\phi,q,\Lambda)$.

Proof. Suppose that $\sup_s \phi_s/\phi_{s-1} < \infty$, then there exists $B > 0$ such that $\phi_s/\phi_{s-1} < B$ for all $s \geq 1$. Let $x \in m(M,\phi,q,\Lambda)$ and $\epsilon > 0$, there exist $R > 0$ such that for every $k \geq R$

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} < \epsilon. \tag{5.5}$$

We can also find a constant $K > 0$ such that

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} < K, \tag{5.6}$$

for all $k \in \mathbb{N}$. Let n be any integer with $\phi_{s-1} < n+1 \leq [\phi_s]$, for every $s > R$. Then

$$\begin{aligned}
&\frac{1}{n+1}\sum_{k=1}^{n} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} \\
&\quad\leq \frac{1}{\phi_{s-1}}\sum_{k=1}^{[\phi_s]} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} \\
&\quad= \frac{1}{\phi_{s-1}}\left(\sum_{k=1}^{[\phi_s]} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} + \sum_{[\phi_s]}^{[\phi_s]} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} + \cdots \right.\\
&\qquad\qquad \left. + \sum_{[\phi_{s-1}]}^{[\phi_s]} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k}\right) \\
&\quad\leq \frac{\phi_1}{\phi_{s-1}}\left(\frac{1}{\phi_1}\sum_{k\in\sigma,\sigma\in P^{(1)}} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k}\right) \\
&\qquad + \frac{\phi_2}{\phi_{s-1}}\left(\frac{1}{\phi_2}\sum_{k\in\sigma,\sigma\in P^{(2)}} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k}\right) + \cdots
\end{aligned}$$

$$+\frac{\phi_R}{\phi_{s-1}}\left(\frac{1}{\phi_R}\sum_{k\in\sigma,\sigma\in P^{(R)}} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k}\right)+\cdots$$

$$+\frac{\phi_s}{\phi_{s-1}}\left(\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P^{(s)}} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k}\right), \tag{5.7}$$

where $P^{(t)}$ are sets of integer numbers which have more than $[\phi_t]$ elements for $t \in \{1,2,\dots,s\}$. Passing by limit on last relation, where $k \to \infty$ (since $s \to \infty$, $\phi_s \to \infty$ and $n \to \infty$), we get that

$$\frac{1}{n+1}\sum_{k=1}^{n} M_k\left(q\left(\frac{|\Lambda_k(x)|}{\varrho}\right)\right)^{p_k} \longrightarrow 0; \tag{5.8}$$

from this, it follows that $x \in m_\theta^c(M,\phi,q,\Lambda)$. □

Theorem 5.3. *Let* $\sup_s(\phi_s/\phi_{s-1}) < \infty$*. Then for any Orlicz function,* M*,* $m(M,\phi,q,\Lambda) = m_\theta^c(M,\phi,q,\Lambda)$*.*

References

[1] M. Mursaleen and A. K. Noman, “On the spaces of λ-convergent and bounded sequences,” *Thai Journal of Mathematics*, vol. 8, no. 2, pp. 311–329, 2010.

[2] W. L. C. Sargent, “Some sequence spaces related to the l_p spaces,” *Journal of the London Mathematical Society*, vol. 35, pp. 161–171, 1960.

[3] D. Rath, “Spaces of r-convex sequences and matrix transformations,” *Indian Journal of Mathematics*, vol. 41, no. 2, pp. 265–280, 1999.

[4] D. Rath and B. C. Tripathy, “Characterization of certain matrix operators,” *Journal of Orissa Mathematical Society*, vol. 8, pp. 121–134, 1989.

[5] B. C. Tripathy and M. Sen, “On a new class of sequences related to the space l_p,” *Tamkang Journal of Mathematics*, vol. 33, no. 2, pp. 167–171, 2002.

[6] B. C. Tripathy and S. Mahanta, “On a class of sequences related to the l_p space defined by Orlicz functions,” *Soochow Journal of Mathematics*, vol. 29, no. 4, pp. 379–391, 2003.

[7] J. Lindenstrauss and L. Tzafriri, “On Orlicz sequence spaces,” *Israel Journal of Mathematics*, vol. 10, pp. 379–390, 1971.

[8] M. A. Krasnoselskii and Y. B. Rutitsky, *Convex Function and Orlicz Spaces*, P.Noordhoff, Groningen, The Netherlands, 1961.

[9] V. K. Bhardwaj and N. Singh, “Some sequence spaces defined by Orlicz functions,” *Demonstratio Mathematica*, vol. 33, no. 3, pp. 571–582, 2000.

[10] M. Güngör, M. Et, and Y. Altin, “Strongly (V_σ,λ,q)-summable sequences defined by Orlicz functions,” *Applied Mathematics and Computation*, vol. 157, no. 2, pp. 561–571, 2004.

[11] A. Esi, “On a class of new type difference sequence spaces related to the space l_p,” *Far East Journal of Mathematical Sciences*, vol. 13, no. 2, pp. 167–172, 2004.

[12] A. Esi and M. Et, “Some new sequence spaces defined by a sequence of Orlicz functions,” *Indian Journal of Pure and Applied Mathematics*, vol. 31, no. 8, pp. 967–973, 2000.

[13] S. D. Parashar and B. Choudhary, “Sequence spaces defined by Orlicz functions,” *Indian Journal of Pure and Applied Mathematics*, vol. 25, no. 4, pp. 419–428, 1994.

[14] Y. Altun and T. Bilgin, “On a new class of sequences related to the l_p space defined by Orlicz function,” *Taiwanese Journal of Mathematics*, vol. 13, no. 4, pp. 1189–1196, 2009.

[15] H. Fast, "Sur la convergence statistique," *Colloquium Mathematicum*, vol. 2, pp. 241–244, 1951.

[16] J. S. Connor, "The statistical and strong p-Cesàro convergence of sequences," *Analysis*, vol. 8, no. 1-2, pp. 47–63, 1988.

[17] A. R. Freedman and J. J. Sember, "Densities and summability," *Pacific Journal of Mathematics*, vol. 95, no. 2, pp. 293–305, 1981.