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Research Article

On Certain Subclasses of Meromorphically p-Valent Functions Associated by the Linear Operator D_λ^n

Amin Saif and Adem Kılıçman

Department of Mathematics, University Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia

Correspondence should be addressed to Adem Kılıçman, akilicman@putra.upm.edu.my

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The purpose of this paper is to introduce two novel subclasses $\Gamma_\lambda(n,\alpha,\beta)$ and $\Gamma_\lambda^*(n,\alpha,\beta)$ of meromorphic p-valent functions by using the linear operator D_λ^n. Then we prove the necessary and sufficient conditions for a function f in order to be in the new classes. Further we study several important properties such as coefficients inequalities, inclusion properties, the growth and distortion theorems, the radii of meromorphically p-valent starlikeness, convexity, and subordination properties. We also prove that the results are sharp for a certain subclass of functions.

1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_k z^k \quad \left(a_k \geq 0;\ p \in N = \{1,2,\ldots\}\right), \tag{1.1}$$

which are meromorphic and p-valent in the punctured unit disc $U^* = \{z \in C : 0 < |z| < 1\} = U - \{0\}$. For the functions f in the class Σ_p, we define a linear operator D_λ^n by the following form:

$$\begin{aligned} D_\lambda f(z) &= \left(1 + p\lambda\right) f(z) + \lambda z f'(z), \quad (\lambda \geq 0),\\ D_\lambda^0 f(z) &= f(z),\\ D_\lambda^1 f(z) &= D_\lambda f(z),\\ D_\lambda^2 f(z) &= D_\lambda\left(D_\lambda^1 f(z)\right), \end{aligned} \tag{1.2}$$

and in general for $n = 0, 1, 2, \ldots$, we can write

$$D_\lambda^n f(z) = \frac{1}{z^p} + \sum_{k=p+1}^{\infty} (1 + p\lambda + k\lambda)^n a_k z^k, \quad (n \in N_0 = N \cup \{0\};\ p \in N). \tag{1.3}$$

Then we can observe easily that for $f \in \Sigma_p$,

$$z\lambda\left(D_\lambda^n f(z)\right)' = D_\lambda^{n+1} f(z) - (1 + p\lambda) D_\lambda^n f(z), \quad (p \in N;\ n \in N_0). \tag{1.4}$$

Recall [1, 2] that a function $f \in \Sigma_p$ is said to be meromorphically starlike of order α if it is satisfying the following condition:

$$\operatorname{Re}\left\{ -\frac{z f'(z)}{f(z)} \right\} > \alpha, \quad (z \in U^*), \tag{1.5}$$

for some α $(0 \leqq \alpha < 1)$. Similarly recall [3] a function $f \in \Sigma_p$ is said to be meromorphically convex of order α if it is satisfying the following condition:

$$\operatorname{Re}\left\{ -1 - \frac{z f''(z)}{f'(z)} \right\} > \alpha, \quad (z \in U^*) \text{ for some } \alpha\ (0 \leq \alpha < 1). \tag{1.6}$$

Let $\Sigma_p(\alpha)$ be a subclass of Σ_p consisting the functions which satisfy the following inequality:

$$\operatorname{Re}\left\{ -\frac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)} \right\} > p\alpha, \quad (z \in U^*;\ \alpha \geq 0). \tag{1.7}$$

In the following definitions, we will define subclasses $\Gamma_\lambda(n, \alpha, \beta)$ and $\Gamma_\lambda^*(n, \alpha, \beta)$ by using the linear operator D_λ^n.

