Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2011, Article ID 479576, 14 pages
doi:10.1155/2011/479576

Research Article

Hypersingular Marcinkiewicz Integrals along Surface with Variable Kernels on Sobolev Space and Hardy-Sobolev Space

Wei Ruiying and Li Yin

School of Mathematics and Information Science, Shaoguan University, Shaoguan 512005, China

Correspondence should be addressed to Wei Ruiying, weiruiying521@163.com

Received 30 June 2010; Revised 5 December 2010; Accepted 20 January 2011

Academic Editor: Andrei Volodin

Let $\alpha \geq 0$, the authors introduce in this paper a class of the hypersingular Marcinkiewicz integrals along surface with variable kernels defined by $\mu^{\Phi}_{\Omega,\alpha}(f)(x) = (\int_0^\infty |\int_{|y|\leq t}(\Omega(x,y)/|y|^{n-1})f(x-\Phi(|y|)y')dy|^2(dt/t^{3+2\alpha}))^{1/2}$, where $\Omega(x,z) \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$ with $q > \max\{1, 2(n-1)/(n+2\alpha)\}$. The authors prove that the operator $\mu^{\Phi}_{\Omega,\alpha}$ is bounded from Sobolev space $L^p_\alpha(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ space for $1 < p \leq 2$, and from Hardy-Sobolev space $H^p_\alpha(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ space for $n/(n+\alpha) < p \leq 1$. As corollaries of the result, they also prove the $\dot{L}^2_\alpha(R^n) - L^2(R^n)$ boundedness of the Littlewood-Paley type operators $\mu^{\Phi}_{\Omega,\alpha,S}$ and $\mu^{*,\Phi}_{\Omega,\alpha,\lambda}$ which relate to the Lusin area integral and the Littlewood-Paley g^*_λ function.

1. Introduction

Let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. For $x \in \mathbb{R}^n \setminus \{0\}$, let $x' = x/|x|$.

Before stating our theorems, we first introduce some definitions about the variable kernel $\Omega(x,z)$. A function $\Omega(x,z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be in $L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$, $q \geq 1$, if $\Omega(x,z)$ satisfies the following two conditions:

(1) $\Omega(x,\lambda z) = \Omega(x,z)$, for any $x, z \in \mathbb{R}^n$ and any $\lambda > 0$;

(2) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})} = \sup_{r\geq 0,\, y\in\mathbb{R}^n}(\int_{\mathbb{S}^{n-1}}|\Omega(rz'+y,z')|^q d\sigma(z'))^{1/q} < \infty$.

In 1955, Calderón and Zygmund [1] investigated the L^p boundedness of the singular integrals T_Ω with variable kernel. They found that these operators connect closely with the problem about the second-order linear elliptic equations with variable coefficients. In 2002, Tang and Yang [2] gave L^p boundedness of the singular integrals with variable kernels associated to surfaces of the form $\{x = \Phi(|y|)y'\}$, where $y' = y/|y|$ for any $y \in \mathbb{R}^n \setminus \{0\}$ ($n \geq 2$). That is, they considered the variable Calderón-Zygmund singular integral operator T_Ω^Φ defined by

$$T_\Omega^\Phi(f)(x) = p \cdot v \cdot \int_{\mathbb{R}^n} \frac{\Omega(x,y)}{|y|^n} f(x - \Phi(|y|)y')dy. \tag{1.1}$$

On the other hand, as a related vector-valued singular integral with variable kernel, the Marcinkiewicz singular with rough variable kernel associated with surfaces of the form $\{x = \Phi(|y|)y'\}$ is considered. It is defined by

$$\mu_\Omega^\Phi(f)(x) = \left(\int_0^\infty \left|F_{\Omega,t}^\Phi(x)\right|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{1.2}$$

where

$$F_{\Omega,t}^\Phi(x) = \int_{|y|\leq t} \frac{\Omega(x,y)}{|y|^{n-1}} f(x - \Phi(|y|)y')dy, \tag{1.3}$$

$$\int_{\mathbb{S}^{n-1}} \Omega(x,z')d\sigma(z') = 0. \tag{1.4}$$

If $\Phi(|y|) = |y|$, we put $\mu_\Omega^\Phi = \mu_\Omega$. Historically, the higher dimension Marcinkiewicz integral operator μ_Ω with convolution kernel, that is $\Omega(x,z) = \Omega(z)$, was first defined and studied by Stein [3] in 1958. See also [4–6] for some further works on μ_Ω with convolution kernel. Recently, Xue and Yabuta [7] studied the L^2 boundedness of the operator μ_Ω^Φ with variable kernel.

Theorem 1.1 (see [7]). *Suppose that $\Omega(x,y)$ is positively homogeneous in y of degree 0, and satisfies* (1.4) *and*

(2′) $\sup_{y\in\mathbb{R}^n}\left(\int_{\mathbb{S}^{n-1}}|\Omega(y,z')|^q d\sigma(z')\right)^{1/q} < \infty$, *for some $q > 2(n-1)/n$. Let Φ be a positive and monotonic (or negative and monotonic) C^1 function on $(0,\infty)$ and let it satisfy the following conditions:*

(i) $\delta \leq |\Phi(t)/(t\Phi'(t))| \leq M$ *for some* $0 < \delta \leq M < \infty$;
(ii) $\Phi'(t)$ *is monotonic on* $(0,\infty)$.

Then there is a constant C such that $\|\mu_\Omega^\Phi(f)\|_2 \leq C\|f\|_2$, where constant C is independent of f.

Since the condition (2) implies (2′), so the $L^2(\mathbb{R}^n)$ boundedness of μ_Ω^Φ holds if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$ with $q > 2(n-1)/n$.

