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Research Article

A Study on the p-Adic q-Integral Representation on $\mathbb{Z}_p$ Associated with the Weighted q-Bernstein and q-Bernoulli Polynomials

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We investigate some interesting properties of the weighted q-Bernstein polynomials related to the weighted q-Bernoulli numbers and polynomials by using p-adic q-integral on $\mathbb{Z}_p$.

1. Introduction and Preliminaries

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers, and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$. Let q be regarded as either a complex number $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1-q|_p < 1$. In this paper, we define the q-number as $[x]_q = (1-q^x)/(1-q)$ (see [1–13]).

Let $C[0,1]$ be the set of continuous functions on $[0,1]$. For $\alpha \in \mathbb{N}$ and $n, k \in \mathbb{Z}_+$, the weighted q-Bernstein operator of order n for $f \in C[0,1]$ is defined by

$$\mathbb{B}_{n,q}^{(\alpha)}(f \mid x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x]_{q^{\alpha}}^{k}[1-x]_{q^{-\alpha}}^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)B_{k,n}^{(\alpha)}(x,q). \tag{1.1}$$

Here $B_{k,n}^{(\alpha)}(x,q)$ is called the weighted q-Bernstein polynomials of degree n (see [2, 5, 6]).

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$, which is called the bosonic q-integral on $\mathbb{Z}_p$, is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \tag{1.2}$$

(see [10]).

The Carlitz's q-Bernoulli numbers are defined by

$$\beta_{0,q} = 1, \quad q(q\beta+1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.3}$$

with the usual convention about replacing β^k by $\beta_{k,q}$ (see [3, 9, 10]). In [3], Carlitz also defined the expansion of Carlitz's q-Bernoulli numbers as follows:

$$\beta_{0,q}^h = \frac{h}{[h]_q}, \quad q^h\left(q\beta^h+1\right)^n - \beta_{n,q}^h = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1.4}$$

with the usual convention about replacing $(\beta^h)^n$ by $\beta_{n,q}^h$.

The weighted q-Bernoulli numbers are constructed in previous paper [6] as follows: for $\alpha \in \mathbb{N}$,

$$\widetilde{\beta}_{0,q}^{(\alpha)} = 1, \quad q\left(q^\alpha\widetilde{\beta}^{(\alpha)}+1\right)^n - \widetilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1.5}$$

with the usual convention about replacing $(\widetilde{\beta}^{(\alpha)})^n$ by $\widetilde{\beta}_{n,q}^{(\alpha)}$. Let $f_n(x) = f(x+n)$. By the definition (1.2) of p-adic q-integral on $\mathbb{Z}_p$, we easily get

$$\begin{aligned} qI_q(f_1) &= q\lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x+1)q^x, \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x + \lim_{N\to\infty} \frac{f(p^N)q^{p^N} - f(0)}{[p^N]_q} \\ &= \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) + (q-1)f(0) + \frac{q-1}{\log q} f'(0), \end{aligned} \tag{1.6}$$

Continuing this process, we obtain easily the relation

$$q^n \int_{\mathbb{Z}_p} f_n(x)\,d\mu_q(x) - \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} q^l f'(l), \tag{1.7}$$

where $n \in \mathbb{N}$ and $f'(l) = df(l)/dx$ (see [6]).

Then by (1.2), applying to the function $x \to [x]_{q^\alpha}^n$, we can see that

$$\widetilde{\beta}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_q(x) = -\frac{n\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{m\alpha+m} [m]_{q^\alpha}^{n-1} + (1-q) \sum_{m=0}^{\infty} q^m [m]_{q^\alpha}^n. \tag{1.8}$$

The weighted q-Bernoulli polynomials are also defined by the generating function as follows:

$$F_q^{(\alpha)}(t,x) = -t\frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{m\alpha+\mathrm{m}} e^{[m+x]_{q^\alpha} t} + (1-q) \sum_{m=0}^{\infty} q^m e^{[m+x]_{q^\alpha} t} = \sum_{n=0}^{\infty} \widetilde{\beta}_{n,q}^{(\alpha)}(x) \frac{t^n}{n!}, \tag{1.9}$$

(see[6]). Thus, we note that

$$\begin{aligned} \widetilde{\beta}_{n,q}^{(\alpha)}(x) &= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \widetilde{\beta}_{l,q}^{(\alpha)} \\ &= -\frac{n\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{m\alpha+m} [m+x]_{q^\alpha}^{n-1} + (1-q) \sum_{m=0}^{\infty} q^m [m+x]_{q^\alpha}^n. \end{aligned} \tag{1.10}$$