Definition 1.1. For fixed parameters $\alpha \geq 0,\ 0 \leq \beta < 1$, the meromorphically p-valent function $f(z) \in \Sigma_p(\alpha)$ will be in the class $\Gamma_\lambda(n, \alpha, \beta)$ if it satisfies the following inequality:

$$\operatorname{Re}\left\{ -\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)} \right\} \geq \alpha \left| \frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)} + 1 \right| + \beta, \quad (n \in N_0). \tag{1.8}$$

Definition 1.2. For fixed parameters $\alpha \geq 1/(2 + \beta);\ 0 \leq \beta < 1$, the meromorphically p-valent function $f(z) \in \Sigma_p(\alpha)$ will be in the class $\Gamma_\lambda^*(n, \alpha, \beta)$ if it satisfies the following inequality:

$$\left| \frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)} + \alpha + \alpha\beta \right| \leq \operatorname{Re}\left\{ -\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)} \right\} + \alpha - \alpha\beta, \quad \forall (n \in N_0). \tag{1.9}$$

Meromorphically multivalent functions have been extensively studied by several authors, see for example, Aouf [4–6], Joshi and Srivastava [7], Mogra [8, 9], Owa et al. [10], Srivastava et al. [11], Raina and Srivastava [12], Uralegaddi and Ganigi [13], Uralegaddi and Somanatha [14], and Yang [15]. Similarly, in [16], some new subclasses of meromorphic functions in the punctured unit disk was considered.

In [17], similar results were proved by using the p-valent functions that satisfy the following differential subordinations:

$$\frac{z\left(\mathcal{O}_p(r,\lambda)f(z)\right)^{(j+1)}}{(p-j)\left(\mathcal{O}_p(r,\lambda)f(z)\right)^{(j)}} \prec \frac{a+\left(aB+(A-B)\beta\right)z}{a(1+Bz)} \tag{1.10}$$

and studied the related coefficients inequalities with β complex number.

This paper is organized as follows. It consists of four sections. Sections 2 and 3 investigate the various important properties and characteristics of the classes $\Gamma_\lambda(n,\alpha,\beta)$ and $\Gamma^*_\lambda(n,\alpha,\beta)$ by giving the necessary and sufficient conditions. Further we study the growth and distortion theorems and determine the radii of meromorphically p-valent starlikeness of order μ $(0 \le \mu < p)$ and meromorphically p-valent convexity of order μ $(0 \le \mu < p)$. In Section 4 we give some results related to the subordination properties.

2. Properties of the Class $\Gamma_\lambda(n,\alpha,\beta)$

We begin by giving the necessary and sufficient conditions for functions f in order to be in the class $\Gamma_\lambda(n,\alpha,\beta)$.

Lemma 2.1 (see [2]). *Let*

$$R_a = \begin{cases} a - \dfrac{\alpha+\beta}{1+\alpha}, & \text{for } a \le 1 + \dfrac{1-\beta}{\alpha(1+\alpha)}, \\ \sqrt{(1-a)^2(1-\alpha^2) - 2(1-\beta)(1-a)}, & \text{for } a \ge 1 + \dfrac{1-\beta}{\alpha(1+\alpha)}. \end{cases} \tag{2.1}$$

Then

$$\{w : |w-a| \le R_a\} \subseteq \{w : \operatorname{Re}(w) \ge \alpha|w-1| + \beta\}. \tag{2.2}$$

Theorem 2.2. *Let $f \in \Sigma_p$. Then f is in the class $\Gamma_\lambda(n,\alpha,\beta)$ if and only if*

$$\sum_{k=p+1}^{\infty} \left[p(\alpha+\beta) + k(1+\alpha)\right](k\lambda + p\lambda + 1)^n a_k \le p(1-\beta) \qquad (\alpha \ge 0;\ 0 \le \beta < 1;\ p \in N;\ n \in N_0). \tag{2.3}$$

Proof. Suppose that $f \in \Gamma_\lambda(n,\alpha,\beta)$. Then by the inequalities (1.3) and (1.8), we get that

$$\operatorname{Re}\left\{-\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)}\right\} \geq \alpha\left|\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)}+1\right|+\beta. \tag{2.4}$$