Our aim of this paper is to study the hypersingular Marcinkiewicz integral $\mu_{\Omega,\alpha}^\Phi$ along surfaces with variable kernel Ω, and with index $\alpha \geq 0$, on the homogeneous Sobolev space

$L^p_\alpha(\mathbb{R}^n)$ for $1 < p \le 2$ and the homogeneous Hardy-Sobolev space $H^p_\alpha(\mathbb{R}^n)$ for some $n/(n+\alpha) < p \le 1$. Let $F^\Phi_{\Omega,t}(x)$ be as above, we then define the operators $\mu^\Phi_{\Omega,\alpha}$ by

$$\mu^\Phi_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty \left|F^\Phi_{\Omega,t}(x)\right|^2 \frac{dt}{t^{3+2\alpha}}\right)^{1/2}, \quad \alpha \ge 0. \tag{1.5}$$

Our main results are as follows.

Theorem 1.2. *Suppose that* $\alpha \ge 0$, $\Omega(x,y)$ *satisfies* (1.4) *and* $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$ *with* $q > \max\{1, 2(n-1)/(n+2\alpha)\}$. *Let* Φ *be a positive and increasing* C^1 *function on* $(0,\infty)$ *and let it satisfy the following conditions:*

(i) $\Phi(t) \simeq t\Phi'(t)$;

(ii) $0 \le \Phi'(t) \le W$ *on* $(0,\infty)$.

Then there is a constant C such that $\|\mu^\Phi_{\Omega,\alpha}(f)\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2_\alpha(\mathbb{R}^n)}$, *where constant C is independent of* f.

Theorem 1.3. *Suppose* $0 < \alpha < n/2$, *and that* $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, *with* $q > \max\{1, 2(n-1)/(n+2\alpha)\}$, *and satisfies* (1.4). *Let* Φ *be a positive and increasing* C^1 *function on* $(0,\infty)$ *and let it satisfy the following conditions:*

(i) $\Phi(t) \simeq t\Phi'(t)$;

(ii) $0 < \Phi'(t) \le 1, \Phi(0) = 0$.

Then, for $n/(n+\alpha) < p \le 1$, *there is a constant C such that* $\|\mu^\Phi_{\Omega,\alpha}(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{H^p_\alpha(\mathbb{R}^n)}$, *where constant C is independent of any* $f \in H^p_\alpha(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$.

Furthermore, our result can be extended to the Littlewood-Paley type operators $\mu^\Phi_{\Omega,\alpha,S}$ and $\mu^{*,\Phi}_{\Omega,\alpha,\lambda}$ with variable kernels and index $\alpha \ge 0$, which relate to the Lusin area integral and the Littlewood-Paley g^*_λ function, respectively. Let $F^\Phi_{\Omega,t}(x)$ be as above, we then define the operators $\mu^\Phi_{\Omega,\alpha,S}$ and $\mu^{*,\Phi}_{\Omega,\alpha,\lambda}$ for $f \in \mathcal{S}(\mathbb{R}^n)$, respectively by

$$\mu^\Phi_{\Omega,\alpha,S}(f)(x) = \left(\iint_{\Gamma(x)} \left|F^\Phi_{\Omega,t}(y)\right|^2 \frac{dy\,dt}{t^{n+3+2\alpha}}\right)^{1/2},$$
$$\mu^{*,\Phi}_{\Omega,\alpha,\lambda}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|F^\Phi_{\Omega,t}(y)\right|^2 \frac{dy\,dt}{t^{n+3+2\alpha}}\right)^{1/2}, \tag{1.6}$$

with $\lambda > 1$, where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$. As an application of Theorem 1.2, we have the following conclusion.

Theorem 1.4. *Under the assumption of Theorem 1.2, then Theorem 1.2 still holds for* $\mu^\Phi_{\Omega,\alpha,S}$ *and* $\mu^{*,\Phi}_{\Omega,\alpha,\lambda}$.

By Theorems 1.2 and 1.3 and applying the interpolation theorem of sublinear operator, we obtain the $L^p_\alpha - L^p$ boundedness of $\mu^\Phi_{\Omega,\alpha}$.

Corollary 1.5. *Suppose $0 < \alpha < n/2$, and that $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q > \max\{1, 2(n-1)/(n+2\alpha)\}$, and satisfies* (1.4). *Let Φ be given as in Theorem 1.3. Then, for $1 < p \le 2$, there exists an absolute positive constant C such that*

$$\left\| \mu^{\Phi}_{\Omega,\alpha}(f) \right\|_{L^p(\mathbb{R}^n)} \le C \|f\|_{L^p_\alpha(\mathbb{R}^n)}, \tag{1.7}$$

for all $f \in L^p_\alpha(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$.

Remark 1.6. It is obvious that the conclusions of Theorem 1.2 are the substantial improvements and extensions of Stein's results in [3] about the Marcinkiewicz integral μ_Ω with convolution kernel, and of Ding's results in [8] about the Marcinkiewicz integral μ_Ω with variable kernels.

Remark 1.7. Recently, the authors in [9] proved the boundedness of hypersingular Marcinkiewicz integral with variable kernels on homogeneous Sobolev space $L^p_\alpha(R^n)$ for $1 < p \le 2$ and $0 < \alpha < 1$ without any smoothness on Ω. So Corollary 1.5 extended the results in [9, Theorem 5].

Throughout this paper, the letter C always remains to denote a positive constant not necessarily the same at each occurrence.

2. The Bounedness on Sobolev Spaces

Before giving the definition of the Sobolev space, let us first recall the Triebel-Lizorkin space.

Fix a radial function $\varphi(x) \in C^\infty$ satisfying $\operatorname{supp}(\varphi) \subseteq \{x : 1/2 < |x| \le 2\}$ and $0 \le \varphi(x) \le 1$, and $\varphi(x) > c > 0$ if $3/5 \le |x| \le 5/3$. Let $\varphi_j(x) = \varphi(2^j x)$. Define the function $\psi_j(x)$ by $\mathcal{F}(\psi_j)(\xi) = \varphi_j(\xi)$, such that $\mathcal{F}(\psi_j * f)(\xi) = \mathcal{F}(f)(\xi)\varphi_j(\xi)$.

