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Research Article

On Some Generalized B^m-Difference Riesz Sequence Spaces and Uniform Opial Property

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We define the new generalized difference Riesz sequence spaces $r_\infty^q(p,B^m)$, $r_c^q(p,B^m)$, and $r_0^q(p,B^m)$ which consist of all the sequences whose B^m-transforms are in the Riesz sequence spaces $r_\infty^q(p)$, $r_c^q(p)$, and $r_0^q(p)$, respectively, introduced by Altay and Başar (2006). We examine some topological properties and compute the α-, β-, and γ-duals of the spaces $r_\infty^q(p,B^m)$, $r_c^q(p,B^m)$, and $r_0^q(p,B^m)$. Finally, we determine the necessary and sufficient conditions on the matrix transformation from the spaces $r_\infty^q(p,B^m)$, $r_c^q(p,B^m)$, and $r_0^q(p,B^m)$ to the spaces l_∞ and c and prove that sequence spaces $r_0^q(p,B^m)$ and $r_c^q(p,B^m)$ have the uniform Opial property for $p_k \geq 1$ for all $k \in \mathbb{N}$.

1. Introduction

Let w be the space of real sequences. We write l_∞, c, c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Also, by bs, cs, and l_1, we denote the sequence spaces of all bounded, convergent and absolutely convergent series, respectively.

A linear topological space X over the real field R is said to be a paranormed space if there is a subadditive function $g : X \to R$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous; that is, $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in R and all x's in X, where θ is the zero vector in the linear space X. Assume here and after that $p = (p_k)$ is a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear space $l_\infty(p)$, $c(p)$, $c_0(p)$, and $l(p)$ were defined by Maddox [1, 2], (Nakano [3] and Simons [4]) as follows:

$$
\begin{aligned}
l_\infty(p) &= \left\{ x = (x_k) \in w : \sup_{k \in \mathbb{N}} |x_k|^{p_k} < \infty \right\}, \\
c(p) &= \left\{ x = (x_k) \in w : \lim_{k \to \infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R} \right\}, \\
c_0(p) &= \left\{ x = (x_k) \in w : \lim_{k \to \infty} |x_k|^{p_k} = 0 \right\},
\end{aligned} \tag{1.1}
$$

which are the complete spaces paranormed by

$$
\begin{gathered}
g_1(x) = \sup_{k \in \mathbb{N}} |x_k|^{p_k/M}, \\
l(p) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty \right\}, \quad \text{with } (0 < p_k \leq H < \infty),
\end{gathered} \tag{1.2}
$$

which is the complete space paranormed by

$$
g_2(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M}. \tag{1.3}
$$

For simplicity notation, here and in what follows, the summation without limits runs from 0 to ∞. We assume throughout that $(p_k)^{-1} + (p_k')^{-1} = 1$ provided $1 < \inf p_k \leq H < \infty$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ and we denote it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$
(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.4}
$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.4) converges for each $n \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

Let (q_k) be a sequence of positive numbers and

$$
Q_n = \sum_{k=0}^{n} q_k, \quad (n \in \mathbb{N}). \tag{1.5}
$$

Then, the matrix $R^q = (r_{nk}^q)$ of the Riesz mean is given by

$$
r_{nk}^q = \begin{cases} \dfrac{q_k}{Q_n}, & (0 \leq k \leq n), \\ 0, & (k > n). \end{cases} \tag{1.6}
$$

The Riesz sequence spaces introduced by Altay and Başar in [5, 6] are

$$\begin{aligned} r^q(p) &= \left\{ x = (x_k) \in w : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} q_j x_j \right|^{p_k} < \infty \right\}, \quad \text{with } (0 < p_k \le H < \infty), \\ r_c^q(p) &= \left\{ x = (x_k) \in w : \lim_{k\to\infty} \left| \frac{1}{Q_k} \sum_{j=0}^{k} q_j x_j - l \right|^{p_k} = 0, \text{ for some } l \in R \right\}, \\ r_0^q(p) &= \left\{ x = (x_k) \in w : \lim_{k\to\infty} \left| \frac{1}{Q_k} \sum_{j=0}^{k} q_j x_j \right|^{p_k} = 0 \right\}, \\ r_\infty^q(p) &= \left\{ x = (x_k) \in w : \sup_{k\in N} \left| \frac{1}{Q_k} \sum_{j=0}^{k} q_j x_j \right|^{p_k} < \infty \right\}, \end{aligned} \tag{1.7}$$

which are the sequence spaces of the sequences x whose R^q-transforms are in $l(p)$, $c(p)$, $c_0(p)$, and $l_\infty(p)$, respectively.

Also, Altay and Başar [7] introduced the generalized difference matrix $B = (b_{nk})$ by

$$b_{nk} = \begin{cases} r, & (k = n), \\ s, & (k = n-1), \\ 0, & (0 \le k < n-1) \text{ or } (k > n) \end{cases} \tag{1.8}$$

for all $k, n \in \mathbb{N}$, $r, s \in \mathbb{R} - \{0\}$. The matrix B can be reduced the difference matrix Δ in case $r = 1$, $s = -1$ by, where Δ denotes the matrix $\Delta = (\Delta_{nk})$ defined by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & (n-1 \le k \le n), \\ 0, & (k < n-1) \text{ or } (k > n). \end{cases} \tag{1.9}$$

The results related to the matrix domain of the matrix B are more general and more comprehensive than the corresponding consequences of matrix domain of Δ and include them [6, 8–13].