That is,

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{1-\sum_{k=p+1}^{\infty}(k/p)(k\lambda+p\lambda+1)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k z^{k+p}}\right\} \\
&\quad\geq \alpha\left|\frac{\sum_{k=p+1}^{\infty}((k/p)+1)(k\lambda+p\lambda+1)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k z^{k+p}}\right|+\beta \\
&\quad\geq \operatorname{Re}\left\{\alpha\cdot\frac{\sum_{k=p+1}^{\infty}((k/p)+1)(k\lambda+p\lambda+1)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k z^{k+p}}+\beta\right\} \\
&\quad= \operatorname{Re}\left\{\frac{\beta+\sum_{k=p+1}^{\infty}[\alpha((k/p)+1)+\beta](k\lambda+p\lambda+1)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k z^{k+p}}\right\},
\end{aligned} \tag{2.5}$$

that is,

$$\operatorname{Re}\left\{\frac{p(1-\beta)-\sum_{k=p+1}^{\infty}(k+k\alpha+p\alpha+p\beta)(k\lambda+p\lambda+1)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k z^{k+p}}\right\} \geq 0. \tag{2.6}$$

Taking z to be real and putting $z \to 1^-$ through real values, then the inequality (2.6) yields

$$\frac{p(1-\beta)-\sum_{k=p+1}^{\infty}(k+k\alpha+p\alpha+p\beta)(k\lambda+p\lambda+1)^n a_k}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k} \geq 0, \tag{2.7}$$

which leads us at once to (2.3).

In order to prove the converse, suppose that the inequality (2.3) holds true. In Lemma 2.1, since $1 \leq 1+((1-\beta)/\alpha(1+\alpha))$, put $a=1$. Then for $p \in N$ and $n \in N_0$, let $w_{np} = -z(D_\lambda^n f(z))'/p(D_\lambda^n f(z))$. If we let $z \in \partial U^* = \{z \in C : |z| = 1\}$, we get from the inequalities (1.3) and (2.3) that $|w_{np}-1| \leq R_1$. Thus by Lemma 2.1 above, we ge that

$$\begin{aligned}
\operatorname{Re}\left\{-\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)}-1\right\} &= \operatorname{Re}\{w_{np}\} \geq \alpha|w_{np}-1|+\beta = \alpha\left|-\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)}-1\right|+\beta \\
&= \alpha\left|\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)}+1\right|+\beta, \quad (\alpha \geq 0;\ 0 \leq \beta < 1;\ p \in N;\ n \in N_0).
\end{aligned} \tag{2.8}$$

Therefore by the maximum modulus theorem, we obtain $f \in \Gamma_\lambda(n,\alpha,\beta)$. □

Corollary 2.3. *If $f \in \Gamma_\lambda(n, \alpha, \beta)$, then*

$$a_k \le \frac{p(1-\beta)}{[p(\alpha+\beta)+k(1+\alpha)](k\lambda+p\lambda+1)^n}, \quad (\alpha \ge 0;\ 0 \le \beta < 1;\ p \in N;\ n \in N_0). \tag{2.9}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} \frac{p(1-\beta)}{[p(\alpha+\beta)+k(1+\alpha)](k\lambda+p\lambda+1)^n} z^k, \quad (\alpha \ge 0;\ 0 \le \beta < 1;\ p \in N;\ n \in N_0). \tag{2.10}$$

Theorem 2.4. *The class $\Gamma_\lambda(n, \alpha, \beta)$ is closed under convex linear combinations.*

Proof. Suppose the function

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_k z^{k,j} \quad (a_{k,j} \ge 0;\ j = 1, 2;\ p \in N), \tag{2.11}$$

be in the class $\Gamma_\lambda(n, \alpha, \beta)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = (1-\delta) f_1(z) + \delta f_2(z) \quad (0 \le \delta \le 1), \tag{2.12}$$

is also in the class $\Gamma_\lambda(n, \alpha, \beta)$. Since

$$h(z) = z^{-p} + \sum_{k=p+1}^{\infty} [(1-\delta) a_{k,1} + \delta a_{k,2}] z^{k,j}, \quad (0 \le \delta \le 1), \tag{2.13}$$

and by Theorem 2.2, we get that

$$\begin{aligned}
&\sum_{k=p+1}^{\infty} [p(\alpha+\beta)+k(1+\alpha)](k\lambda+p\lambda+1)^n [(1-\delta)a_{k,1} + \delta a_{k,2}] \\
&\quad = \sum_{k=p+1}^{\infty} (1-\delta)[p(\alpha+\beta)+k(1+\alpha)](k\lambda+p\lambda+1)^n a_{k,1} \\
&\qquad + \sum_{k=p+1}^{\infty} \delta [p(\alpha+\beta)+k(1+\alpha)](k\lambda+p\lambda+1)^n a_{k,2} \\
&\quad \le (1-\delta)p(1-\beta) + \delta p(1-\beta) = p(1-\beta), \quad (\alpha \ge 0;\ 0 \le \beta < 1;\ p \in N;\ n \in N_0).
\end{aligned} \tag{2.14}$$

