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Research Article

Subordination and Superordination for Multivalent Functions Associated with the Dziok-Srivastava Operator

Nak Eun Cho,[1] Oh Sang Kwon,[2] Rosihan M. Ali,[3] and V. Ravichandran[3,4]

[1] *Department of Applied Mathematics, Pukyong National University, Busan 608-737, Republic of Korea*
[2] *Department of Mathematics, Kyungsung University, Busan 608-736, Republic of Korea*
[3] *School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Penang, Malaysia*
[4] *Department of Mathematics, University of Delhi, Delhi 110007, India*

Correspondence should be addressed to Rosihan M. Ali, rosihan@cs.usm.my

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Subordination and superordination preserving properties for multivalent functions in the open unit disk associated with the Dziok-Srivastava operator are derived. Sandwich-type theorems for these multivalent functions are also obtained.

1. Introduction

Let $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $\mathscr{H} := \mathscr{H}(\mathbb{U})$ denote the class of analytic functions defined in $\mathbb{U}$. For $n \in \mathbb{N} := \{1, 2, \ldots\}$ and $a \in \mathbb{C}$, let $\mathscr{H}[a, n]$ consist of functions $f \in \mathscr{H}$ of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let f and F be members of $\mathscr{H}$. The function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, if there exists a function w analytic in $\mathbb{U}$, with $|w(z)| \leq |z|$ and such that $f(z) = F(w(z))$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function F is univalent in $\mathbb{U}$, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$ (cf. [1, 2]). Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$, and let h be univalent in $\mathbb{U}$. The subordination $\varphi(p(z), zp'(z)) \prec h(z)$ is called a first-order differential subordination. It is of interest to determine conditions under which $p \prec q$ arises for a prescribed univalent function q. The theory of differential subordination in $\mathbb{C}$ is a generalization of a differential inequality in $\mathbb{R}$, and this theory of differential subordination was initiated by the works of Miller, Mocanu, and Reade in 1981. Recently, Miller and Mocanu [3] investigated the dual problem of differential superordination. The monograph by Miller and Mocanu [1] gives a good introduction to the theory of differential subordination, while the book by Bulboacă [4] investigates both subordination and superordination. Related results on superordination can be found in [5–23].

By using the theory of differential subordination, various subordination preserving properties for certain integral operators were obtained by Bulboacă [24], Miller et al. [25], and Owa and Srivastava [26]. The corresponding superordination properties and sandwich-type results were also investigated, for example, in [4]. In the present paper, we investigate subordination and superordination preserving properties of functions defined through the use of the Dziok-Srivastava linear operator $H_{p,q,s}(\alpha_1)$ (see (1.9) and (1.10)), and also obtain corresponding sandwich-type theorems.

The Dziok-Srivastava linear operator is a particular instance of a linear operator defined by convolution. For $p \in \mathbb{N}$, let $\mathcal{A}_p$ denote the class of functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \tag{1.1}$$

that are analytic and p-valent in the open unit disk $\mathbb{U}$ with $f^{(p+1)}(0) \neq 0$. The Hadamard product (or convolution) $f * g$ of two analytic functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k \tag{1.2}$$

is defined by the series

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k. \tag{1.3}$$

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \neq 0, -1, -2, \ldots;\ j = 1, \ldots, s$), the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q;\ \beta_1, \ldots, \beta_s; z)$ is given by

$$\begin{gathered} {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}, \\ (q \leq s+1;\ q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \in \mathbb{U}), \end{gathered} \tag{1.4}$$

where $(\nu)_n$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\nu)_n := \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & \text{if } n = 0,\ \nu \in \mathbb{C} \setminus \{0\}, \\ \nu(\nu+1)\cdots(\nu+n-1) & \text{if } n \in \mathbb{N},\ \nu \in \mathbb{C}. \end{cases} \tag{1.5}$$

To define the Dziok-Srivastava operator

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \mathcal{A}_p \longrightarrow \mathcal{A}_p \tag{1.6}$$

via the Hadamard product given by (1.3), we consider a corresponding function

$$\mathcal{F}_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \tag{1.7}$$

defined by

$$\mathcal{F}_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) := z^p {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z). \tag{1.8}$$

The Dziok-Srivastava linear operator is now defined by the Hadamard product

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) := \mathcal{F}_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z). \tag{1.9}$$

This operator was introduced and studied in a series of recent papers by Dziok and Srivastava ([27–29]; see also [30, 31]). For convenience, we write

$$H_{p,q,s}(\alpha_1) := H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s). \tag{1.10}$$

The importance of the Dziok-Srivastava operator from the general convolution operator rests on the relation

$$z(H_{p,q,s}(\alpha_1) f(z))' = \alpha_1 H_{p,q,s}(\alpha_1 + 1) f(z) - (\alpha_1 - p) H_{p,q,s}(\alpha_1) f(z) \tag{1.11}$$

that can be verified by direct calculations (see, e.g., [27]). The linear operator $H_{p,q,s}(\alpha_1)$ includes various other linear operators as special cases. These include the operators introduced and studied by Carlson and Shaffer [32], Hohlov ([33], also see [34, 35]), and Ruscheweyh [36], as well as works in [27, 37].