For $0 < p,\ q < \infty$, and $\alpha \in \mathbb{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}^{\alpha,q}_p$ is the set of all distributions f satisfying

$$\dot{F}^{\alpha,q}_p(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}^{\alpha,q}_p} = \left\| \left(\sum_k \left| 2^{-\alpha k} \psi_k * f \right|^q \right)^{1/q} \right\|_p < \infty \right\}. \tag{2.1}$$

For $p \ge 1$, the homogeneous Sobolev spaces $L^p_\alpha(\mathbb{R}^n)$ is defined by $L^p_\alpha(\mathbb{R}^n) = \dot{F}^{\alpha,2}_p(\mathbb{R}^n)$, namely $\|f\|_{L^p_\alpha} = \|f\|_{\dot{F}^{\alpha,2}_p}$. From [10] we know that for any $f \in L^2_\alpha(\mathbb{R}^n)$

$$\|f\|_{L^2_\alpha(\mathbb{R}^n)} \cong \left(\int_{\mathbb{R}^n} \left| \mathcal{F}(f)(\xi) \right|^2 |\xi|^{2\alpha} d\xi \right)^{1/2}, \tag{2.2}$$

and if α is a nonnegative integer, then for any $f \in L^p_\alpha(\mathbb{R}^n)$

$$\|f\|_{L^p_\alpha(\mathbb{R}^n)} \cong \sum_{|\tau| = \alpha} \|D^\tau f\|_{L^p(\mathbb{R}^n)}. \tag{2.3}$$

For $0 < p \le 1$, we define the homogeneous Hardy-Sobolev space $H_\alpha^p(\mathbb{R}^n)$ by $H_\alpha^p(\mathbb{R}^n) = \dot{F}_p^{\alpha,2}(\mathbb{R}^n)$. It is well known that $H^p(\mathbb{R}^n) = \dot{F}_p^{0,2}(\mathbb{R}^n)$ for $0 < p \le 1$, one can refer [10] for the details.

Next, let us give the main lemmas we will use in proving theorems.

Lemma 2.1 (see [11]). *Suppose that $n \ge 2$ and $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ has the form $f(x) = f_0(|x|)P(x)$ where $P(x)$ is a solid spherical harmonic polynomial of degree m. Then the Fourier transform of f has the form $\mathcal{F}(f)(x) = F_0(|x|)P(x)$, where*

$$F_0(r) = 2\pi i^{-m} r^{-((n+2m-2)/2)} \int_0^\infty f_0(s) J_{(n+2m-2)/2}(2\pi r s) s^{(n+2m)/2} ds, \tag{2.4}$$

and $r = |\xi|$, $J_m(s)$ is the Bessel function.

Lemma 2.2 (see [12]). *For $\lambda = (n-2)/2$, and $-\lambda \le \alpha \le 1$, there exists $C > 0$ such that for any $h \ge 0$ and $m = 1, 2, \ldots,$*

$$\left| \int_0^h \frac{J_{m+\lambda}(t)}{t^{\lambda+\alpha}} dt \right| \le \frac{C}{m^{\lambda+\alpha}}. \tag{2.5}$$

Lemma 2.3. *Let $\alpha \ge 0$, $\lambda = (n-2)/2$, Φ is a C^1 function on $(0, \infty)$ and let it satisfy the conditions (i) and (ii) in Theorem 1.2.*

Denote $g_\alpha(f)(x) = \left(\int_0^{+\infty} |N_\varepsilon f(x)|^2 (d\varepsilon/\varepsilon^{1+2\alpha})\right)^{1/2}$, if

$$\mathcal{F}(N_\varepsilon f)(\xi) = \int_0^{\Phi(\varepsilon)|\xi|} \frac{J_{m+\lambda}(t)}{t^{\lambda+1}} dt \cdot \mathcal{F}(f)(\xi). \tag{2.6}$$

Then there exists a constant C independent of m, such that $\|g_\alpha(f)\|_{L^2} \le C/m^{\lambda+1+\alpha} \|f\|_{L^2_\alpha}$ for every integer $m \in \mathbb{N}, m > \alpha$.

Proof. Let $\eta(|x|) = \int_0^{|x|} (J_{m+\lambda}(t)/t^{\lambda+1}) dt$, then we have

$$\begin{aligned} \|g_\alpha(f)\|_2^2 &= \int_{\mathbb{R}^n} \int_0^{+\infty} |N_\varepsilon f(x)|^2 \frac{d\varepsilon}{\varepsilon^{1+2\alpha}} dx \\ &= \int_0^{+\infty} \int_{\mathbb{R}^n} |\eta(\Phi(\varepsilon)|\xi|) \mathcal{F}(f)(\xi)|^2 d\xi \frac{d\varepsilon}{\varepsilon^{1+2\alpha}} \\ &= \int_0^{+\infty} \int_{\mathbb{R}^n} \left| \eta\left(\Phi\left(\frac{\beta}{|\xi|} \right) |\xi| \right) \mathcal{F}(f)(\xi) \right|^2 |\xi|^{2\alpha} d\xi \frac{d\beta}{\beta^{1+2\alpha}} \\ &= \int_{\mathbb{R}^n} \int_0^{+\infty} \left| \eta\left(\Phi\left(\frac{\beta}{|\xi|} \right) |\xi| \right) \right|^2 \frac{d\beta}{\beta^{1+2\alpha}} |\mathcal{F}(f)(\xi)|^2 |\xi|^{2\alpha} d\xi. \end{aligned} \tag{2.7}$$

So it suffices to show $\int_0^{+\infty} \eta(\Phi(\beta/|\xi|)|\xi|)^2 (d\beta/\beta^{1+2\alpha}) \le (C/m^{\lambda+1+\alpha})^2$.

Decompose this integral into two parts $\int_0^{+\infty} = \int_0^{m/2} + \int_{m/2}^{+\infty} =: I_1 + I_2$.