Hence $f \in \Gamma_\lambda(n, \alpha, \beta)$. □

The following are the growth and distortion theorems for the class $\Gamma_\lambda(n, \alpha, \beta)$.

Theorem 2.5. *If $f \in \Gamma_\lambda(n,\alpha,\beta)$, then*

$$\left\{\frac{(p+m-1)!}{(p-1)!}-\frac{(1-\beta)}{(2\alpha+\beta+1)(p+1)^{n-1}}\cdot\frac{p!2^{-n}}{(p-m)!}r^{2p}\right\}r^{-(p+m)} \le \left|f^{(m)}(z)\right|$$
$$\le \left\{\frac{(p+m-1)!}{(p-1)!}+\frac{(1-\beta)}{(2\alpha+\beta+1)(p+1)^{n-1}}\cdot\frac{p!2^{-n}}{(p-m)!}r^{2p}\right\}r^{-(p+m)} \tag{2.15}$$
$$(0<|z|=r<1;\ \alpha\ge 0;\ 0\le\beta<1;\ p\in N;\ n,m\in N_0;\ p>m).$$

The result is sharp for the function f given by

$$f(z)=z^{-p}+\sum_{k=p+1}^{\infty}\frac{(1-\beta)}{(2\alpha+\beta+1)(2p+2)^{n}}z^{p},\quad (n\in N_0;\ p\in N). \tag{2.16}$$

Proof. From Theorem 2.2, we get that

$$\frac{p(2\alpha+\beta+1)(2p+2)^{n}}{(p+1)!}\sum_{k=p+1}^{\infty}k!a_k \le \sum_{k=p+1}^{\infty}\left[p(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^{n}a_k$$
$$\le p(1-\beta), \tag{2.17}$$

that is,

$$\sum_{k=p+1}^{\infty}k!a_k \le \frac{p(1-\beta)(p+1)!}{p(2\alpha+\beta+1)(2p+2)^{n}}=\frac{(1-\beta)p!2^{-n}}{(2\alpha+\beta+1)(p+1)^{n-1}}. \tag{2.18}$$

By the differentiating the function f in the form (1.1) m times with respect to z, we get that

$$f^{m}(z)=(-1)^{m}\frac{(p+m-1)!}{(p-1)!}z^{-(p+m)}+\sum_{k=p+1}^{\infty}\frac{k!}{(k-m)!}a_k z^{k-m},\quad (m\in N_0;\ p\in N) \tag{2.19}$$

and Theorem 2.5 follows easily from (2.18) and (2.19). Finally, it is easy to see that the bounds in (2.15) are attained for the function f given by (2.18). □

Next we determine the radii of meromorphically p-valent starlikeness of order μ $(0\le\mu<p)$ and meromorphically p-valent convexity of order μ $(0\le\mu<p)$ for the class $\Gamma_\lambda(n,\alpha,\beta)$.