2. Definitions and Lemmas

Recall that a domain $D \subset \mathbb{C}$ is *convex* if the line segment joining any two points in D lies entirely in D, while the domain is *starlike* with respect to a point $w_0 \in D$ if the line segment joining any point in D to w_0 lies inside D. An analytic function f is convex or starlike if $f(\mathbb{D})$ is, respectively, *convex* or *starlike* with respect to 0. For $f \in \mathcal{A} := \mathcal{A}_1$, analytically, these functions are described by the conditions $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ or $\operatorname{Re}(zf'(z)/f(z)) > 0$, respectively. More generally, for $0 \leq \alpha < 1$, the classes of convex functions of order α and starlike functions of order α are, respectively, defined by $\operatorname{Re}(1 + zf''(z)/f'(z)) > \alpha$ or $\operatorname{Re}(zf'(z)/f(z)) > \alpha$. A function f is close-to-convex if there is a convex function g (not necessarily normalized) such that $\operatorname{Re}(f'(z)/g'(z)) > 0$. Close-to-convex functions are known to be univalent.

The following definitions and lemmas will also be required in our present investigation.

Definition 2.1 (see [1, page 16]). Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$, and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination

$$\varphi(p(z), zp'(z)) \prec h(z), \tag{2.1}$$

then p is called a solution of differential subordination (2.1). A univalent function q is called a dominant of the solutions of differential subordination (2.1), or more simply a dominant, if $p \prec q$ for all p satisfying (2.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (2.1) is said to be the best dominant of (2.1).

Definition 2.2 (see [3, Definition 1, pages 816-817]). Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$, and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination

$$h(z) \prec \varphi(p(z), zp'(z)), \tag{2.2}$$

then p is called a solution of differential superordination (2.2). An analytic function q is called a subordinant of the solutions of differential superordination (2.2), or more simply a subordinant, if $q \prec p$ for all p satisfying (2.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2.2) is said to be the best subordinant of (2.2).

Definition 2.3 (see [1, Definition 2.2b, page 21]). Denote by Q the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} f(z) = \infty \right\}, \tag{2.3}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(f)$.

Lemma 2.4 (cf. [1, Theorem 2.3i, page 35]). *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition*

$$\operatorname{Re} H(is, t) \leq 0, \tag{2.4}$$

for all real s and $t \leq -n(1 + s^2)/2$, where n is a positive integer. If the function $p(z) = 1 + p_n z^n + \cdots$ is analytic in $\mathbb{U}$ and

$$\operatorname{Re} H\left(p(z), zp'(z)\right) > 0 \quad (z \in \mathbb{U}), \tag{2.5}$$

then $\operatorname{Re} p(z) > 0$ *in* $\mathbb{U}$.

One of the points of importance of Lemma 2.4 was its use in showing that every convex function is starlike of order $1/2$ (see e.g., [38, Theorem 2.6a, page 57]). In this paper, we take an opportunity to use the technique in the proof of Theorem 3.1.

Lemma 2.5 (see [39, Theorem 1, page 300]). *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, and let $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If $\operatorname{Re}(\beta h(z) + \gamma) > 0$ for $z \in \mathbb{U}$, then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathbb{U}) \tag{2.6}$$

with $q(0) = c$ is analytic in $\mathbb{U}$ and satisfies $\operatorname{Re}(\beta q(z) + \gamma) > 0$ $(z \in \mathbb{U})$.

Lemma 2.6 (see [1, Lemma 2.2d, page 24]). *Let $p \in Q$ with $p(0) = a$, and let $q(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{U}$ with $q(z) \not\equiv a$ and $n \geq 1$. If q is not subordinate to p, then there exists points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus E(p)$, for which $q(\mathbb{U}_{r_0}) \subset p(\mathbb{U})$,*

$$q(z_0) = p(\zeta_0), \qquad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n). \tag{2.7}$$

A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is a subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$, and $L(z,s) \prec L(z,t)$ for $0 \leq s < t$.

Lemma 2.7 (see [3, Theorem 7, page 822]). *Let $q \in \mathcal{H}[a,1]$, $\varphi : \mathbb{C}^2 \to \mathbb{C}$, and set $h(z) \equiv \varphi(q(z), zq'(z))$. If $L(z,t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $p \in \mathcal{H}[a,1] \cap Q$, then*

$$h(z) \prec \varphi\big(p(z), zp'(z)\big) \tag{2.8}$$

implies that

$$q(z) \prec p(z). \tag{2.9}$$

Furthermore, if $\varphi(q(z), zp'(z)) = h(z)$ has a univalent solution $q \in Q$, then q is the best subordinant.