Başarır and Kayikçi [14] defined the matrix $B^m = (b_{nk}^m)$ which reduced the difference matrix $\Delta^m = \Delta(\Delta^{m-1})$ in case $r = 1$, $s = -1$ by

$$b_{nk}^m = \begin{cases} \begin{pmatrix} m \\ n-k \end{pmatrix} r^{m-n+k} s^{n-k}; & (\max\{0, n-m\} \le k \le n), \\ 0; & (0 \le k < \max\{0, n-m\}) \text{ or } (k > n), \end{cases} \tag{1.10}$$

and introduced the generalized B^m-difference Riesz sequence space which is the sequence space of the sequences x whose $R^q B^m$-transforms are in $l(p)$.

The main purpose of this paper is to introduce the B^m-difference Riesz sequence spaces $r^q_\infty(p, B^m)$, $r^q_c(p, B^m)$, and $r^q_0(p, B^m)$ of the sequences whose $R^q B^m$-transform are in $l_\infty(p)$, $c(p)$, and $c_0(p)$, respectively, and to investigate some topological and geometric properties of them. For simplicity, we take the matrix $R^q B^m = T$.

2. B^m-Difference Riesz Sequence Spaces

Let us define the sequence $y = \{y_n(q)\}$, which is used, as the $R^q B^m = T$-transform of a sequence $x = (x_k)$, that is,

$$y_n(q) = (Tx)_n = \frac{1}{Q_n}\sum_{k=0}^{n-1}\left[\sum_{i=k}^{n}\binom{m}{i-k} r^{m-i+k} s^{i-k} q_i x_k\right] + \frac{r^m}{Q_n} q_n x_n, \quad (n \in \mathbb{N}). \tag{2.1}$$

We define the B^m-difference Riesz sequence spaces $r^q_\infty(p, B^m)$, $r^q_c(p, B^m)$, and $r^q_0(p, B^m)$ by

$$\begin{aligned}
r^q_\infty(p, B^m) &= \left\{x = (x_j) \in w : ((Tx)_n) \in l_\infty(p)\right\}, \\
r^q_c(p, B^m) &= \left\{x = (x_j) \in w : ((Tx)_n) \in c(p)\right\}, \\
r^q_0(p, B^m) &= \left\{x = (x_j) \in w : ((Tx)_n) \in c_0(p)\right\}.
\end{aligned} \tag{2.2}$$

If $m = 1$ then they are reduced the spaces $r^q_\infty(p, B)$, $r^q_c(p, B)$, and $r^q_0(p, B)$ defined by Başarır in [15]. If we take $B = \Delta$ then we have $r^q_\infty(p, \Delta^m)$, $r^q_c(p, \Delta^m)$, and $r^q_0(p, \Delta^m)$. If we take $B = \Delta$ and $m = 1$ then we have $r^q_\infty(p, \Delta)$, $r^q_c(p, \Delta)$, and $r^q_0(p, \Delta)$. If we take $p_k = p$ for all k then we have $r^q_\infty(B^m)$, $r^q_c(B^m)$, and $r^q_0(B^m)$.

We have the following.

Theorem 2.1. *$r^q_0(p, B^m)$ is a complete linear metric space paranormed by g_B, defined by*

$$g_B(x) = \sup_{k\in\mathbb{N}} |(Tx)_k|^{p_k/M}, \tag{2.3}$$

g_B is a paranorm for the spaces $r^q_\infty(p, B^m)$ and $r^q_c(p, B^m)$ only in the trivial case with $\inf p_k > 0$ *when $r^q_\infty(p, B^m) = r^q_\infty(B^m)$ and $r^q_c(p, B^m) = r^q_c(B^m)$.*

Proof. We prove the theorem for the space $r_0^q(p, B^m)$. The linearity of $r_0^q(p, B^m)$ with respect to the coordinatewise addition and scalar multiplication that follow from the inequalities which are satisfied for $u, v \in r_0^q(p, B^m)$ [16]

$$\begin{aligned}
&\sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i(u_j+v_j)\right]+\frac{r^m q_k}{Q_k}(u_k+v_k)\right]\right|^{p_k/M}\\
&\quad\le \sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i u_j\right]+\frac{r^m q_k}{Q_k}u_k\right]\right|^{p_k/M}\\
&\quad+\sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i v_j\right]+\frac{r^m q_k}{Q_k}v_k\right]\right|^{p_k/M},
\end{aligned} \tag{2.4}$$

and for any $\alpha \in \mathbb{R}$ [1],

$$|\alpha|^{p_k} \le \max\left\{1, |\alpha|^M\right\}. \tag{2.5}$$

It is clear that $g_B(\theta) = 0$ and $g_B(-x) = g_B(x)$ for all $x \in r_0^q(p, B^m)$. Again, the inequalities (2.4) and (2.5) yield the subadditivity of g_B and

$$g_B(\alpha u) \le \max\{1, |\alpha|\} g_B(u). \tag{2.6}$$

Let $\{x^n\}$ be any sequence of the elements of the space $r_0^q(p, B^m)$ such that

$$g_B(x^n - x) \longrightarrow 0, \tag{2.7}$$

and (λ_n) also be any sequence of scalars such that $\lambda_n \to \lambda$, as $n \to \infty$. Then, since the inequality

$$g_B(x^n) \le g_B(x) + g_B(x^n - x) \tag{2.8}$$

holds by subadditivity of g_B, $\{g_B(x^n)\}$ is bounded, and thus we have

$$\begin{aligned}
&g_B(\lambda_n x^n - \lambda x)\\
&= \sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i\left(\lambda_n x_j^n - \lambda x_j\right)\right] + \frac{r^m q_k}{Q_k}(\lambda_n x_k^n - \lambda x_k)\right]\right|^{p_k/M}\\
&= |\lambda_n - \lambda|^{1/M}\sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i x_j^n\right] + \frac{r^m q_k}{Q_k}x_k^n\right]\right|^{p_k/M}\\
&\quad + |\lambda|^{1/M}\sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i\left(x_j^n - x_j\right)\right] + \frac{r^m q_k}{Q_k}(x_k^n - x_k)\right]\right|^{p_k/M}\\
&\leq |\lambda_n - \lambda|^{1/M}g_B(x^n) + |\lambda|^{1/M}g_B(x^n - x),
\end{aligned} \tag{2.9}$$