From (1.2) and the previous equalities, we obtain the Witt's formula for the weighted q-Bernoulli polynomials as follows:

$$\widetilde{\beta}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n d\mu_q(y) = \sum_{l=0}^{n} \binom{n}{l} q^{\alpha l x} [x]_{q^\alpha}^{n-l} \int_{\mathbb{Z}_p} [y]_{q^\alpha}^l d\mu_q(y). \tag{1.11}$$

By using (1.2) and the weighted q-Bernoulli polynomials, we easily get

$$q^n \widetilde{\beta}_{m,q}^{(\alpha)}(n) - \widetilde{\beta}_{m,q}^{(\alpha)} = (q-1) \sum_{l=0}^{n-1} q^l [l]_{q^\alpha}^m + \frac{m\alpha}{[\alpha]_q} \sum_{l=0}^{n-1} q^{\alpha l + l} [l]_{q^\alpha}^{m-1}, \tag{1.12}$$

where $n, \alpha \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ (see [6]).

In this paper, we consider the weighted q-Bernstein polynomials to express the bosonic q-integral on $\mathbb{Z}_p$ and investigate some properties of the weighted q-Bernstein polynomials associated with the weighted q-Bernoulli polynomials by using the expression of p-adic q-integral on $\mathbb{Z}_p$ of those polynomials.

2. Weighted q-Bernstein Polynomials and q-Bernoulli Polynomials

In this section, we assume that $\alpha \in \mathbb{N}$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$.

Now we consider the p-adic weighted q-Bernstein operator as follows:

$$\mathbb{B}_{n,q}^{(\alpha)}(f \mid x)(fx) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}^{(\alpha)}(x,q). \tag{2.1}$$

The p-adic q-Bernstein polynomials with weight α of degree n are given by

$$B_{k,n}^{(\alpha)}(x,q) = \binom{n}{k}[x]_{q^{\alpha}}^{k}[1-x]_{q^{-\alpha}}^{n-k}, \tag{2.2}$$

where $x \in \mathbb{Z}_p$, $\alpha \in \mathbb{N}$, and $n, k \in \mathbb{Z}_+$ (see [6, 7]). Note that $B_{k,n}^{(\alpha)}(x,q) = B_{n-k,n}^{(\alpha)}(1-x,1/q)$. That is, the weighted q-Bernstein polynomials are symmetric.

From the definition of the weighted q-Bernoulli polynomials, we have

$$\widetilde{\beta}_{n,q^{-1}}^{(\alpha)}(1-x) = (-1)^n q^{\alpha n}\widetilde{\beta}_{n,q}^{(\alpha)}(x). \tag{2.3}$$

By the definition of p-adic q-integral on $\mathbb{Z}_p$, we get

$$\begin{aligned}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n}d\mu_q(x) &= q^{\alpha n}(-1)^n\int_{\mathbb{Z}_p}[-1+x]_{q^{\alpha}}^{n}d\mu_q(x) \\ &= \int_{\mathbb{Z}_p}\left(1-[x]_{q^{\alpha}}\right)^n d\mu_q(x).\end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we have

$$\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n}d\mu_q(x) = \sum_{l=0}^{n}\binom{n}{l}(-1)^l\widetilde{\beta}_{l,q}^{(\alpha)} = q^{\alpha n}(-1)^n\widetilde{\beta}_{n,q}^{(\alpha)}(-1) = \widetilde{\beta}_{n,q}^{(\alpha)}(2). \tag{2.5}$$

Therefore, we obtain the following lemma.

Lemma 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$\begin{gathered}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n}d\mu_q(x) = \sum_{l=0}^{n}\binom{n}{l}(-1)^l\widetilde{\beta}_{l,q}^{(\alpha)} = q^{\alpha n}(-1)^n\widetilde{\beta}_{n,q}^{(\alpha)}(-1) = \widetilde{\beta}_{n,q}^{(\alpha)}(2), \\ \widetilde{\beta}_{n,q^{-1}}^{(\alpha)}(1-x) = (-1)^n q^{\alpha n}\widetilde{\beta}_{n,q}^{(\alpha)}(x).\end{gathered} \tag{2.6}$$

By (2.2), (2.3), and (2.4), we get

$$q^2\widetilde{\beta}_{n,q}^{(\alpha)}(2) = n\frac{\alpha}{[\alpha]_q}q^{1+\alpha} + q^2 - q + \widetilde{\beta}_{n,q}^{(\alpha)}, \quad \text{if } n > 1. \tag{2.7}$$

Thus, we have

$$\widetilde{\beta}_{n,q}^{(\alpha)}(2) = \frac{1}{q^2}\widetilde{\beta}_{n,q}^{(\alpha)} + \frac{n\alpha}{[\alpha]_q}q^{\alpha-1} + 1 - \frac{1}{q}, \quad \text{if } n > 1. \tag{2.8}$$

Therefore, by (2.8), we obtain the following proposition.