Lemma 2.8 (see [3, Lemma B, page 822]). *The function $L(z,t) = a_1(t)z + \cdots$, with $a_1(t) \neq 0$ and $\lim_{t\to\infty}|a_1(t)| = \infty$, is a subordination chain if and only if*

$$\operatorname{Re}\left(\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right) > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty). \tag{2.10}$$

3. Main Results

We first prove the following subordination theorem involving the operator $H_{p,q,s}(\alpha_1)$ defined by (1.10).

Theorem 3.1. *Let $f, g \in \mathcal{A}_p$. For $\alpha_1 > 0$, $0 \leq \lambda < p$, let*

$$\varphi(z) := \frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)g(z)}{z^p} + \frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)g(z)}{z^p} \quad (z \in \mathbb{U}). \tag{3.1}$$

Suppose that

$$\operatorname{Re}\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right)>-\delta, \quad z\in\mathbb{U}, \tag{3.2}$$

where

$$\delta=\frac{(p-\lambda)^2+p^2\alpha_1^2-\left|(p-\lambda)^2-p^2\alpha_1^2\right|}{4p(p-\lambda)\alpha_1}. \tag{3.3}$$

Then the subordination condition

$$\frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p}+\frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}\prec\varphi(z) \tag{3.4}$$

implies that

$$\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}\prec\frac{H_{p,q,s}(\alpha_1)g(z)}{z^p}. \tag{3.5}$$

Moreover, the function $H_{p,q,s}(\alpha_1)g(z)/z^p$ is the best dominant.

Proof. Let us define the functions F and G, respectively, by

$$F(z):=\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}, \qquad G(z):=\frac{H_{p,q,s}(\alpha_1)g(z)}{z^p}. \tag{3.6}$$

We first show that if the function q is defined by

$$q(z):=1+\frac{zG''(z)}{G'(z)}, \tag{3.7}$$

then

$$\operatorname{Re} q(z)>0 \quad (z\in\mathbb{U}). \tag{3.8}$$

Logarithmic differentiation of both sides of the second equation in (3.6) and using (1.11) for $g\in\mathcal{A}_p$ yield

$$\frac{p\alpha_1}{p-\lambda}\varphi(z)=\frac{p\alpha_1}{p-\lambda}G(z)+zG'(z). \tag{3.9}$$

Now, differentiating both sides of (3.9) results in the following relationship:

$$
\begin{aligned}
1+\frac{z\varphi''(z)}{\varphi'(z)} &= 1+\frac{zG''(z)}{G'(z)}+\frac{zq'(z)}{q(z)+p\alpha_1/(p-\lambda)} \\
&= q(z)+\frac{zq'(z)}{q(z)+p\alpha_1/(p-\lambda)} \equiv h(z).
\end{aligned}
\tag{3.10}
$$

We also note from (3.2) that

$$
\operatorname{Re}\left(h(z)+\frac{p\alpha_1}{p-\lambda}\right)>0 \quad (z\in\mathbb{U}),
\tag{3.11}
$$

and, by using Lemma 2.5, we conclude that differential equation (3.10) has a solution $q \in \mathscr{H}(\mathbb{U})$ with $q(0)=h(0)=1$. Let us put

$$
H(u,v)=u+\frac{v}{u+p\alpha_1/(p-\lambda)}+\delta,
\tag{3.12}
$$

where δ is given by (3.3). From (3.2), (3.10), and (3.12), it follows that

$$
\operatorname{Re}\left(H\left(q(z),zq'(z)\right)\right)>0 \quad (z\in\mathbb{U}).
\tag{3.13}
$$

In order to use Lemma 2.4, we now proceed to show that $\operatorname{Re} H(is,t)\le 0$ for all real s and $t\le -(1+s^2)/2$. Indeed, from (3.12),

$$
\begin{aligned}
\operatorname{Re} H(is,t) &= \operatorname{Re}\left(is+\frac{t}{is+p\alpha_1/(p-\lambda)}+\delta\right) \\
&= \frac{tp\alpha_1/(p-\lambda)}{\left|p\alpha_1/(p-\lambda)+is\right|^2}+\delta \\
&\le -\frac{E_\delta(s)}{2\left|p\alpha_1/(p-\lambda)+is\right|^2},
\end{aligned}
\tag{3.14}
$$

where

$$
E_\delta(s) := \left(\frac{p\alpha_1}{p-\lambda}-2\delta\right)s^2-\frac{p\alpha_1}{p-\lambda}\left(2\delta\frac{p\alpha_1}{p-\lambda}-1\right).
\tag{3.15}
$$

For δ given by (3.3), we can prove easily that the expression $E_\delta(s)$ given by (3.15) is positive or equal to zero. Hence, from (3.14), we see that $\operatorname{Re} H(is,t)\le 0$ for all real s and $t\le -(1+s^2)/2$. Thus, by using Lemma 2.4, we conclude that $\operatorname{Re} q(z)>0$ for all $z\in\mathbb{U}$. That is,