Theorem 2.6. *If $f\in\Gamma_\lambda(n,\alpha,\beta)$, then f is meromorphically p-valent starlike of order $\mu(0\le\mu<1)$ in the disk $|z|<r_1$, that is,*

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\}>\mu\quad (0\le\mu<p;\ |z|<r_1;\ p\in N), \tag{2.20}$$

where

$$r_1 = \inf_{k\geq p+1}\left\{\frac{(p-\mu)\left[p(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{p(k+\mu)(1-\beta)}\right\}^{1/(k+p)}. \tag{2.21}$$

Proof. By the form (1.1), we get that

$$\begin{aligned}\left|\frac{(zf'(z)/f(z))+p}{(zf'(z)/f(z))-p+2\mu}\right| &= \left|\frac{\sum_{k=p+1}^{\infty}(k+p)a_k z^k}{2(p-\mu)z^{-p}+\sum_{k=p+1}^{\infty}(k-p+2\mu)a_k z^k}\right| \\ &\leq \frac{\sum_{k=p+1}^{\infty}(k+p)|z|^k}{2(p-\mu)a_k|z|^{-p}+\sum_{k=p+1}^{\infty}(k-p+2\mu)a_k|z|^k} \\ &= \frac{\sum_{k=p+1}^{\infty}(k+p)a_k|z|^{k+p}}{2(p-\mu)+\sum_{k=p+1}^{\infty}(k-p+2\mu)a_k|z|^{k+p}}.\end{aligned} \tag{2.22}$$

Then the following incurability

$$\left|\frac{(zf'(z)/f(z))+p}{(zf'(z)/f(z))-p+2\mu}\right| \leq 1, \quad (0\leq\mu<p;\ p\in N) \tag{2.23}$$

also holds if

$$\sum_{k=p+1}^{\infty}\frac{(k+\mu)}{(p-\mu)}a_k|z|^{k+p} \leq 1, \quad (0\leq\mu<p;\ p\in N). \tag{2.24}$$

Then by Corollary 2.3 the inequality (2.24) will be true if

$$\frac{(k+\mu)}{(p-\mu)}|z|^{k+p} \leq \frac{\left[p(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{p(1-\beta)}, \quad (0\leq\mu<p;\ p\in N), \tag{2.25}$$

that is,

$$|z|^{k+p} \leq \frac{(p-\mu)\left[p(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{p(k+\mu)(1-\beta)}, \quad (0\leq\mu<p;\ p\in N). \tag{2.26}$$

Therefore the inequality (2.26) leads us to the disc $|z| < r_1$, where r_1 is given by the form (2.21). □

Theorem 2.7. *If $f\in\Gamma_\lambda(n,\alpha,\beta)$, then f is meromorphically p-valent convex of order μ $(0\leq\mu<1)$ in the disk $|z|<r_2$, that is,*

$$\operatorname{Re}\left\{-1-\frac{zf''(z)}{f'(z)}\right\} > \mu \quad (0\leq\mu<p;\ |z|<r_2;\ p\in N), \tag{2.27}$$

where

$$r_2 = \inf_{k \geq p+1} \left\{ \frac{(p-\mu)\left[(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{k(k+\mu)(1-\beta)} \right\}^{1/(k+p)}. \tag{2.28}$$

Proof. By the form (1.1), we get that

$$\begin{aligned}\left| \frac{1+(zf''(z)/f'(z))+p}{1+(zf''(z)/f'(z))-p+2\mu} \right| &= \left| \frac{\sum_{k=p+1}^{\infty} k(k+p)a_k z^k}{2p(p-\mu)z^{-p}+\sum_{k=p+1}^{\infty} k(k-p+2\mu)a_k z^k} \right| \\ &\leq \frac{\sum_{k=p+1}^{\infty} k(k+p)|z|^k}{2p(p-\mu)a_k|z|^{-p}+\sum_{k=p+1}^{\infty} k(k-p+2\mu)a_k|z|^k} \\ &= \frac{\sum_{k=p+1}^{\infty} k(k+p)a_k|z|^{k+p}}{2p(p-\mu)+\sum_{k=p+1}^{\infty} k(k-p+2\mu)a_k|z|^{k+p}}.\end{aligned} \tag{2.29}$$

Then the following incurability:

$$\left| \frac{1+(zf''(z)/f'(z))+p}{1+(zf''(z)/f'(z))-p+2\mu} \right| \leq 1, \quad (0 \leq \mu < p;\ p \in N) \tag{2.30}$$

will hold if

$$\sum_{k=p+1}^{\infty} \frac{k(k+\mu)}{p(p-\mu)} a_k |z|^{k+p} \leq 1, \quad (0 \leq \mu < p;\ p \in N). \tag{2.31}$$