which tends to zero as $n \to \infty$. Hence, the scalar multiplication is continuous. Finally, it is clear to say that g_B is a paranorm on the space $r_0^q(p, B^m)$. Moreover, we will prove the completeness of the space $r_0^q(p, B^m)$. Let x^i be a Cauchy sequence in the space $r_0^q(p, B^m)$, where $x^i = \{x_k^{(i)}\} = \{x_0^i, x_1^i, x_2^i, \ldots\} \in r_0^q(p, B^m)$. Then, for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$g_B\left(x^i - x^j\right) < \varepsilon, \tag{2.10}$$

for all $i, j \geq n_0(\varepsilon)$. If we use the definition of g_B, we obtain for each fixed $k \in \mathbb{N}$ that

$$\left|\left(Tx^i\right)_k - \left(Tx^j\right)_k\right| \leq \sup_{k\in\mathbb{N}}\left|\left(Tx^i\right)_k - \left(Tx^j\right)_k\right|^{p_k/M} < \varepsilon, \tag{2.11}$$

for $i, j \geq n_0(\varepsilon)$ which leads us to the fact that

$$\left\{\left(Tx^0\right)_k, \left(Tx^1\right)_k, \left(Tx^2\right)_k, \ldots\right\} \tag{2.12}$$

is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, so we write $(Tx^i)_k \to (Tx)_k$ as $i \to \infty$. Hence, by using these infinitely many limits $(Tx)_0, (Tx)_1, (Tx)_2, \ldots$, we define the sequence $\{(Tx)_0, (Tx)_1, (Tx)_2, \ldots\}$. From (2.11) with $j \to \infty$, we have

$$\left|\left(Tx^i\right)_k - (Tx)_k\right| < \varepsilon, \tag{2.13}$$

$i \geq n_0(\varepsilon)$ for every fixed $k \in \mathbb{N}$. Since $x^i = \{x_k^{(i)}\} \in r_0^q(p, B^m)$,

$$\left|\left(Tx^i\right)_k\right|^{p_k/M} < \varepsilon, \tag{2.14}$$

for all $k \in \mathbb{N}$. Therefore, by (2.13), we obtain that

$$|(Tx)_k|^{p_k/M} \leq \left|(Tx)_k - \left(Tx^i\right)_k\right|^{p_k/M} + \left|\left(Tx^i\right)_k\right|^{p_k/M} < \varepsilon, \tag{2.15}$$

for all $i \geq n_0(\varepsilon)$. This shows that the sequence Tx belongs to the space $c_0(p)$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $r_0^q(p, B^m)$ is complete. □

Theorem 2.2. *Let $\sum_{i=j}^{k}\binom{m}{i-j} r^{m-i+j} s^{i-j} q_i \neq 0$ for all k, m and $0 \leq j \leq k-1$. Then the B^m-difference Riesz sequence spaces $r_\infty^q(p, B^m)$, $r_c^q(p, B^m)$, and $r_0^q(p, B^m)$ are linearly isomorphic to the spaces $l_\infty(p)$, $c(p)$, and $c_0(p)$, respectively, where $0 < p_k \leq H < \infty$.*

Proof. We establish this for the space $r_\infty^q(p, B^m)$. For the proof of the theorem, we should show the existence of a linear bijection between the space $r_\infty^q(p, B^m)$ and $l_\infty(p)$ for $0 < p_k \leq H < \infty$. With the notation of (2.1), define the transformation S from $r_\infty^q(p, B^m)$ to $l_\infty(p)$ by $x \mapsto y = Sx.S$ is a linear transformation, morever; it is obviuos that $x = \theta$ whenever $Sx = \theta$ and hence S is injective.

Let $y = (y_k) \in l_\infty(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{n=0}^{k-1}\left[\sum_{i=n}^{n+1}(-1)^{k-n}\frac{s^{k-i}}{r^{m+k-i}}\binom{m+k-i-1}{k-i}\frac{1}{q_i}Q_n y_n\right] + \frac{Q_k}{r^m q_k}y_k, \quad \forall k \in \mathbb{N}. \tag{2.16}$$

Then,

$$\begin{aligned} g_B(x) &= \sup_{k\in\mathbb{N}}\left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i x_j\right] + \frac{r^m q_k}{Q_k}x_k\right|^{p_k/M} \\ &= \sup_{k\in\mathbb{N}}\left|\sum_{j=0}^{k}\delta_{kj}y_j\right|^{p_k/M} = \sup_{k\in\mathbb{N}}|y_k|^{p_k/M} = g_1(y) < \infty, \end{aligned} \tag{2.17}$$

where

$$\delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{2.18}$$

Thus, we have that $x \in r_\infty^q(\mathrm{p}, B^m)$. Consequently, S is surjective and is paranorm preserving. Hence, S is linear bijection, and this explains that the spaces $r_\infty^q(p, B^m)$ and $l_\infty(p)$ are linearly isomorphic. □

3. The Basis for the Spaces $r_c^q(p, B^m)$ and $r_0^q(p, B^m)$

In this section, we give two sequences of the points of the spaces $r_0^q(p, B^m)$ and $r_c^q(p, B^m)$ which form the basis for those spaces.

If a sequence space λ paranormed by h contains a sequence (b_n) with the property that for every $x \in \lambda$, there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} h\left(x - \sum_{k=0}^{n} \alpha_k \beta_k \right) = 0, \tag{3.1}$$

then (b_n) is called a Schauder basis (or briefly basis) for λ. The series $\sum \alpha_k \beta_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k \beta_k$.