G defined by (3.6) is convex in $\mathbb{U}$. Next, we prove that subordination condition (3.4) implies that

$$F(z) \prec G(z) \tag{3.16}$$

for the functions F and G defined by (3.6). Without loss of generality, we also can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. For this purpose, we consider the function $L(z,t)$ given by

$$L(z,t) := G(z) + \frac{(p-\lambda)(1+t)}{p\alpha_1} zG'(z) \quad (z \in \mathbb{U};\ 0 \leq t < \infty). \tag{3.17}$$

Note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(\frac{p\alpha_1 + (p-\lambda)(1+t)}{p\alpha_1}\right) \neq 0 \quad (0 \leq t < \infty;\ \alpha_1 > 0;\ 0 \leq \lambda < p). \tag{3.18}$$

This shows that the function

$$L(z,t) = a_1(t)z + \cdots \tag{3.19}$$

satisfies the condition $a_1(t) \neq 0$ for all $t \in [0,\infty)$. Furthermore,

$$\operatorname{Re}\left(\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right) = \operatorname{Re}\left(\frac{p\alpha_1}{p-\lambda} + (1+t)\left(1 + \frac{zG''(z)}{G'(z)}\right)\right) > 0. \tag{3.20}$$

Therefore, by virtue of Lemma 2.8, $L(z,t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \varphi(\mathbb{U}) \quad (\zeta \in \partial\mathbb{U};\ 0 \leq t < \infty). \tag{3.21}$$

Now suppose that F is not subordinate to G; then, by Lemma 2.6, there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\zeta_0), \qquad z_0F(z_0) = (1+t)\zeta_0G'(\zeta_0) \quad (0 \leq t < \infty). \tag{3.22}$$

Hence,

$$\begin{aligned} L(\zeta_0, t) &= G(\zeta_0) + \frac{(p-\lambda)(1+t)}{p\alpha_1}\zeta_0 G'(\zeta_0) \\ &= F(z_0) + \frac{(p-\lambda)}{p\alpha_1} z_0 F'(z_0) \\ &= \frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)f(z_0)}{z_0^p} + \frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)f(z_0)}{z_0^p} \in \varphi(\mathbb{U}), \end{aligned} \tag{3.23}$$

by virtue of subordination condition (3.4). This contradicts the above observation that $L(\zeta_0, t) \notin \varphi(\mathbb{U})$. Therefore, subordination condition (3.4) must imply the subordination given by (3.16). Considering $F(z) = G(z)$, we see that the function G is the best dominant. This evidently completes the proof of Theorem 3.1. □

We next prove a dual result to Theorem 3.1, in the sense that subordinations are replaced by superordinations.

Theorem 3.2. *Let $f, g \in \mathcal{A}_p$. For $\alpha_1 > 0$, $0 \le \lambda < p$, let*

$$\varphi(z) := \frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)g(z)}{z^p} + \frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)g(z)}{z^p} \quad (z \in \mathbb{U}). \tag{3.24}$$

Suppose that

$$\operatorname{Re}\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\delta, \quad z \in \mathbb{U}, \tag{3.25}$$

where δ is given by (3.3). *Further, suppose that*

$$\frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p} + \frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} \tag{3.26}$$

is univalent in $\mathbb{U}$ and $H_{p,q,s}(\alpha_1)f(z)/z^p \in \mathcal{H}[1,1] \cap Q$. Then the superordination

$$\varphi(z) \prec \frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p} + \frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} \tag{3.27}$$

implies that

$$\frac{H_{p,q,s}(\alpha_1)g(z)}{z^p} \prec \frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}. \tag{3.28}$$

Moreover, the function $H_{\lambda,q,s}(\alpha_1)g(z)/z^p$ is the best subordinant.

Proof. The first part of the proof is similar to that of Theorem 3.1 and so we will use the same notation as in the proof of Theorem 3.1.

Now let us define the functions F and G, respectively, by (3.6). We first note that if the function q is defined by (3.7), then (3.9) becomes

$$\varphi(z) = G(z) + \frac{p-\lambda}{p\alpha_1} zG'(z). \tag{3.29}$$

After a simple calculation, (3.29) yields the relationship

$$1 + \frac{z\varphi''(z)}{\varphi'(z)} = q(z) + \frac{zq'(z)}{q(z) + p\alpha_1/(p-\lambda)}. \tag{3.30}$$

Then by using the same method as in the proof of Theorem 3.1, we can prove that $\operatorname{Re} q(z) > 0$ for all $z \in \mathbb{U}$. That is, G defined by (3.6) is convex (univalent) in $\mathbb{U}$. Next, we prove that the subordination condition (3.27) implies that

$$G(z) \prec F(z) \tag{3.31}$$

for the functions F and G defined by (3.6). Now considering the function $L(z,t)$ defined by

$$L(z,t) := G(z) + \frac{(p-\lambda)t}{p\alpha_1} zG'(z) \quad (z \in \mathbb{U};\ 0 \le t < \infty), \tag{3.32}$$

we can prove easily that $L(z,t)$ is a subordination chain as in the proof of Theorem 3.1. Therefore according to Lemma 2.7, we conclude that superordination condition (3.27) must imply the superordination given by (3.31). Furthermore, since the differential equation (3.29) has the univalent solution G, it is the best subordinant of the given differential superordination. This completes the proof of Theorem 3.2. □

Combining Theorems 3.1 and 3.2, we obtain the following sandwich-type theorem.