For I_2, by using Lemma 2.2 and $\Phi(t) \simeq t\Phi'(t)$, we can get

$$I_2 = \int_{m/2}^{+\infty} \left(\int_0^{\Phi(\beta/|\xi|)|\xi|} \frac{J_{m+\lambda}(t)}{t^{\lambda+1}} dt \right)^2 \frac{d\beta}{\beta^{1+2\alpha}} \leq \frac{C}{m^{2\lambda+2}} \int_{m/2}^{+\infty} \frac{d\beta}{\beta^{1+2\alpha}} \leq \frac{C}{m^{2\lambda+2+2\alpha}}. \tag{2.8}$$

For the other part I_1, applying Stirling's formula, we have

$$\sqrt{2\pi} x^{x-1/2} e^{-x} \leq \Gamma(x) \leq 2\sqrt{2\pi} x^{x-1/2} e^{-x}. \tag{2.9}$$

Also in [13], the authors proved the following inequality

$$|J_\nu(t)| \leq \frac{(t/2)^\nu}{\Gamma(\nu+1)}. \tag{2.10}$$

So by (2.9) and (2.10), $0 \leq \alpha < [\alpha]+1 \leq m$, and noting that $\Phi(t) \leq Wt$, we have

$$\begin{aligned}
I_1 &= \int_0^{m/2} \left(\int_0^{\Phi(\beta/|\xi|)|\xi|} \frac{J_{m+\lambda}(t)}{t^{\lambda+1}} dt \right)^2 \frac{d\beta}{\beta^{1+2\alpha}} \\
&\leq \int_0^{m/2} \left(\int_0^{\Phi(\beta/|\xi|)|\xi|} \frac{|J_{m+\lambda}(t)|}{t^{\lambda+1}} dt \right)^2 \frac{d\beta}{\beta^{1+2\alpha}} \\
&\leq \frac{1}{2^{2m+2\lambda}\Gamma^2(m+\lambda+1)} \int_0^{m/2} \left(\int_0^{\Phi(\beta/|\xi|)|\xi|} \frac{t^{m+\lambda}}{t^{\lambda+1}} dt \right)^2 \frac{d\beta}{\beta^{1+2\alpha}} \\
&\leq \frac{1}{2^{2m+2\lambda}\Gamma^2(m+\lambda+1)} \int_0^{m/2} \left(\Phi'\left(\frac{\beta}{|\xi|} \right) \right)^{2m} \frac{d\beta}{\beta^{1+2\alpha}} \\
&\leq \frac{e^{2m+2\lambda+2}}{2\pi 2^{2m+2\lambda}(m+\lambda+1)^{2m+2\lambda+1}} \int_0^{m/2} \left(\Phi'\left(\frac{\beta}{|\xi|} \right)^{2m} \right) \frac{d\beta}{\beta^{1+2\alpha-2m}} \\
&\leq C \frac{e^{2m+2\lambda+2}}{2\pi 2^{2m+2\lambda}(m+\lambda+1)^{2m+2\lambda+1}} \int_0^{m/2} \frac{d\beta}{\beta^{1+2\alpha-2m}} \\
&\leq C\left(\frac{e}{4}\right)^{2m+2\lambda+2} \frac{1}{m^{2\alpha+2\lambda+2}} \\
&\leq \frac{C}{m^{2\alpha+2\lambda+2}}.
\end{aligned} \tag{2.11}$$

So far we can deduce the desired conclusion of Lemma 2.3. □

Proof of Theorem 1.2. The basic idea of proof can go back to [14], for recently papers, one see [8, 15]. By the same argument as in [1], let $\{Y_{m,j}\}$ $(m \geq 1,\ j = 1, 2, \ldots, D_m)$ denote the complete system of normalized surface spherical harmonics. See [14] for instance, we can decompose $\Omega(x, y')$ as following:

$$\Omega(x, y') = \sum_{m=1}^{+\infty} \sum_{j=1}^{D_m} a_{m,j}(x) Y_{m,j}(y') \text{ is a finite sum.} \tag{2.12}$$

Denote

$$a_m(x) = \left(\sum_{j=1}^{D_m} \left| a_{m,j}(x) \right|^2 \right)^{1/2}, \qquad b_{m,j}(x) = \frac{a_{m,j}(x)}{a_m(x)}, \tag{2.13}$$

then we get

$$\sum_{j=1}^{D_m} b_{m,j}^2(x) = 1, \qquad \Omega(x, y') = \sum_{m=1}^{+\infty} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) Y_{m,j}(y'). \tag{2.14}$$

Then, applying Hölder inequality twice, we have for any $0 < \varepsilon < 1$ that

$$\begin{aligned}
\left| \mu_{\Omega,\alpha}^{\Phi} f(x) \right|^2 &= \int_0^{\infty} \left| \int_{|y| \leq t} \sum_{m=1}^{+\infty} b_{m,j}(x) \frac{Y_{m,j}(y')}{|y|^{n-1}} f(x - \Phi(|y|) y') dy \right|^2 \frac{dt}{t^{3+2\alpha}} \\
&\leq \left(\sum_{m=1}^{+\infty} a_m^2(x) m^{-\varepsilon(1+2\alpha)} \right) \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)} \\
&\quad \times \int_0^{+\infty} \left| \int_{|y| \leq t} \sum_{j=1}^{D_m} b_{m,j}(x) \frac{Y_{m,j}(y')}{|y|^{n-1}} f(x - \Phi(|y|) y') dy \right|^2 \frac{dt}{t^{3+2\alpha}} \\
&\leq \left(\sum_{m=1}^{+\infty} a_m^2(x) m^{-\varepsilon(1+2\alpha)} \right) \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)} \int_0^{+\infty} \left(\sum_{j=1}^{D_m} b_{m,j}^2(x) \right) \\
&\quad \times \sum_{j=1}^{D_m} \left| \int_{|y| \leq t} \frac{Y_{m,j}(y')}{|y|^{n-1}} f(x - \Phi(|y|) y') dy \right|^2 \frac{dt}{t^{3+2\alpha}} \\
&= \left(\sum_{m=1}^{+\infty} a_m^2(x) m^{-\varepsilon(1+2\alpha)} \right) \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)} \\
&\quad \times \int_0^{+\infty} \sum_{j=1}^{D_m} \left| \int_{|y| \leq t} \frac{Y_{m,j}(y')}{|y|^{n-1}} f(x - \Phi(|y|) y') dy \right|^2 \frac{dt}{t^{3+2\alpha}}.
\end{aligned} \tag{2.15}$$

By [14, page 230, equation (4.4)], we can observe that the series in the first parenthesis on the right-hand side of the inequality above, for each x fixed, is equal to $\|\Omega(x,\cdot)\|^2_{L^2_{-\gamma}(\mathbb{S}^{n-1})}$, where $L^2_{-\gamma}(\mathbb{S}^{n-1})$ is the Sobolev space on $\mathbb{S}^{n-1}$ with $\gamma = \varepsilon((1/2)+\alpha)$ for any $0<\varepsilon<1$. So if we take ε sufficiently close to 1, then by the Sobolev imbedding theorem $L^q \subset L^2_{-\gamma}$, we have

$$\left(\sum_m a_m^2(x) m^{-\varepsilon(1+2\alpha)}\right)^{1/2} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})} := C\|\Omega\| \tag{2.16}$$

with $q > \max\{1, 2(n-1)/(n+2\alpha)\}$.