Because of the isomorphism S is onto, defined in the proof of Theorem 2.2, the inverse image of the basis of the spaces, $c_0(p)$ and $c(p)$ are the basis of the new spaces $r_0^q(p, B^m)$ and $r_c^q(p, B^m)$, respectively.

We have the following.

Theorem 3.1. *Let $\mu_k(q) = (Tx)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \le H < \infty$. Define the sequence $b^{(k)}(q) = \{b_n^{(k)}(q)\}_{n\in\mathbb{N}}$ of the elements of the space $r_0^q(p, B^m)$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(q) = \begin{cases} \sum_{i=k}^{k+1} (-1)^{n-k} \dfrac{s^{n-i}}{r^{m+n-i}} \dbinom{m+n-i-1}{n-i} \dfrac{1}{q_i} Q_k, & (n > k), \\ \dfrac{Q_n}{r^m q_n}, & (k = n), \\ 0, & (k > n). \end{cases} \tag{3.2}$$

Then, one has the following.

(a) *The sequence $\{b^{(k)}(q)\}_{k\in\mathbb{N}}$ is a basis for the space $r_0^q(p, B^m)$, and any $x \in r_0^q(p, B^m)$ has a unique representation of the form*

$$x = \sum_k \mu_k(q) b^{(k)}(q). \tag{3.3}$$

(b) *The set $\{z = (T)^{-1}e, b^{(k)}(q)\}$ is a basis for the space $r_c^q(p, B^m)$ and any $x \in r_c^q(p, B^m)$ has a unique representation of the form*

$$x = le + \sum_k \left| \mu_k(q) - l \right| b^{(k)}(q), \tag{3.4}$$

where

$$l = \lim_{k\to\infty} (Tx)_k. \tag{3.5}$$

Proof. It is clear that $\{b^{(k)}(q)\} \subset r_0^q(p, B^m)$, since

$$Tb^{(k)}(q) = e^{(k)} \in c_0(p), \quad (\text{for } k \in \mathbb{N}), \tag{3.6}$$

for $0 < p_k \leq H < \infty$, where $e^{(k)}$ is the sequence whose only non-zero term is 1 in kth place for each $k \in \mathbb{N}$. Let $x \in r_0^q(p, B^m)$ be given. For every nonnegative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \mu_k(q) b^{(k)}(q). \tag{3.7}$$

Then, we obtain by applying T to (3.7) with (3.6) that

$$\begin{gathered} Tx^{[m]} = \sum_{k=0}^{m} \mu_k(q) Tb^{(k)}(q) = \sum_{k=0}^{m} (T)_k e^{(k)}, \\ \left(R^q\left(x - x^{[M]}\right)\right)_i = \begin{cases} 0, & (0 \leq i \leq m), \\ (Tx)_i, & (i > m). \end{cases} \end{gathered} \tag{3.8}$$

Given $\varepsilon > 0$, then there exists an integer m_0 such that

$$\sup_{i \geq m} |(Tx)_i|^{p_k/M} < \frac{\varepsilon}{2}, \tag{3.9}$$

for all $m \geq m_0$. Hence,

$$g_B\left(x - x^{[m]}\right) = \sup_{i \geq m} |(Tx)_i|^{p_k/M} \leq \sup_{i \geq m_0} |(Tx)_i|^{p_k/M} < \frac{\varepsilon}{2} < \varepsilon, \tag{3.10}$$

for all $m \geq m_0$, which proves that $x \in r_0^q(p, B^m)$ is represented as in (3.3).

To show the uniqueness of this representation, we suppose that there exists a representation

$$x = \sum_k \lambda_k(q) b^{(k)}(q). \tag{3.11}$$

Since the linear transformation S from $r_0^q(p, B^m)$ to $c_0(p)$, used in Theorem 2.2, is continuous we have

$$(Tx)_n = \sum_k \lambda_k(q) \left\{Tb^{(k)}(q)\right\}_n = \sum_k \lambda_k(q) e_n^{(k)} = \lambda_n(q); \quad n \in \mathbb{N}, \tag{3.12}$$

which contradicts the fact that $(Tx)_n = \mu_k(q)$ for all $n \in \mathbb{N}$. Hence, the representation (3.3) of $x \in r_0^q(p, B^m)$ is unique. Thus, the proof of the part (a) of theorem is completed.

(b) Since $\{b^{(k)}(q)\} \subset r_0^q(p, B^m)$ and $e \in c(p)$, the inclusion $\{e, b^{(k)}(q)\} \subset r_c^q(p, B^m)$ trivially holds. Let us take $x \in r_c^q(p, B^m)$. Then, there uniquely exists an l satisfying (3.5). We thus have the fact that $u \in r_0^q(p, B^m)$ whenever we set $u = x - le$. Therefore, we deduce by part (a) of the present theorem that the representation of x given by (3.4) is unique and this step concludes the proof of the part (b) of theorem. □

4. The α-, β-, and γ-Duals of the Spaces $r_c^q(p,B^m)$, $r_0^q(p,B^m)$, and $r_\infty^q(p,B^m)$

In this section, we prove the theorems determining the α-, β-, and γ-duals of the sequence spaces $r_c^q(p,B^m)$ and $r_0^q(p,B^m)$.