Theorem 3.3. *Let $f, g_k \in \mathcal{A}_p$ $(k = 1,2)$. For $k = 1,2$, $\alpha_1 > 0$, $0 \le \lambda < p$, let*

$$\varphi_k(z) := \frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)g_k(z)}{z^p} + \frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)g_k(z)}{z^p} \quad (z \in \mathbb{U}). \tag{3.33}$$

Suppose that

$$\operatorname{Re}\left(1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\delta, \tag{3.34}$$

where δ is given by (3.2). *Further, suppose that*

$$\frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p} + \frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} \tag{3.35}$$

is univalent in $\mathbb{U}$ *and* $H_{\lambda,q,s}(\alpha_1)f(z)/z^p \in \mathcal{H}[1,1] \cap Q$. *Then*

$$\varphi_1(z) \prec \frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p} + \frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} \prec \varphi_2(z) \tag{3.36}$$

implies that

$$\frac{H_{p,q,s}(\alpha_1)g_1(z)}{z^p} \prec \frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} \prec \frac{H_{p,q,s}(\alpha_1)g_2(z)}{z^p}. \tag{3.37}$$

Moreover, the functions $H_{p,q,s}(\alpha_1)g_1(z)/z^p$ *and* $H_{p,q,s}(\alpha_1)g_2(z)/z^p$ *are the best subordinant and the best dominant, respectively.*

The assumption of Theorem 3.3 that the functions

$$\frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p} + \frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}, \qquad \frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} \tag{3.38}$$

need to be univalent in $\mathbb{U}$ may be replaced by another condition in the following result.

Corollary 3.4. *Let* $f, g_k \in \mathcal{A}_p$ $(k = 1,2)$. *For* $\alpha_1 > 0,\ 0 \le \lambda < p$, *let*

$$\psi(z) := \frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p} + \frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} \quad (z \in \mathbb{U}), \tag{3.39}$$

and φ_1, φ_2 *be as in* (3.33). *Suppose that condition* (3.34) *is satisfied and*

$$\operatorname{Re}\left(1 + \frac{z\psi''(z)}{\psi'(z)}\right) > -\delta, \quad z \in \mathbb{U}, \tag{3.40}$$

where δ *is given by* (3.3). *Then*

$$\varphi_1(z) \prec \frac{p-\lambda}{p}\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p} + \frac{\lambda}{p}\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} \prec \varphi_2(z) \tag{3.41}$$

implies that

$$\frac{H_{p,q,s}(\alpha_1)g_1(z)}{z^p} \prec \frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} \prec \frac{H_{p,q,s}(\alpha_1)g_2(z)}{z^p}. \tag{3.42}$$

Moreover, the functions $H_{p,q,s}(\alpha_1)g_1(z)/z^p$ *and* $H_{p,q,s}(\alpha_1)g_2(z)/z^p$ *are the best subordinant and the best dominant, respectively.*

Proof. In order to prove Corollary 3.4, we have to show that condition (3.40) implies the univalence of $\varphi(z)$ and

$$F(z) := \frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}. \tag{3.43}$$

Since δ given by (3.3) in Theorem 3.1 satisfies the inequality $0 < \delta \leq 1/2$, condition (3.40) means that φ is a close-to-convex function in $\mathbb{U}$ (see [40]) and hence φ is univalent in $\mathbb{U}$. Furthermore, by using the same techniques as in the proof of Theorem 3.1, we can prove the convexity (univalence) of F and so the details may be omitted. Therefore, from Theorem 3.3, we obtain Corollary 3.4. □

By taking $q = s + 1$, $\alpha_1 = \beta_1 = p$, $\alpha_i = \beta_i$ $(i = 2, 3, \ldots, s)$, $\alpha_{s+1} = 1$, and $\lambda = 0$ in Theorem 3.3, we have the following result.

Corollary 3.5. *Let $f, g_k \in \mathcal{A}_p$. Let*

$$\varphi_k(z) := \frac{g_k'(z)}{pz^{p-1}} \quad (k = 1, 2). \tag{3.44}$$

Suppose that

$$\operatorname{Re}\left(1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\frac{1}{2p} \quad (z \in \mathbb{U}) \tag{3.45}$$

and $f'(z)/pz^{p-1}$ is univalent in $\mathbb{U}$ and $f(z) \in \mathcal{H}[1,1] \cap Q$. Then

$$\frac{g_1'(z)}{pz^{p-1}} \prec \frac{f'(z)}{pz^{p-1}} \prec \frac{g_2'(z)}{pz^{p-1}} \tag{3.46}$$

implies that

$$\frac{g_1(z)}{z^p} \prec \frac{f(z)}{z^p} \prec \frac{g_2(z)}{z^p}. \tag{3.47}$$

Moreover, the functions $g_1(z)/z^p$ and $g_2(z)/z^p$ are the best subordinant and the best dominant, respectively.