By Fourier transform and (2.16), we get

$$\begin{aligned}\left\|\mu^{\Phi}_{\Omega,\alpha}(f)\right\|_2^2 &\leq C\|\Omega\|^2 \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)} \int_0^{+\infty}\int_{\mathbb{R}^n} \sum_{j=1}^{D_m}\left|\int_{|y|\leq t} \frac{Y_{m,j}(y')}{|y|^{n-1}} f(x-\Phi(|y|)y')dy\right|^2 dx \frac{dt}{t^{3+2\alpha}} \\ &\leq C\|\Omega\|^2 \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)} \sum_{j=1}^{D_m} \int_0^{+\infty}\int_{\mathbb{R}^n}\left|\mathcal{F}\left(\int_{|y|\leq t}\frac{Y_{m,j}(y')}{|y|^{n-1}} f(\cdot-\Phi(|y|)y')dy\right)(\xi)\right|^2 d\xi\frac{dt}{t^{3+2\alpha}} \\ &=: C\|\Omega\|^2 \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)} \sum_{j=1}^{D_m}\left\|\mu^{\Phi}_{\Omega,j,\alpha}(f)\right\|_2^2.\end{aligned} \tag{2.17}$$

For $\mu^{\Phi}_{\Omega,j,\alpha}(f)$, we have

$$\begin{aligned}\left\|\mu^{\Phi}_{\Omega,j,\alpha}(f)\right\|_2^2 &= \int_0^{+\infty}\int_{\mathbb{R}^n}\left|\int_{|y|\leq t}\int_{\mathbb{R}^n}\frac{Y_{m,j}(y')}{|y|^{n-1}} f(x-\Phi(|y|)y')e^{-2\pi i x\cdot\xi}dx\,dy\right|^2 d\xi\frac{dt}{t^{3+2\alpha}} \\ &= \int_0^{+\infty}\int_{\mathbb{R}^n}\left|\int_{|y|\leq t}\frac{Y_{m,j}(y')}{|y|^{n-1}}e^{-2\pi i\Phi(|y|)y'\cdot\xi}\right. \\ &\qquad \left.\times\int_{\mathbb{R}^n} f(x-\Phi(|y|)y')e^{-2\pi i(x-\Phi(|y|)y')\cdot\xi}dx\,dy\right|^2 d\xi\frac{dt}{t^{3+2\alpha}} \\ &= \int_0^{+\infty}\int_{\mathbb{R}^n}\left|\int_{|y|\leq t}\frac{Y_{m,j}(y')}{|y|^{n-1}}e^{-2\pi i\Phi(|y|)y'\cdot\xi}dy\right|^2 |\mathcal{F}(f)(\xi)|^2 d\xi\frac{dt}{t^{3+2\alpha}} \\ &= \int_{\mathbb{R}^n}\int_0^{+\infty}\left|\frac{1}{t^{1+\alpha}}\int_{|y|\leq t}\frac{Y_{m,j}(y')}{|y|^{n-1}}e^{-2\pi i\Phi(|y|)y'\cdot\xi}dy\right|^2\frac{dt}{t}|\mathcal{F}(f)(\xi)|^2 d\xi.\end{aligned} \tag{2.18}$$

For the integral on the right hand side of the above inequality, by changing of variable, we can get

$$\begin{aligned}
&\frac{1}{t^{1+\alpha}}\int_{|y|\le t}\frac{\Upsilon_{m,j}(y')}{|y|^{n-1}}e^{-2\pi i\Phi(|y|)y'\cdot\xi}dy\\
&=\frac{1}{t^{1+\alpha}}\int_0^t\int_{\mathbb{S}^{n-1}}\Upsilon_{m,j}(y')e^{-2\pi i\Phi(s)y'\cdot\xi}dy'ds\\
&=\frac{1}{t^{1+\alpha}}\int_0^{\Phi(t)}\int_{\mathbb{S}^{n-1}}\Upsilon_{m,j}(y')e^{-2\pi i\gamma y'\cdot\xi}\left(\Phi^{-1}(\gamma)\right)'dy'd\gamma\\
&=\frac{1}{t^{1+\alpha}}\int_{|y|\le\Phi(t)}\frac{\Upsilon_{m,j}(y')}{|y|^{n-1}}e^{-2\pi i y\cdot\xi}\left(\Phi^{-1}(|y|)\right)'dy.
\end{aligned}\tag{2.19}$$

So we have

$$\left\|\mu^{\Phi}_{\Omega,j,\alpha}(f)\right\|_2^2=\int_{\mathbb{R}^n}\int_0^{+\infty}\left|\frac{1}{t^{1+\alpha}}\int_{|y|\le\Phi(t)}\frac{\Upsilon_{m,j}(y')}{|y|^{n-1}}e^{-2\pi i y\cdot\xi}\left(\Phi^{-1}(|y|)\right)'dy\right|^2\frac{dt}{t}\left|\mathcal{F}(f)(\xi)\right|^2d\xi.\tag{2.20}$$