Then by Corollary 2.3 the inequality (2.31) will be true if

$$\frac{k(k+\mu)}{p(p-\mu)}|z|^{k+p} \leq \frac{\left[p(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{p(1-\beta)}, \quad (0 \leq \mu < p;\ p \in N), \tag{2.32}$$

that is,

$$|z|^{k+p} \leq \frac{(p-\mu)\left[(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{k(k+\mu)(1-\beta)}, \quad (0 \leq \mu < p;\ p \in N). \tag{2.33}$$

Therefore the inequality (2.33) leads us to the disc $|z| < r_2$, where r_2 is given by the form (2.28). □

3. Properties of the Class $\Gamma_\lambda^*(n, \alpha, \beta)$

We first give the necessary and sufficient conditions for functions f in order to be in the class $\Gamma_\lambda^*(n, \alpha, \beta)$.

Lemma 3.1 (see [2]). *Let $\mu > \delta$ and*

$$R_a = \begin{cases} a-\delta, & \text{for } a \le 2\mu+\delta, \\ 2\sqrt{\mu(a-\mu-\delta)}, & \text{for } a \ge 2\mu+\delta. \end{cases} \tag{3.1}$$

Then

$$\{w : |w-a| \le R_a\} \subseteq \{w : |w-(\mu+\delta)| \le \operatorname{Re}\{w+\mu-\delta\}\}. \tag{3.2}$$

Lemma 3.2. *Let $\alpha \ge 0$ and $0 \le \beta < 1$*

$$R_a = \begin{cases} a-\alpha\beta, & \text{for } a \le 2\alpha+\alpha\beta, \\ 2\sqrt{\alpha(a-\alpha-\alpha\beta)}, & \text{for } a \ge 2\alpha+\alpha\beta. \end{cases} \tag{3.3}$$

Then

$$\{w : |w-a| \le R_a\} \subseteq \{w : |w-(\alpha+\alpha\beta)| \le \operatorname{Re}\{w+\alpha-\alpha\beta\}\}. \tag{3.4}$$

Proof. Since $\alpha \ge 0$ and $0 \le \beta < 1$, then $\alpha > \alpha\beta$. Then in Lemma 3.1, put $\mu = \alpha$ and $\delta = \alpha\beta$. □

Theorem 3.3. *Let $f \in \Sigma_p$. Then f is in the class $\Gamma_\lambda^*(n,\alpha,\beta)$ if and only if*

$$\sum_{k=p+1}^{\infty} (k+p\alpha\beta)(k\lambda+p\lambda+1)^n a_k \le p(1-\alpha\beta) \quad \left(\alpha \ge \frac{1}{2+\beta};\ 0 \le \beta < 1;\ p \in N;\ n \in N_0\right). \tag{3.5}$$

Proof. Suppose that $f \in \Gamma_\lambda^*(n,\alpha,\beta)$. Then by the inequality (1.9), we get that

$$\left|\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))} + \alpha + \alpha\beta\right| \le \operatorname{Re}\left\{-\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))}\right\} + \alpha - \alpha\beta. \tag{3.6}$$

That is,

$$\begin{aligned} \operatorname{Re}\left\{\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))} + \alpha + \alpha\beta\right\} &\le \left|\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))} + \alpha + \alpha\beta\right| \\ &\le \operatorname{Re}\left\{-\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))}\right\} + \alpha - \alpha\beta, \end{aligned} \tag{3.7}$$

that is,

$$\operatorname{Re}\left\{\frac{2z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)}+2\alpha\beta\right\}\le 0. \tag{3.8}$$

Hence by the inequality (1.3),

$$\operatorname{Re}\left\{\frac{-2p(1-\alpha\beta)+\sum_{k=p+1}^{\infty}2(k+p\alpha\beta)(k\lambda+p\lambda+1)^n a_k z^{k+p}}{p+\sum_{k=p+1}^{\infty}p(k\lambda+p\lambda+1)^n a_k z^{k+p}}\right\}\le 0. \tag{3.9}$$

Taking z to be real and putting $z\to 1^-$ through real values, then the inequality (3.9) yields

$$\frac{-2p(1-\alpha\beta)+\sum_{k=p+1}^{\infty}2(k+p\alpha\beta)(k\lambda+p\lambda+1)^n a_k}{p+\sum_{k=p+1}^{\infty}p(k\lambda+p\lambda+1)^n a_k}\le 0, \tag{3.10}$$

which leads us at once to (3.5).