Next consider the generalized Libera integral operator F_μ $(\mu > -p)$ defined by (cf. [37, 41–43])

$$F_\mu(f)(z) := \frac{\mu + p}{z^\mu} \int_0^z t^{\mu-1} f(t)\,dt \quad (f \in \mathcal{A}_p;\ \mu > -p). \tag{3.48}$$

For the choice $p = 1$, with $\mu \in \mathbb{N}$, (3.48) reduces to the well-known Bernardi integral operator [41]. The following is a sandwich-type result involving the generalized Libera integral operator F_μ.

Theorem 3.6. *Let $f, g_k \in \mathcal{A}_p$ $(k = 1,2)$. Let*

$$\varphi_k(z) := \frac{H_{p,q,s}(\alpha_1)g_k(z)}{z^p} \quad (k = 1,2). \tag{3.49}$$

Suppose that

$$\operatorname{Re}\left(1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\delta, \quad z \in \mathbb{U}, \tag{3.50}$$

where

$$\delta = \frac{1 + (\mu + p)^2 - \left|1 - (\mu + p)^2\right|}{4(\mu + p)} \quad (\mu > -p). \tag{3.51}$$

If $H_{p,q,s}(\alpha_1)f(z)/z^p$ is univalent in $\mathbb{U}$ and $H_{p,q,s}(\alpha_1)F_\mu(f)(z) \in \mathcal{H}[1,1] \cap Q$, then

$$\varphi_1(z) \prec \frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} \prec \varphi_2(z) \tag{3.52}$$

implies that

$$\frac{H_{p,q,s}(\alpha_1)F_\mu(g_1)(z)}{z^p} \prec \frac{H_{p,q,s}(\alpha_1)F_\mu(f)(z)}{z^p} \prec \frac{H_{p,q,s}(\alpha_1)F_\mu(g_2)(z)}{z^p}. \tag{3.53}$$

Moreover, the functions $H_{p,q,s}(\alpha_1)F_\mu(g_1)(z)/z^p$ and $H_{p,q,s}(\alpha_1)F_\mu(g_2)(z)/z^p$ are the best subordinant and the best dominant, respectively.

Proof. Let us define the functions F and G_k $(k = 1,2)$ by

$$F(z) := \frac{H_{p,q,s}(\alpha_1)F_\mu(f)(z)}{z^p}, \qquad G_k(z) := \frac{H_{p,q,s}(\alpha_1)F_\mu(g_k)(z)}{z^p}, \tag{3.54}$$

respectively. From the definition of the integral operator F_μ given by (3.48), it follows that

$$z\left(H_{p,q,s}(\alpha_1)F_\mu(f)(z)\right)' = (\mu + p)H_{p,q,s}(\alpha_1)f(z) - \mu H_{p,q,s}(\alpha_1)F_\mu(f)(z). \tag{3.55}$$

Then, from (3.49) and (3.55),

$$(\mu + p)\varphi_k(z) = (\mu + p)G_k(z) + zG_k'(z). \tag{3.56}$$

Setting

$$q_k(z) = 1 + \frac{zG_k''(z)}{G_k'(z)} \quad (k = 1,2;\ z \in \mathbb{U}), \tag{3.57}$$

and differentiating both sides of (3.51) result in

$$1+\frac{z\varphi_k''(z)}{\varphi_k'(z)}=q_k(z)+\frac{zq_k'(z)}{q_k(z)+\mu+p}. \tag{3.58}$$

The remaining part of the proof is similar to that of Theorem 3.3 (a combined proof of Theorems 3.1 and 3.2) and is therefore omitted. □

By using the same methods as in the proof of Corollary 3.4, the following result is obtained.

Corollary 3.7. *Let* $f, g_k \in \mathcal{A}_p$ $(k=1,2)$ *and*

$$\psi(z):=\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}. \tag{3.59}$$

Suppose that condition (3.50) *is satisfied and*

$$\operatorname{Re}\left(1+\frac{z\psi''(z)}{\psi'(z)}\right)>-\delta, \quad z\in\mathbb{U}, \tag{3.60}$$

where δ *is given by* (3.51). *Then*

$$\varphi_1(z)\prec\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}\prec\varphi_2(z) \tag{3.61}$$

implies that

$$\frac{H_{p,q,s}(\alpha_1)F_\mu(g_1)(z)}{z^p}\prec\frac{H_{p,q,s}(\alpha_1)F_\mu(f)(z)}{z^p}\prec\frac{H_{p,q,s}(\alpha_1)F_\mu(g_2)(z)}{z^p}. \tag{3.62}$$

Moreover, the functions $H_{p,q,s}(\alpha_1)F_\mu(g_1)(z)/z^p$ *and* $H_{p,q,s}(\alpha_1)F_\mu(g_2)(z)/z^p$ *are the best subordinant and the best dominant, respectively.*

Taking $q=s+1$, $\alpha_1=\beta_1=p$, $\alpha_i=\beta_i$ $(i=2,3,\ldots,s)$, and $\alpha_{s+1}=1$ in Corollary 3.7, we have the following result.