Put $P_{m,j}(x)=\Upsilon_{m,j}(x')|x|^m$ and $\varphi^{\Phi,m,j}_{t,\alpha}(x)=P_{m,j}(x)\cdot|x|^{-n-m+1}\chi_{|x|\le\Phi(t)}(x)(\Phi^{-1}(|x|))'t^{-1-\alpha}$, we can deduce from Lemma 2.1 that

$$\mathcal{F}\left(\varphi^{\Phi,m,j}_{t,\alpha}\right)(\xi)=P_{m,j}(|\xi|)\cdot F_0(|\xi|)=\Upsilon_{m,j}(\xi')\cdot|\xi|^mF_0(|\xi|),\tag{2.21}$$

where

$$\begin{aligned}
F_0(r)&=2\pi i^{-m}r^{-(n/2)-m+1}\int_0^{\Phi(t)}t^{-1-\alpha}s^{-n-m+1}\left(\Phi^{-1}(s)\right)'J_{(n/2)+m-1}(2\pi rs)s^{(n/2)+m}ds\\
&=2\pi i^{-m}r^{-(n/2)-m+1}t^{-1-\alpha}\int_0^{\Phi(t)}s^{-(n/2)+1}J_{(n/2)+m-1}(2\pi rs)d\left(\Phi^{-1}(s)\right)\\
&=2\pi i^{-m}r^{-(n/2)-m+1}t^{-1-\alpha}\int_0^t\frac{J_{(n/2)+m-1}(2\pi r\Phi(\beta))}{(\Phi(\beta))^{(n/2)-1}}d\beta\\
&=(2\pi)^{(n/2)}i^{-m}r^{-m}\frac{t^{-\alpha}}{t}\int_0^t\frac{J_{(n/2)+m-1}(2\pi r\Phi(\beta))}{(2\pi r\Phi(\beta))^{(n/2)-1}}d\beta.
\end{aligned}\tag{2.22}$$

Hence, we have

$$\begin{aligned}
&\left\|\mu^{\Phi}_{\Omega,j,\alpha}(f)\right\|_2^2\\
&=\int_0^{+\infty}\int_{\mathbb{R}^n}\left|\varphi^{\Phi,m,j}_{t,\alpha}*f(x)\right|^2dx\frac{dt}{t}\\
&=\int_0^{+\infty}\int_{\mathbb{R}^n}\left|\mathcal{F}\left(\varphi^{\Phi,m,j}_{t,\alpha}*f\right)(\xi)\right|^2d\xi\frac{dt}{t}\\
&\leq\int_{\mathbb{R}^n}\int_0^{+\infty}\left|Y_{m,j}(\xi')|\xi|^m i^{-m}|\xi|^{-m}(2\pi)^{n/2}\frac{t^{-\alpha}}{t}\int_0^t\frac{J_{(n/2)+m-1}(2\pi|\xi|\Phi(\beta))}{(2\pi|\xi|\Phi(\beta))^{(n/2)-1}}d\beta\right|^2\frac{dt}{t}|\mathcal{F}(f)(\xi)|^2d\xi\\
&\leq C\int_{\mathbb{R}^n}\int_0^{+\infty}\left|Y_{m,j}(\xi')\frac{1}{t}\int_0^t\frac{J_{(n/2)+m-1}(2\pi|\xi|\Phi(\beta))}{(2\pi|\xi|\Phi(\beta))^{(n/2)-1}}d\beta\right|^2\frac{dt}{t^{1+2\alpha}}|\mathcal{F}(f)(\xi)|^2d\xi.
\end{aligned}\tag{2.23}$$

By [14], we know that $\sum_{j=1}^{D_m}|Y_{m,j}(z')|^2\cong m^{n-2}$.

So we can get

$$\sum_{j=1}^{D_m}\left\|\mu^{\Phi}_{\Omega,j,\alpha}(f)\right\|_2^2\leq Cm^{n-2}\int_{\mathbb{R}^n}\int_0^{+\infty}\left|\frac{1}{t}\int_0^t\frac{J_{(n/2)+m-1}(2\pi|\xi|\Phi(\beta))}{(2\pi|\xi|\Phi(\beta))^{(n/2)-1}}d\beta\right|^2\frac{dt}{t^{1+2\alpha}}|\mathcal{F}(f)(\xi)|^2d\xi.\tag{2.24}$$

Set $\lambda=(n/2)-1,\rho=2\pi|\xi|\Phi(\beta)$ and note that $\Phi(t)\simeq t\Phi'(t)$, we can deduce that

$$\begin{aligned}
U&:=\frac{1}{t}\int_0^t\frac{J_{(n/2)+m-1}(2\pi|\xi|\Phi(\beta))}{(2\pi|\xi|\Phi(\beta))^{(n/2)-1}}d\beta\\
&=\frac{1}{t}\int_0^{2\pi|\xi|\Phi(t)}\frac{J_{m+\lambda}(\rho)}{\rho^{\lambda}}\frac{1}{2\pi|\xi|\Phi'(\Phi^{-1}(\rho/2\pi|\xi|))}d\rho\\
&=\frac{1}{t}\int_0^{2\pi|\xi|\Phi(t)}\frac{J_{m+\lambda}(\rho)}{\rho^{\lambda+1}}\Phi^{-1}\left(\frac{\rho}{2\pi|\xi|}\right)d\rho.
\end{aligned}\tag{2.25}$$

Noting that $\Phi(t)$ is increasing, by using the second mean-value theorem, we get, for some $0\leq\eta<2\pi|\xi|\Phi(t)$,

$$\begin{aligned}
|U|&\leq\left|\frac{1}{t}\Phi^{-1}(\Phi(t))\int_{\eta}^{2\pi|\xi|\Phi(t)}\frac{J_{m+\lambda}(\rho)}{\rho^{\lambda+1}}d\rho\right|\\
&\leq\left|\int_0^{2\pi|\xi|\Phi(t)}\frac{J_{m+\lambda}(\rho)}{\rho^{\lambda+1}}d\rho\right|.
\end{aligned}\tag{2.26}$$

From (2.26), it follows that

$$\sum_{j=1}^{D_m}\left\|\mu^{\Phi}_{m,j,\alpha}(f)\right\|_2^2 \leq Cm^{n-2}\int_{\mathbb{R}^n}\int_0^{+\infty}\left|\int_0^{2\pi|\xi|\Phi(t)}\frac{J_{m+\lambda}(\rho)}{\rho^{\lambda+1}}d\rho\cdot\mathcal{F}(f)(\xi)\right|^2\frac{dt}{t^{1+2\alpha}}d\xi. \tag{2.27}$$