In order to prove the converse, consider that the inequality (3.5) holds true. In Lemma 3.2 above, since $\alpha>\alpha\beta$ and $\alpha\ge 1/(2+\beta)$, that is, $1\le 2\alpha+\alpha\beta$, we can put $a=1$. Then for $p\in N$ and $n\in N_0$, let $w_{np}=-z(D_\lambda^n f(z))'/p(D_\lambda^n f(z))$. Now, if we let $z\in\partial U^*=\{z\in C:|z|=1\}$, we get from the inequalities (1.3) and (3.5) that $|w_{np}-1|\le R_1$. Thus by Lemma 3.2 above, we ge that

$$\begin{aligned}
&\left|\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)}+\alpha+\alpha\beta\right|\\
&\quad=\left|-\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)}-(\alpha+\alpha\beta)\right|\\
&\quad=\left|w-(\alpha+\alpha\beta)\right|\\
&\quad\le \operatorname{Re}\{w+\alpha-\alpha\beta\}=\operatorname{Re}\{w\}+\alpha-\alpha\beta\\
&\quad=\left\{-\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)}\right\}+\alpha-\alpha\beta,\quad \left(\alpha\ge\frac{1}{2+\beta};\ 0\le\beta<1;\ p\in N;\ n\in N_0\right).
\end{aligned} \tag{3.11}$$

Therefore by the maximum modulus theorem, we obtain $f\in\Gamma_\lambda^*(n,\alpha,\beta)$. □

Corollary 3.4. *If $f\in\Gamma_\lambda^*(n,\alpha,\beta)$, then*

$$a_k\le\frac{p(1-\alpha\beta)}{(k+p\alpha\beta)(k\lambda+p\lambda+1)^n}\quad\left(\alpha\ge\frac{1}{2+\beta};\ 0\le\beta<1;\ p\in N;\ n\in N_0\right). \tag{3.12}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} \frac{p(1-\alpha\beta)}{(k+p\alpha\beta)(k\lambda+p\lambda+1)^n} z^k \quad \left(\alpha \geq \frac{1}{2+\beta};\ 0 \leq \beta < 1;\ p \in N;\ n \in N_0\right). \tag{3.13}$$

Theorem 3.5. *The class $\Gamma_\lambda^*(n,\alpha,\beta)$ is closed under convex linear combinations.*

Proof. This proof is similar as the proof of Theorem 2.4. □

The following are the growth and distortion theorems for the class $\Gamma_\lambda^*(n,\alpha,\beta)$.

Theorem 3.6. *If $f \in \Gamma_\lambda^*(n,\alpha,\beta)$, then*

$$\begin{aligned} &\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(1-\alpha\beta)}{(1+\alpha\beta)(p+1)^{n-1}} \cdot \frac{p!2^{-n}}{(p-m)!} r^{2p}\right\} r^{-(p+m)} \leq \left|f^{(m)}(z)\right| \\ &\quad \leq \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{(1-\alpha\beta)}{(1+\alpha\beta)(p+1)^{n-1}} \cdot \frac{p!2^{-n}}{(p-m)!} r^{2p}\right\} r^{-(p+m)} \\ &\quad \left(0 < |z| = r < 1;\ \alpha \geq \frac{1}{2+\beta};\ 0 \leq \beta < 1;\ p \in N;\ n, m \in N_0;\ p > m\right). \end{aligned} \tag{3.14}$$

The result is sharp for the function f given by

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} \frac{(1-\alpha\beta)}{(1+\alpha\beta)(2p+2)^n} z^p, \quad (n \in N_0;\ p \in N). \tag{3.15}$$

Next we determine the radii of meromorphically p-valent starlikeness of order μ $(0 \leq \mu < p)$ and meromorphically p-valent convexity of order μ $(0 \leq \mu < p)$ for the class $\Gamma_\lambda^*(n,\alpha,\beta)$.