For the sequence spaces λ and μ, define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \ \forall x \in \lambda\}. \tag{4.1}$$

With the notation (4.1), the α-, β-, γ-duals of a sequence space λ, which are, respectively, denoted by λ^α, λ^β, and λ^γ are defined by

$$\lambda^\alpha = S(\lambda, l_1), \qquad \lambda^\beta = S(\lambda, cs), \qquad \lambda^\gamma = S(\lambda, bs). \tag{4.2}$$

Now, we give some lemmas which we need to prove our theorems

Lemma 4.1 (see [17]). *$A \in (l_\infty(p) : l_1)$ if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} K^{1/p_k} \right| < \infty, \quad \forall \text{ integers } K > 1. \tag{4.3}$$

Lemma 4.2 (see [18]). *Let $p_k > 0$ for every $k \in \mathbb{N}$. Then, $A \in (l_\infty(p) : l_\infty)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| K^{1/p_k} < \infty, \quad \forall \text{ integers } K > 1. \tag{4.4}$$

Lemma 4.3 (see [18]). *Let $p_k > 0$ for every $k \in \mathbb{N}$. Then, $A \in (l_\infty(p) : c)$ if and only if*

$$\sum_k |a_{nk}| K^{1/p_k} \text{ converges uniformly in } n, \quad \forall \text{ integers } K > 1, \tag{4.5}$$

$$\lim_{n\to\infty} \alpha_{nk}, \quad \forall k \in \mathbb{N}. \tag{4.6}$$

Theorem 4.4. *For each $m \in \mathbb{N}$, define the sets $R_1(p)$, $R_2(p)$, $R_3(p)$, $R_4(p)$, $R_5(p)$, and $R_6(p)$ as follows:*

$$R_1(p) = \bigcap_{K>1}\left\{a=(a_k)\in w : \sup_{N\in\mathcal{F}}\sum_n\left|\sum_{k\in N}\left[\sum_{i=k}^{k+1}\nabla(i,n,k)Q_k a_n + \frac{Q_n a_n}{r^m q_n}\right]K^{1/p_k}\right| < \infty,\ \forall K>1\right\},$$

$$R_2(p) = \bigcap_{K>1}\left\{a=(a_k)\in w : \sum_k\left|\left(\frac{a_k}{r^m q_k} + \nabla(i,n,k)\sum_{j=k+1}^{n} a_j\right)Q_k\right|K^{1/p_k}\ \text{converges}\right.$$
$$\left.\text{uniformly in } n \text{ and}\left(\frac{a_k Q_k}{r^m q_k}K^{1/p_k}\right)\in c_0,\ \forall K>1\right\},$$

$$R_3(p) = \bigcap_{K>1}\left\{a=(a_k)\in w : \sum_k\left|\left(\frac{a_k}{r^m q_k} + \nabla(i,n,k)\sum_{j=k+1}^{n} a_j\right)Q_k\right|K^{1/p_k} < \infty,\ \forall K>1,\right.$$
$$\left.\left\{\left(\frac{a_k}{r^m q_k} + \nabla(i,n,k)\sum_{i=k+1}^{n} a_i\right)Q_k\right\}\in l_\infty\right\},$$

$$R_4(p) = \bigcup_{K>1}\left\{a=(a_k)\in w : \sup_{N\in\mathcal{F}}\sum_n\left|\sum_{k\in N}\left[\sum_{i=k}^{k+1}\nabla(i,n,k)Q_k a_i + \frac{Q_n a_n}{r^m q_n}\right]K^{-1/p_k}\right| < \infty,\ \forall K>1\right\},$$

$$R_5(p) = \left\{a=(a_k)\in w : \sum_n\left|\sum_k\left[\sum_{i=k}^{k+1}\nabla(i,n,k)Q_k a_i + \frac{Q_n a_n}{r^m q_n}\right]\right| < \infty\right\},$$

$$R_6(p) = \bigcup_{K>1}\left\{a=(a_k)\in w : \sum_k\left|\left(\frac{a_k}{r^m q_k} + \nabla(i,n,k)\sum_{j=k+1}^{n} a_j\right)Q_k\right|K^{-1/p_k} < \infty, \forall K>1\right\}, \tag{4.7}$$

where

$$\nabla(i,n,k) = (-1)^{n-k}\frac{s^{n-i}}{r^{m+n-i}}\binom{m+n-i-1}{n-i}\frac{1}{q_i}. \tag{4.8}$$

Then,

$$\begin{gathered}
\left\{r_\infty^q(p,B^m)\right\}^\alpha = R_1(p), \qquad \left\{r_\infty^q(p,B^m)\right\}^\beta = R_2(p), \qquad \left\{r_\infty^q(p,B^m)\right\}^\gamma = R_3(p),\\
\left\{r_c^q(p,B^m)\right\}^\alpha = R_4(p)\cap R_5(p), \qquad \left\{r_c^q(p,B^m)\right\}^\beta = R_6(p)\cap cs,\\
\left\{r_c^q(p,B^m)\right\}^\gamma = R_6(p)\cap bs,
\end{gathered} \tag{4.9}$$

$$\left\{r_0^q(p,B^m)\right\}^\alpha = R_4(p), \qquad \left\{r_0^q(p,B^m)\right\}^\beta = \left\{r_0^q(p,B^m)\right\}^\gamma = R_6(p).$$

Proof. We give the proof for the space $r_\infty^q(p, B^m)$. Let us take any $a = (a_n) \in w$. We easily derive with the notation

$$y_k = \frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i x_j\right] + \frac{r^m}{Q_k}q_k x_k, \tag{4.10}$$

that

$$\begin{aligned} a_n x_n &= \sum_{k=0}^{n-1}\left[\sum_{i=k}^{k+1}\nabla(i,n,k)a_n Q_k y_k\right] + \frac{a_n Q_n y_n}{r^m q_n} \\ &= \sum_{k=0}^{n} u_{nk} y_k = (Uy)_n, \end{aligned} \tag{4.11}$$