Corollary 3.8. *Let* $f, g_k \in \mathcal{A}_p$ $(k=1,2)$. *Let*

$$\varphi_k(z):=\frac{g_k(z)}{z^p} \quad (k=1,2). \tag{3.63}$$

Suppose that

$$\operatorname{Re}\left(1+\frac{z\varphi_k''(z)}{\varphi_k'(z)}\right)>-\delta, \quad z\in\mathbb{U}, \tag{3.64}$$

where δ *is given by* (3.51), *and* $f(z)/z^p$ *is univalent in* $\mathbb{U}$ *and* $F_\mu(f)(z)/z^p \in \mathcal{H}[1,1] \cap Q$. *Then,*

$$\frac{g_1(z)}{z^p} \prec \frac{f(z)}{z^p} \prec \frac{g_2(z)}{z^p} \tag{3.65}$$

implies that

$$\frac{F_\mu(g_1)(z)}{z^p} \prec \frac{F_\mu(f)(z)}{z^p} \prec \frac{F_\mu(g_2)(z)}{z^p}. \tag{3.66}$$

Moreover, the functions $F_\mu(g_1)(z)/z^p$ *and* $F_\mu(g_2)(z)/z^p$ *are the best subordinant and the best dominant, respectively.*

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References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.

[2] H. M. Srivastava and S. Owa, Eds., *Current Topics in Analytic Function Theory*, World Scientific, River Edge, NJ, USA, 1992.

[3] S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations," *Complex Variables. Theory and Application*, vol. 48, no. 10, pp. 815–826, 2003.

[4] T. Bulboacă, *Differential Subordinations and Superordinations: New Results*, House of Science Book Publ., Cluj-Napoca, Romania, 2005.

[5] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Subordination and superordination of the Liu-Srivastava linear operator on meromorphic functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 31, no. 2, pp. 193–207, 2008.

[6] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Subordination and superordination on Schwarzian derivatives," *Journal of Inequalities and Applications*, vol. 2008, Article ID 712328, 18 pages, 2008.

[7] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Differential subordination and superordination of analytic functions defined by the multiplier transformation," *Mathematical Inequalities & Applications*, vol. 12, no. 1, pp. 123–139, 2009.

[8] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "On subordination and superordination of the multiplier transformation for meromorphic functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 33, no. 2, pp. 311–324, 2010.

[9] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Differential subordination and superordination of analytic functions defined by the Dziok-Srivastava linear operator," *Journal of The Franklin Institute*, vol. 347, no. 9, pp. 1762–1781, 2010.

[10] R. M. Ali, V. Ravichandran, M. H. Khan, and K. G. Subramanian, "Differential sandwich theorems for certain analytic functions," *Far East Journal of Mathematical Sciences (FJMS)*, vol. 15, no. 1, pp. 87–94, 2004.

[11] R. M. Ali, V. Ravichandran, M. H. Khan, and K. G. Subramanian, "Applications of first order differential superordinations to certain linear operators," *Southeast Asian Bulletin of Mathematics*, vol. 30, no. 5, pp. 799–810, 2006.

[12] R. M. Ali and V. Ravichandran, "Classes of meromorphic α-convex functions," *Taiwanese Journal of Mathematics*, vol. 14, no. 4, pp. 1479–1490, 2010.

[13] T. Bulboacă, “A class of superordination-preserving integral operators,” *Indagationes Mathematicae*, vol. 13, no. 3, pp. 301–311, 2002.

[14] T. Bulboacă, “Classes of first-order differential superordinations,” *Demonstratio Mathematica*, vol. 35, no. 2, pp. 287–292, 2002.

[15] T. Bulboacă, “Generalized Briot-Bouquet differential subordinations and superordinations,” *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 47, no. 5-6, pp. 605–620, 2002.

[16] T. Bulboacă, “Sandwich-type theorems for a class of integral operators,” *Bulletin of the Belgian Mathematical Society. Simon Stevin*, vol. 13, no. 3, pp. 537–550, 2006.

[17] N. E. Cho and H. M. Srivastava, “A class of nonlinear integral operators preserving subordination and superordination,” *Integral Transforms and Special Functions*, vol. 18, no. 1-2, pp. 95–107, 2007.