Thus using Lemma 2.3, we can deduce the desired conclusion of Theorem 1.2. □

Proof of Theorem 1.4. First, we know that $\mu^{\Phi}_{\Omega,\alpha,S}(f)(x) \leq 2^{\lambda n}\mu^{*,\Phi}_{\Omega,\alpha,\lambda}(f)(x)$. On the other hand,

$$\begin{aligned}
&\left\|\mu^{*,\Phi}_{\Omega,\alpha,\lambda}(f)\right\|_2^2 \\
&\quad= \int_{\mathbb{R}^n}\iint_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t}\int_{|z|\leq t}\frac{\Omega(x,z)}{|z|^{n-1}}f(x-\Phi(|z|)z')dz\right|^2\frac{dzdt}{t^{n+1+2\alpha}}dx \\
&\quad= \int_0^{\infty}\int_{\mathbb{R}^n}\left(\frac{1}{t^n}\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}dx\right)\left|\frac{1}{t}\int_{|z|\leq t}\frac{\Omega(x,z)}{|z|^{n-1}}f(x-\Phi(|z|)z')dz\right|^2\frac{dzdt}{t^{1+2\alpha}} \\
&\quad\leq C\left\|\mu^{\Phi}_{\Omega,\alpha}(f)\right\|_2^2.
\end{aligned} \tag{2.28}$$

Thus, using Theorem 1.2, we can finish Theorem 1.4. □

3. The Bounedness on Hardy-Sobolev Spaces

In order to prove the boundedness for operator $\mu^{\Phi}_{\Omega,\alpha}$ on Hardy-Sobolev spaces and prove Theorem 1.3, we first introduce a new kind of atomic decomposition for Hardy-Sobolev space as following which will be used next.

Definition 3.1 (see [16]). For $\alpha \geq 0$, the function $a(x)$ is called a $(p,2,\alpha)$ atom if it satisfies the following three conditions:

(1) $\operatorname{supp}(a) \subset B$ with a ball $B \subset \mathbb{R}^n$;

(2) $\|a\|_{L^2_\alpha} \leq |B|^{(1/2)-(1/p)}$;

(3) $\int_{\mathbb{R}^n} a(x)P(x) = 0$, for any polynomial $P(x)$ of degree $\leq N = [n((1/p)-1)\alpha]$.

By [16], we have that every $f \in H^p_\alpha(\mathbb{R}^n)$ can be written as a sum of $(p,2,\alpha)$ atoms $a_j(x)$, that is,

$$f = \sum_j \lambda_j a_j. \tag{3.1}$$

Proof of Theorem 1.3. Similar to the argument of Lemma 3.3 in [17] and using above atomic decomposition, it suffices to show that

$$\left\|\mu_{\Omega,\alpha}^{\Phi}(a)\right\|_{L^p}^{p} \leq C, \tag{3.2}$$

with the constant C independent of any $(p,2,\alpha)$ atom a.

Assume $\operatorname{supp}(a) \subset B(0,R)$. We first note that

$$\begin{aligned}\left\|\mu_{\Omega,\alpha}^{\Phi}(a)\right\|_{L^p}^{p} &\leq \int_{|x|\leq 8R}\left|\mu_{\Omega,\alpha}^{\Phi}(a)(x)\right|^{p}dx + \int_{|x|>8R}\left|\mu_{\Omega,\alpha}^{\Phi}(a)(x)\right|^{p}dx \\ &=: U_1 + U_2.\end{aligned} \tag{3.3}$$

For U_1, using Theorem 1.2, it is not difficult to deduce that

$$\begin{aligned}U_1 &\leq C\left\|\mu_{\Omega,\alpha}^{\Phi}(a)\right\|_{L^2}^{p} R^{n(1-(p/2))} \leq C\|a\|_{L_\alpha^2}^{p} R^{n(1-(p/2))} \\ &\leq CR^{n((p/2)-1)} R^{n(1-(p/2))} \leq C.\end{aligned} \tag{3.4}$$

For U_2, we first consider the case $n/(n+\alpha) < p < 1$, according to [15, Lemma 5.5], for $0 < \alpha < n/2$ and $(p,2,\alpha)$ atom a with support $B = B(0,R)$, one has

$$\int_B |a(x)|dx \leq CR^{n-(n/p)+\alpha}. \tag{3.5}$$

Using Minkowski inequality and Hölder inequality for integrals, and (3.5), we can get

$$\begin{aligned}U_2 &= \int_{|x|>8R}\left|\mu_{\Omega,\alpha}^{\Phi}(a)(x)\right|^{p}dx \\ &= \int_{|x|>8R}\left(\int_0^{+\infty}\left|\int_{|y|\leq t}\frac{\Omega(x,y)}{|y|^{n-1}}a(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t^{3+2\alpha}}\right)^{p/2}dx \\ &\leq \int_{|x|>8R}\left|\int_{\mathbb{R}^n}\frac{|\Omega(x,y)|}{|y|^{n+\alpha}}|a(x-\Phi(|y|)y')|dy\right|^{p}dx.\end{aligned} \tag{3.6}$$