$(n \in \mathbb{N})$, where $U = (u_{nk})$ is defined by

$$u_{nk} = \begin{cases} \displaystyle\sum_{i=k}^{k+1}\nabla(i,n,k)a_n Q_i, & (0 \le k \le n-1), \\ \dfrac{a_n Q_n}{r^m q_n}, & (k = n), \\ 0, & (k > n), \end{cases} \tag{4.12}$$

for all $k, n \in \mathbb{N}$. Thus we deduce from (4.6) that $ax = (a_n x_n) \in l_1$ whenever $x = (x_k) \in r_\infty^q(p, B^m)$ if and only if $Uy \in l_1$ whenever $y = (y_k) \in l_\infty(p)$. From Lemma 4.1, we obtain the desired result that

$$\left[r^q(p, B^m)\right]^\alpha = R_1(p). \tag{4.13}$$

Consider the equation

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n-1}\left(\frac{a_k}{r^m q_k} + \sum_{i=k}^{k+1}\nabla(i,n,k)\sum_{j=k+1}^{n} a_j\right)Q_k y_k + \frac{a_k Q_k y_k}{r^m q_k} \\ &= (Vy)_n, \quad (n \in \mathbb{N}), \end{aligned} \tag{4.14}$$

where $V = (v_{nk})$ defined by

$$v_{nk} = \begin{cases} \left(\dfrac{a_k}{r^m q_k} + \displaystyle\sum_{i=k}^{k+1}\nabla(i,n,k)\sum_{j=k+1}^{n} a_j\right)Q_k, & (0 \le k \le n-1), \\ \dfrac{a_k Q_k}{r^m q_k}, & (k = n), \\ 0, & (k > n), \end{cases} \tag{4.15}$$

for all $k, n \in \mathbb{N}$. Thus, we deduce by with (4.11) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in r_\infty^q(p, B^m)$ if and only if $Vy \in c$ whenever $y = (y_k) \in l_\infty(p)$. Therefor, we derive from Lemma 4.3 that

$$\sum_k \left| \left(\frac{a_k}{r^m q_k} + \nabla(i,n,k) \sum_{j=k+1}^{n} a_j \right) Q_k \right| K^{1/p_k} \text{ converges uniformly in } n \ \forall K > 1,$$
$$\lim_{k\to\infty} \frac{a_k Q_k}{r^m q_k} K^{1/p_k} = 0, \tag{4.16}$$

which shows that $[r^q(p, B^m)]^\beta = R_2(p)$.

As this, we deduce by (4.11) that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in r_\infty^q(p, B^m)$ if and only if $Vy \in l_\infty$ whenever $y = (y_k) \in l_\infty(p)$. Therefore, we obtain by Lemma 4.2 that $[r^q(p, B^m)]^\gamma = R_3(p)$ and this completes proof. □

Now we characterize the matrix mappings from the spaces $r_\infty^q(p, B^m)$, $r_c^q(p, B^m)$, and $r_0^q(p, B^m)$ to the spaces l_∞ and c. Since the following theorems can be proved by using standart methods, we omit the detail.

Theorem 4.5. *(i) $A \in (r_\infty^q(p, B^m) : l_\infty)$ if and only if*

$$\lim_{k\to\infty} \frac{a_{nk}}{q_k} Q_k M^{1/p_k} = 0, \quad (\forall n, M \in \mathbb{N}), \tag{4.17}$$

$$\sup_{n\in\mathbb{N}} \sum_k \left| \frac{a_{nk}}{r^m q_k} + \nabla(i,n,k) \sum_{j=k+1}^{n} a_{nj} \right| Q_k M^{1/p_k} < \infty, \quad (\forall M \in \mathbb{N}) \tag{4.18}$$

hold.

(ii) $A \in (r_c^q(p, B^m) : l_\infty)$ if and only if (4.14),

$$\sup_{n\in\mathbb{N}} \sum_k \left| \left(\frac{a_{nk}}{r^m q_k} + \nabla(i,n,k) \sum_{j=k+1}^{n} a_{nj} \right) Q_k \right| M^{1/p_k} = 0, \quad (\exists M \in \mathbb{N}),$$
$$\sup_{n\in\mathbb{N}} \sum_k \left| \left(\frac{a_{nk}}{r^m q_k} + \nabla(i,n,k) \sum_{j=k+1}^{n} a_{nj} \right) Q_k \right| < \infty \tag{4.19}$$

hold.

(iii) $A \in (r_0^q(p, B^m) : l_\infty)$ if and only if (4.14) *and* (4.18) *hold.*

Theorem 4.6. *(i)* $A \in (r_{\infty}^{q}(p, B^m) : c)$ *if and only if* (4.14),

$$\sup_{n\in\mathbb{N}}\sum_{k}\left|\left(\frac{a_{nk}}{r^m q_k} + \nabla(i,n,k)\sum_{j=k+1}^{n} a_{nj}\right)Q_k\right|M^{1/p_k} < \infty, \quad (\forall M \in \mathbb{N}),$$
$$\exists(\alpha_k) \subset \mathbb{R} \quad \textit{such that} \lim_{n\to\infty}\left[\sum_{k}\left|\left(\frac{a_{nk}}{r^m q_k} + \nabla(i,n,k)\sum_{j=k+1}^{n} a_{nj}\right)Q_k - \alpha_k\right|M^{1/p_k}\right] = 0, \tag{4.20}$$

$(\forall M \in \mathbb{N})$ *hold.*

(ii) $A \in (r_{c}^{q}(p, B^m) : c)$ *if and only if* (4.14), (4.18),

$$\exists\alpha \subset \mathbb{R} \quad \textit{such that} \lim_{n\to\infty}\left|\left(\frac{a_{nk}}{r^m q_k} + \nabla(i,n,k)\sum_{j=k+1}^{n} a_{nj}\right)Q_k - \alpha\right| = 0, \tag{4.21}$$

$$\exists(\alpha_k) \subset \mathbb{R} \quad \textit{such that} \lim_{n\to\infty}\left|\left(\frac{a_{nk}}{r^m q_k} + \nabla(i,n,k)\sum_{j=k+1}^{n} a_{nj}\right)Q_k - \alpha_k\right| = 0, \quad (\forall k \in \mathbb{N}), \tag{4.22}$$

$$\exists(\alpha_k) \subset \mathbb{R} \quad \textit{such that} \sup_{n\in\mathbb{N}}\left|\left(\frac{a_{nk}}{r^m q_k} + \nabla(i,n,k)\sum_{j=k+1}^{n} a_{nj}\right)Q_k - \alpha_k\right|M^{-1/p_k} < \infty, \tag{4.23}$$

$(\exists M \in \mathbb{N})$ *hold.*

(iii) $A \in (r_{0}^{q}(p, B^m) : c)$ *if and only if* (4.14), (4.18), (4.21), *and* (4.22) *hold.*

5. Uniform Opial Property of B^m-Difference Riesz Sequence Spaces

In this section, we investigate the uniform Opial property of the sequence spaces $r_0^q(p, B^m)$ and $r_c^q(p, B^m)$.