[18] N. E. Cho, O. S. Kwon, S. Owa, and H. M. Srivastava, “A class of integral operators preserving subordination and superordination for meromorphic functions,” *Applied Mathematics and Computation*, vol. 193, no. 2, pp. 463–474, 2007.

[19] N. E. Cho and I. H. Kim, “A class of integral operators preserving subordination and superordination,” *Journal of Inequalities and Applications*, vol. 2008, Article ID 859105, 14 pages, 2008.

[20] N. E. Cho and O. S. Kwon, “A class of integral operators preserving subordination and superordination,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 33, no. 3, pp. 429–437, 2010.

[21] O. S. Kwon and N. E. Cho, “A class of nonlinear integral operators preserving double subordinations,” *Abstract and Applied Analysis*, vol. 2008, Article ID 792160, 10 pages, 2008.

[22] S. S. Miller and P. T. Mocanu, “Briot-Bouquet differential superordinations and sandwich theorems,” *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 327–335, 2007.

[23] R.-G. Xiang, Z.-G. Wang, and M. Darus, “A family of integral operators preserving subordination and superordination,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 33, no. 1, pp. 121–131, 2010.

[24] T. Bulboacă, “Integral operators that preserve the subordination,” *Bulletin of the Korean Mathematical Society*, vol. 34, no. 4, pp. 627–636, 1997.

[25] S. S. Miller, P. T. Mocanu, and M. O. Reade, “Subordination-preserving integral operators,” *Transactions of the American Mathematical Society*, vol. 283, no. 2, pp. 605–615, 1984.

[26] S. Owa and H. M. Srivastava, “Some subordination theorems involving a certain family of integral operators,” *Integral Transforms and Special Functions*, vol. 15, no. 5, pp. 445–454, 2004.

[27] J. Dziok and H. M. Srivastava, “Classes of analytic functions associated with the generalized hypergeometric function,” *Applied Mathematics and Computation*, vol. 103, no. 1, pp. 1–13, 1999.

[28] J. Dziok and H. M. Srivastava, “Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric function,” *Advanced Studies in Contemporary Mathematics*, vol. 5, no. 2, pp. 115–125, 2002.

[29] J. Dziok and H. M. Srivastava, “Certain subclasses of analytic functions associated with the generalized hypergeometric function,” *Integral Transforms and Special Functions*, vol. 14, no. 1, pp. 7–18, 2003.

[30] J.-L. Liu and H. M. Srivastava, “Certain properties of the Dziok-Srivastava operator,” *Applied Mathematics and Computation*, vol. 159, no. 2, pp. 485–493, 2004.

[31] J.-L. Liu and H. M. Srivastava, “Classes of meromorphically multivalent functions associated with the generalized hypergeometric function,” *Mathematical and Computer Modelling*, vol. 39, no. 1, pp. 21–34, 2004.

[32] B. C. Carlson and D. B. Shaffer, “Starlike and prestarlike hypergeometric functions,” *SIAM Journal on Mathematical Analysis*, vol. 15, no. 4, pp. 737–745, 1984.

[33] Ju. E. Hohlov, “Operators and operations on the class of univalent functions,” *Izvestiya Vysshikh Uchebnykh Zavedeniĭ. Matematika*, vol. 10 (197), pp. 83–89, 1978.

[34] H. Saitoh, “A linear operator and its applications of first order differential subordinations,” *Mathematica Japonica*, vol. 44, no. 1, pp. 31–38, 1996.

[35] H. M. Srivastava and S. Owa, “Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions,” *Nagoya Mathematical Journal*, vol. 106, pp. 1–28, 1987.

[36] S. Ruscheweyh, “New criteria for univalent functions,” *Proceedings of the American Mathematical Society*, vol. 49, pp. 109–115, 1975.

[37] S. Owa and H. M. Srivastava, “Some applications of the generalized Libera integral operator,” *Proceedings of the Japan Academy, Series A*, vol. 62, no. 4, pp. 125–128, 1986.

[38] S. S. Miller and P. T. Mocanu, “Differential subordinations and univalent functions,” *The Michigan Mathematical Journal*, vol. 28, no. 2, pp. 157–172, 1981.

[39] S. S. Miller and P. T. Mocanu, “Univalent solutions of Briot-Bouquet differential equations,” *Journal of Differential Equations*, vol. 56, no. 3, pp. 297–309, 1985.

[40] W. Kaplan, “Close-to-convex schlicht functions,” *The Michigan Mathematical Journal*, vol. 1, pp. 169–185, 1952.

[41] S. D. Bernardi, “Convex and starlike univalent functions,” *Transactions of the American Mathematical Society*, vol. 135, pp. 429–446, 1969.

[42] R. M. Goel and N. S. Sohi, “A new criterion for p-valent functions,” *Proceedings of the American Mathematical Society*, vol. 78, no. 3, pp. 353–357, 1980.

[43] R. J. Libera, “Some classes of regular univalent functions,” *Proceedings of the American Mathematical Society*, vol. 16, pp. 755–758, 1965.