For the integral on the right hand side of the above inequality, by changing of variable and noting that $0 < \Phi^{'}(t) \le 1, \Phi(0) = 0$, we can get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|\Omega(x,y)|}{|y|^{n+\alpha}} |a(x - \Phi(|y|)y')| dy \\
&= \int_{\mathbb{S}^{n-1}} \int_0^R \frac{|\Omega(x,y')|}{r^{1+\alpha}} |a(x - \Phi(r)y')| dr\, dy' \\
&= \int_{\mathbb{S}^{n-1}} \int_0^{\Phi(R)} \frac{|\Omega(x,y')|}{(\Phi^{-1}(\gamma))^{1+\alpha}} |a(x - \gamma y')| \frac{1}{\Phi'(\Phi^{-1}(\gamma))} d\gamma\, dy' \\
&= \int_{\mathbb{S}^{n-1}} \int_0^{\Phi(R)} \frac{|\Omega(x,y')|}{(\Phi^{-1}(\gamma))^{1+\alpha}} |a(x - \gamma y')| \frac{\Phi^{-1}(\gamma)}{\gamma} d\gamma\, dy' \\
&= \int_{\mathbb{S}^{n-1}} \int_0^{\Phi(R)} \frac{|\Omega(x,y')|}{(\Phi^{-1}(\gamma))^{\alpha}\gamma} |a(x - \gamma y')| d\gamma\, dy' \\
&= \int_{|y| \le \Phi(R)} \frac{|\Omega(x,y)|}{|y|^n (\Phi^{-1}(|y|))^{\alpha}} |a(x - y)| dy \\
&= \int_{|x-y| \le \Phi(R)} \frac{|\Omega(x,x-y)|}{|x-y|^n (\Phi^{-1}(|x-y|))^{\alpha}} |a(y)| dy \\
&\le \int_{|x-y| \le \Phi(R)} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}} |a(y)| dy.
\end{aligned} \tag{3.7}$$

By (3.7), we can get

$$\begin{aligned}
U_2 &\le \sum_{j=3}^{+\infty} \int_{2^j R < |x| < 2^{j+1} R} \left| \int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}} |a(y)| dy \right|^p dx \\
&\le \sum_{j=3}^{+\infty} \left(2^j R\right)^{n(1-p)} \left(\int_{2^j R < |x| < 2^{j+1} R} \int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}} |a(y)| dy\, dx \right)^p \\
&\le \sum_{j=3}^{+\infty} \left(2^j R\right)^{n(1-p)} \left(\int_B |a(y)| \int_{2^j R < |x| < 2^{j+1} R} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}} dx\, dy \right)^p \\
&\le C \|\Omega\|^p_{L^\infty \times L^1} \left(\int_B |a(y)| dy \right)^p \cdot \sum_{j=3}^{+\infty} \left(2^j R\right)^{-\alpha p} \left(2^j R\right)^{n(1-p)}.
\end{aligned} \tag{3.8}$$

Thus by (3.5) and the condition $p > n/(n+\alpha)$,

$$U_2 \le C \|\Omega\|^p_{L^\infty \times L^1} \sum_{j=3}^{+\infty} 2^{j(n-np-\alpha p)} \le C. \tag{3.9}$$

As for $p = 1$, similar to the argument of $n/(n+\alpha) < p < 1$, we can easily get $U_2 \leq C$. So far the proof of Theorem 1.3 has been finished. □

Acknowledgments

This project supported by the National Natural Science Foundation of China under Grant no. 10747141, Zhejiang Provincial National Natural Science Foundation of China under Grant no. Y604056, and Science Foundation of Shaoguan University under Grant no. 200915001.

References

[1] A. P. Calderón and A. Zygmund, "On a problem of Mihlin," *Transactions of the American Mathematical Society*, vol. 78, pp. 209–224, 1955.
[2] L. Tang and D. Yang, "Boundedness of singular integrals of variable rough Calderón-Zygmund kernels along surfaces," *Integral Equations and Operator Theory*, vol. 43, no. 4, pp. 488–502, 2002.
[3] E. M. Stein, "On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz," *Transactions of the American Mathematical Society*, vol. 88, pp. 430–466, 1958.
[4] X. X. Tao and R. Y. Wei, "Boundedness of commutators related to Marcinkiewicz integrals with variable kernels in Herz-type Hardy spaces," *Acta Mathematica Scientia*, vol. 29, no. 6, pp. 1508–1517, 2009.
[5] D. Fan and S. Sato, "Weak type $(1,1)$ estimates for Marcinkiewicz integrals with rough kernels," *The Tohoku Mathematical Journal*, vol. 53, no. 2, pp. 265–284, 2001.
[6] M. Sakamoto and K. Yabuta, "Boundedness of Marcinkiewicz functions," *Studia Mathematica*, vol. 135, no. 2, pp. 103–142, 1999.
[7] Q. Xue and K. Yabuta, "L^2-boundedness of Marcinkiewicz integrals along surfaces with variable kernels: another sufficient condition," *Journal of Inequalities and Applications*, vol. 2007, Article ID 26765, 14 pages, 2007.
[8] Y. Ding, C.-C. Lin, and S. Shao, "On the Marcinkiewicz integral with variable kernels," *Indiana University Mathematics Journal*, vol. 53, no. 3, pp. 805–821, 2004.
[9] J. Chen, Y. Ding, and D. Fan, "Littlewood-Paley operators with variable kernels," *Science in China Series A*, vol. 49, no. 5, pp. 639–650, 2006.
[10] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, vol. 79 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Washington, DC, USA, 1991.
[11] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, USA, 1971, Princeton Mathematical Series, No. 3.
[12] Y. Ding and R. Li, "An estimate of Bessel function and its application," *Science in China Series A*, vol. 38, no. 1, pp. 78–87, 2008.
[13] N. E. Aguilera and E. O. Harboure, "Some inequalities for maximal operators," *Indiana University Mathematics Journal*, vol. 29, no. 4, pp. 559–576, 1980.
[14] A.-P. Calderón and A. Zygmund, "On singular integrals with variable kernels," *Applicable Analysis*, vol. 7, no. 3, pp. 221–238, 1977-1978.
[15] J. Chen, D. Fan, and Y. Ying, "Certain operators with rough singular kernels," *Canadian Journal of Mathematics*, vol. 55, no. 3, pp. 504–532, 2003.
[16] Y.-S. Han, M. Paluszyński, and G. Weiss, "A new atomic decomposition for the Triebel-Lizorkin spaces," in *Harmonic Analysis and Operator Theory (Caracas, 1994)*, vol. 189 of *Contemporary Mathematics*, pp. 235–249, American Mathematical Society, Providence, RI, USA, 1995.
[17] S. Hofmann and S. Mayboroda, "Hardy and BMO spaces associated to divergence form elliptic operators," *Mathematische Annalen*, vol. 344, no. 1, pp. 37–116, 2009.