The Opial property plays an important role in the study of weak convergence of iterates of mapping of Banach spaces and of the asymptotic behavior of nonlinear semigroup. The Opial property is important because Banach spaces with this property have the weak fixed point property [19] (see [20, 21]).

We give the definition of uniform Opial property in a linear metric space and use the method in [22], and obtain that $r_0^q(p, B^m)$ and $r_c^q(p, B^m)$ have uniform Opial property for $p_k \geq 1$.

For a sequence $x = (x_n) \in r_0^q(p, B^m)$ or $x = (x_n) \in r_c^q(p, B^m)$ and for $i \in \mathbb{N}$, we use the notation $x_{|i} = (x(1), x(2), \ldots, x(i), 0, 0, \ldots)$ and $x_{|\mathbb{N}-i} = (0, 0, \ldots, 0, x(i+1), x(i+2), \ldots)$.

We know that every total paranormed space becomes a linear metric space with the metric given by $d(x, y) = g(x - y)$. It is clear that $r_\infty^q(p, B^m)$, $r_0^q(p, B^m)$, and $r_c^q(p, B^m)$ are total paranormed spaces with $d(x, y) = g_B(x - y)$.

Now, we can give the definition of uniform Opial property in a linear metric space.

A linear metric space (X,d) has the uniform Opial property if for each $\varepsilon > 0$ there exists $\tau > 0$ such that for any weakly null sequence $\{x_n\}$ in $S(0,r)$ and $x \in X$ with $d(x,0) \geq \varepsilon$ the following inequality holds:

$$r + \tau \leq \lim_{n\to\infty}\inf d(x_n + x, 0). \tag{5.1}$$

Now, we give following lemma which we need it to prove our main theorem. It can be proved by using same method in [14] we omit the detail.

Lemma 5.1. *If* $\lim\inf_{k\to\infty} p_k > 0$ *then for any* $L > 0$ *and* $\varepsilon > 0$*, there exists* $\delta = \delta(\varepsilon, L) > 0$ *for* $u, v \in X$ *such that*

$$d^M(u+v,0) < d^M(u,0) + \varepsilon \tag{5.2}$$

whenever $d^M(u,0) \leq L$ *and* $d^M(v,0) \leq \delta$*, where* $X = r_0^q(p,B^m)$ *or* $r_c^q(p,B^m)$*.*

Theorem 5.2. *If* $p_k \geq 1$*, then* $r_0^q(p,B^m)$ *and* $r_c^q(p,B^m)$ *have uniform Opial property.*

Proof. We prove the theorem for $r_0^q(p,B^m)$. $r_c^q(p,B^m)$ can be proved by similiar way. For any $\varepsilon > 0$, we can find a positive number $\varepsilon_0 \in (0,\varepsilon)$ such that

$$r^M + \frac{\varepsilon^M}{4} > (r+\varepsilon_0)^M. \tag{5.3}$$

Take any $x \in r_0^q(p,B^m)$ with $d^M(x,0) \geq \varepsilon^M$ and (x_n) to be weakly null sequence in $S(0,r)$. By this, we write

$$d^M(x_n,0) = r^M. \tag{5.4}$$

There exists $q_0 \in \mathbb{N}$ such that

$$d^M\left(x_{|\mathbb{N}-q_0}, 0\right) = \sum_{k=q_0+1}^{\infty} |Tx(k)|^{p_k} < \left(\frac{\varepsilon_0}{4}\right)^M < \frac{\varepsilon^M}{4}. \tag{5.5}$$

Furthermore, we have

$$\begin{aligned}
\varepsilon^M &\leq d^M(x,0) = \sum_{k=0}^{q_0}|Tx(k)|^{p_k} + \sum_{k=q_0+1}^{\infty}|Tx(k)|^{p_k},\\
\varepsilon^M &\leq \sum_{k=0}^{q_0}|Tx(k)|^{p_k} + \frac{\varepsilon^M}{4},\\
\frac{3\varepsilon^M}{4} &\leq \sum_{k=0}^{q_0}|Tx(k)|^{p_k}.
\end{aligned} \tag{5.6}$$

By $x_n \to 0$, weakly, this implies that $x_n \to 0$, coordinatewise, hence there exists $n_0 \in \mathbb{N}$ such that with (5.6)

$$\frac{3\varepsilon^M}{4} \leq \sum_{k=0}^{q_0} |T(x_n(k) + x(k))|^{p_k}, \tag{5.7}$$

for all $n \geq n_0$. Lemma 5.1 asserts that

$$d^M(y + z, 0) \leq d^M(y, 0) + \frac{\varepsilon^M}{4}, \tag{5.8}$$

whenever $d^M(y,0) \leq r^M$ and $d^M(z,0) \leq \varepsilon_0$. Again by $x_n \to 0$, weakly, there exists $n_1 > n_0$ such that $d^M(x_{n|q_0}, 0) < \varepsilon_0$ for all $n > n_1$, so by (5.8), we obtain that