Proposition 2.2. *For $n \in \mathbb{N}$ with $n > 1$, one has*

$$\tilde{\beta}_{n,q}^{(\alpha)}(2) = \frac{1}{q^2}\tilde{\beta}_{n,q}^{(\alpha)} + \frac{n\alpha}{[\alpha]_q}q^{\alpha-1} + 1 - \frac{1}{q}. \tag{2.9}$$

By using Proposition 2.2 and Lemma 2.1, we obtain the following corollary.

Corollary 2.3. *For $n \in \mathbb{N}$ with $n > 1$, one has*

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n d\mu_q(x) = q^2\tilde{\beta}_{n,q^{-1}}^{(\alpha)} + \frac{n\alpha}{[\alpha]_q} + 1 - q, \tag{2.10}$$

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n d\mu_q(x) = \frac{n\alpha}{[\alpha]_q} + 1 - q + q^2\int_{\mathbb{Z}_p} [x]_{q^{-\alpha}}^n d\mu_{q^{-1}}(x) = \int_{\mathbb{Z}_p} \left(1 - [x]_{q^{\alpha}}\right)^n d\mu_q(x). \tag{2.11}$$

Taking the bosonic q-integral on $\mathbb{Z}_p$ for one weighted q-Bernstein polynomials in (2.1), we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)d\mu_q(x) &= \binom{n}{k}\int_{\mathbb{Z}_p} [x]_{q^{\alpha}}^k [1-x]_{q^{-\alpha}}^{n-k} d\mu_q(x) \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\int_{\mathbb{Z}_p} [x]_{q^{\alpha}}^{k+l} d\mu_q(x) \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\tilde{\beta}_{k+l,q}^{(\alpha)}.
\end{aligned} \tag{2.12}$$

By the symmetry of q-Bernstein polynomials, we get

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)d\mu_q(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}^{(\alpha)}\left(1-x,\frac{1}{q}\right)d\mu_q(x) \\
&= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n-l} d\mu_q(x).
\end{aligned} \tag{2.13}$$

For $n > k+1$, by (2.11) and (2.13), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)\,d\mu_q(x) &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\left(\frac{n\alpha}{[\alpha]_q}+1-q+q^2\int_{\mathbb{Z}_p}[x]_{q^{-\alpha}}^{n-l}\,d\mu_{q^{-1}}(x)\right)\\ &= \begin{cases}\dfrac{n\alpha}{[\alpha]_q}+1-q+q^2\widetilde{\beta}_{n,q^{-1}}^{(\alpha)}, & \text{if } k=0,\\ \dbinom{n}{k}q^2\displaystyle\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\widetilde{\beta}_{n-l,q^{-1}}^{(\alpha)}, & \text{if } k>0.\end{cases}\end{aligned} \tag{2.14}$$

By comparing the coefficients on the both sides of (2.12) and (2.14), we obtain the following theorem.

Theorem 2.4. *For $n,k \in \mathbb{Z}_+$ with $n > k+1$, one has*

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\widetilde{\beta}_{k+l,q}^{(\alpha)} = q^2\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\widetilde{\beta}_{n-l,q^{-1}}^{(\alpha)}, \quad \text{if } k\neq 0. \tag{2.15}$$

In particular, when $k=0$, one has

$$\frac{n\alpha}{[\alpha]_q}+1-q+q^2\widetilde{\beta}_{n,q^{-1}}^{(\alpha)} = \sum_{l=0}^{n}\binom{n}{l}(-1)^l\widetilde{\beta}_{l,q}^{(\alpha)}. \tag{2.16}$$

Let $m,n,k \in \mathbb{Z}_+$ with $m+n > 2k+1$. Then we see that

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)B_{k,m}^{(\alpha)}(x,q)\,d\mu_q(x)\\ &\quad= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[x]_{q^{\alpha}}^{2k}[1-x]_{q^{-\alpha}}^{n+m-2k}\,d\mu_q(x)\\ &\quad= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n+m-l}\,d\mu_q(x).\\ &\quad= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(\frac{n\alpha}{[\alpha]_q}+1-q+q^2\int_{\mathbb{Z}_p}[x]_{q^{-\alpha}}^{n+m-l}\,d\mu_{q^{-1}}(x)\right)\\ &\quad= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(\frac{n\alpha}{[\alpha]_q}+1-q+q^2\widetilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)}\right).\end{aligned} \tag{2.17}$$

Therefore, by (2.17), we obtain the following theorem.