$$d^M\left(x_{n|\mathbb{N}-q_0} + x_{n|q_0}, 0\right) < d^M\left(x_{n|\mathbb{N}-q_0}, 0\right) + \frac{\varepsilon^M}{4}, \tag{5.9}$$

hence,

$$\begin{aligned} d^M(x_n, 0) - \frac{\varepsilon^M}{4} &< d^M\left(x_{n|\mathbb{N}-q_0}, 0\right) = \sum_{k=q_0+1}^{\infty} |Tx_n(k)|^{p_k}, \\ r^M - \frac{\varepsilon^M}{4} &< \sum_{k=q_0+1}^{\infty} |Tx_n(k)|^{p_k}, \end{aligned} \tag{5.10}$$

for all $n > n_1$. This, together with (5.5), (5.6), implies that for any $n > n_1$,

$$\begin{aligned} d^M(x_n + x, 0) &= \sum_{k=0}^{q_0} |T(x_n(k) + x(k))|^{p_k} \\ &\quad + \sum_{k=q_0+1}^{\infty} |T(x_n(k) + x(k))|^{p_k} \\ &\geq \sum_{k=0}^{q_0} |T(x_n(k) + x(k))|^{p_k} + \left| \sum_{k=q_0+1}^{\infty} |Tx_n(k)|^{p_k} \right| \\ &\quad - \left| \sum_{k=q_0+1}^{\infty} |Tx(k)|^{p_k} \right| \\ &> \frac{3\varepsilon^M}{4} + \left(r^M + \frac{\varepsilon^M}{4} \right) - \frac{\varepsilon^M}{4} = r^M + \frac{\varepsilon^M}{4} \\ &> (r + \varepsilon_0)^M. \end{aligned} \tag{5.11}$$

This means that $d^M(x_n + x, 0) > (r + \varepsilon_0)$, so we get that the sequence space $r_0^q(p, B^m)$ has uniform Opial property for $p_k \geq 1$. □

References

[1] I. J. Maddox, "Paranormed sequence spaces generated by infinite matrices," *Proceedings of the Cambridge Philosophical Society*, vol. 64, pp. 335–340, 1968.

[2] I. J. Maddox, "Spaces of strongly summable sequences," *Quarterly Journal of Mathematics*, vol. 18, no. 2, pp. 345–355, 1967.

[3] H. Nakano, "Modulared sequence spaces," *Proceedings of the Japan Academy*, vol. 27, pp. 508–512, 1951.

[4] S. Simons, "The sequence spaces $l(p_v)$ and $m(p_v)$," *Proceedings of the London Mathematical Society. Third Series*, vol. 15, no. 3, pp. 422–436, 1965.

[5] B. Altay and F. Başar, "On the paranormed Riesz sequence spaces of non-absolute type," *Southeast Asian Bulletin of Mathematics*, vol. 26, no. 5, pp. 701–715, 2003.

[6] B. Altay and F. Başar, "Some paranormed Riesz sequence spaces of non-absolute type," *Southeast Asian Bulletin of Mathematics*, vol. 30, no. 4, pp. 591–608, 2006.

[7] B. Altay and F. Başar, "On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces c_0 and c," *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 18, pp. 3005–3013, 2005.

[8] B. Altay and H. Polat, "On some new Euler difference sequence spaces," *Southeast Asian Bulletin of Mathematics*, vol. 30, no. 2, pp. 209–220, 2006.

[9] H. Kızmaz, "On certain sequence spaces," *Canadian Mathematical Bulletin*, vol. 24, no. 2, pp. 169–176, 1981.

[10] M. Başarır and M. Öztürk, "On the Riesz difference sequence space," *Rendiconti del Circolo Matematico di Palermo. Second Series*, vol. 57, no. 3, pp. 377–389, 2008.

[11] C. Aydin and F. Basar, "Some generalizations of the sequence space a_p^r," *Iranian Journal of Science and Technology. Transaction A. Science*, vol. 30, no. 2, pp. 175–190, 2006.

[12] H. Polat and F. Başar, "Some Euler spaces of difference sequences of order m," *Acta Mathematica Scientia B*, vol. 27, no. 2, pp. 254–266, 2007.

[13] F. Başar, B. Altay, and M. Mursaleen, "Some generalizations of the space bv_p of p-bounded variation sequences," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 2, pp. 273–287, 2008.

[14] M. Başarır and M. Kayikçi, "On the generalized B^m-Riesz difference sequence space and β-property," *Journal of Inequalities and Applications*, vol. 2009, Article ID 385029, 18 pages, 2009.

[15] M. Başarır, "On the generalized Riesz B-difference sequence spaces," *Filomat*, vol. 24, no. 4, pp. 35–52, 2010.

[16] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, London, UK, 1970.

[17] K.-G. Grosse-Erdmann, "Matrix transformations between the sequence spaces of Maddox," *Journal of Mathematical Analysis and Applications*, vol. 180, no. 1, pp. 223–238, 1993.

[18] C. G. Lascarides and I. J. Maddox, "Matrix transformations between some classes of sequences," *Proceedings of the Cambridge Philosophical Society*, vol. 68, pp. 99–104, 1970.

[19] J.-P. Gossez and E. Lami Dozo, "Some geometric properties related to the fixed point theory for nonexpansive mappings," *Pacific Journal of Mathematics*, vol. 40, pp. 565–573, 1972.

[20] J. Diestel, *Geometry of Banach Spaces—Selected Topics*, vol. 485 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1975.

[21] L. Maligranda, *Orlicz Spaces and Interpolation*, vol. 5 of *Seminars in Mathematics*, Universidade Estadual de Campinas, Departamento de Matemática, Campinas, Brazil, 1989.

[22] K. Khompurngson, *Geometric properties of some paranormed sequence spaces*, M.S. thesis, Chiang Mai University, 2004.