Theorem 2.5. *For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k + 1$, one has*

$$\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q) B_{k,m}^{(\alpha)}(x,q)\, d\mu_q(x) = \begin{cases} \dfrac{n\alpha}{[\alpha]_q} + 1 - q + q^2 \widetilde{\beta}_{n+m,q^{-1}}^{(\alpha)}, & \text{if } k = 0, \\ \dbinom{n}{k}\dbinom{m}{k} q^2 \displaystyle\sum_{l=0}^{2k} \dbinom{2k}{l} (-1)^{l+2k} \widetilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)}, & \text{if } k \neq 0. \end{cases} \tag{2.18}$$

For $m, n, k \in \mathbb{Z}_+$, we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q) B_{k,m}^{(\alpha)}(x,q)\, d\mu_q(x) \\ &= \binom{n}{k}\binom{m}{k} \int_{\mathbb{Z}_p} [x]_{q^{\alpha}}^{2k} [1-x]_{q^{-\alpha}}^{n+m-2k}\, d\mu_q(x) \\ &= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_{q^{\alpha}}^{2k+l}\, d\mu_q(x) \\ &= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \widetilde{\beta}_{l+2k,q}^{(\alpha)}. \end{aligned} \tag{2.19}$$

Therefore, by (2.18) and (2.19), we obtain the following theorem.

Theorem 2.6. *For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k + 1$, one has*

$$\frac{n\alpha}{[\alpha]_q} + 1 - q + q^2 \widetilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)} = \sum_{l=0}^{n+m} \binom{n+m}{l} (-1)^l \widetilde{\beta}_{l,q}^{(\alpha)}. \tag{2.20}$$

Furthermore, for $k \neq 0$, one has

$$\sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \widetilde{\beta}_{l+2k,q}^{(\alpha)} = q^2 \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \widetilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)}. \tag{2.21}$$

By the induction hypothesis, we obtain the following theorem.

Theorem 2.7. *For $s \in \mathbb{N}$ and $k, n_1, \ldots, n_s \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk + 1$, one has*

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}^{(\alpha)}(x,q) \right) d\mu_q(x) = \begin{cases} \dfrac{n\alpha}{[\alpha]_q} + 1 - q + q^2 \widetilde{\beta}_{n_1+\cdots+n_s,q^{-1}}^{(\alpha)}, & \text{if } k = 0, \\ \left(\displaystyle\prod_{i=1}^{s} \dbinom{n_i}{k} \right) \displaystyle\sum_{l=0}^{sk} \dbinom{sk}{l} (-1)^{l+sk} \widetilde{\beta}_{n_1+\cdots+n_s-l,q^{-1}}^{(\alpha)}, & \text{if } k \neq 0. \end{cases} \tag{2.22}$$

For $s \in \mathbb{N}$, let $k, n_1, \ldots, n_s \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk + 1$. Then we show that

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}^{(\alpha)}(x,q) \right) d\mu_q(x) = \left(\prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l \widetilde{\beta}_{l+sk,q}^{(\alpha)}. \tag{2.23}$$

Therefore, by Theorem 2.7 and (2.23), we obtain the following theorem.

Theorem 2.8. *For $s \in \mathbb{N}$, let $k, n_1, \ldots, n_s \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk + 1$. Then one sees that for $k = 0$*

$$\sum_{l=0}^{n_1+\cdots+n_s} \binom{n_1+\cdots+n_s}{l} (-1)^l \widetilde{\beta}_{l,q}^{(\alpha)} = \frac{n\alpha}{[\alpha]_q} + 1 - q + q^2 \widetilde{\beta}_{n_1+\cdots+n_s,q^{-1}}^{(\alpha)}. \tag{2.24}$$

For $k \neq 0$, one has

$$\sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \widetilde{\beta}_{n_1+\cdots+n_s-l,q^{-1}}^{(\alpha)} = \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l \widetilde{\beta}_{l+sk,q}^{(\alpha)}. \tag{2.25}$$

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