Theorem 3.7. *If $f \in \Gamma_\lambda^*(n,\alpha,\beta)$, then f is meromorphically p-valent starlike of order μ $(0 \leq \mu < 1)$ in the disk $|z| < r_1$, that is,*

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \mu \quad (0 \leq \mu < p;\ |z| < r_1;\ p \in N), \tag{3.16}$$

where

$$r_1 = \inf_{k \geq p+1} \left\{\frac{(p-\mu)(k+p\alpha\beta)(k\lambda+p\lambda+1)^n}{p(k+\mu)(1-\alpha\beta)}\right\}^{1/(k+p)}. \tag{3.17}$$

Proof. This proof is similar to the proof of Theorem 2.6. □

Theorem 3.8. *If $f \in \Gamma_{\lambda}^{*}(n,\alpha,\beta)$, then f is meromorphically p-valent convex of order μ $(0 \le \mu < 1)$ in the disk $|z| < r_2$, that is,*

$$\operatorname{Re}\left\{-1-\frac{zf''(z)}{f'(z)}\right\} > \mu \quad (0 \le \mu < p;\ |z| < r_2;\ p \in N), \tag{3.18}$$

where

$$r_2 = \inf_{k \ge p+1}\left\{\frac{(p-\mu)(k+p\alpha\beta)(k\lambda+p\lambda+1)^n}{k(k+\mu)(1-\alpha\beta)}\right\}^{1/(k+p)}. \tag{3.19}$$

Proof. This proof is similar to the proof of Theorem 2.7. □

4. Subordination Properties

If f and g are analytic functions in U, we say that f is *subordinate* to g, written symbolically as follows:

$$f \prec g \quad \text{in } U \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U) \tag{4.1}$$

if there exists a function w which is analytic in U with

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in U), \tag{4.2}$$

such that

$$f(z) = g(w(z)) \quad (z \in U). \tag{4.3}$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in U) \Longrightarrow f(0) = g(0), \quad f(U) \subset g(U). \tag{4.4}$$

In particular, if the function g is univalent in U we have the following equivalence (see [18]):

$$f(z) \prec g(z) \quad (z \in U) \Longleftrightarrow f(0) = g(0), \quad f(U) \subset g(U). \tag{4.5}$$

Let $\phi : C^2 \to C$ be a function and let h be univalent in U. If J is analytic function in U and satisfied the differential subordination $\phi(J(z), J'(z)) \prec h(z)$ then J is called a *solution of the differential subordination* $\phi(J(z), J'(z)) \prec h(z)$. The univalent function q is called a *dominant* of the solution of the differential subordination, $J \prec q$.

Lemma 4.1 (see [19]). *Let $q(z)\neq 0$ be univalent in U. Let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w)\neq 0$ when $w\in q(U)$. Set*

$$Q(z)=zq'(z)\phi(q(z)),\qquad h(z)=\theta(q(z))+Q(z). \tag{4.6}$$

Suppose that

(i) *$Q(z)$ is starlike univalent in U,*

(ii) $\operatorname{Re}\{zh'(z)/Q(z)\}>0$ *for $z\in U$.*

If J is analytic function in U and

$$\theta(J(z))+zJ'(z)\phi(J(z))\prec\theta(q(z))+zq'(z)\phi(q(z)), \tag{4.7}$$

then $J(z)\prec q(z)$ and q is the best dominant.

Lemma 4.2 (see [20]). *Let $w,\gamma\in C$ and ϕ is convex and univalent in U with $\phi(0)=1$ and* $\operatorname{Re}\{w\phi(z)+\gamma\}>0$ *for all $z\in U$. If q is analytic in U with $q(0)=1$ and*

$$q(z)+\frac{zq'(z)}{wq(z)+\gamma}\prec\phi(z)\quad(z\in U), \tag{4.8}$$

then $q(z)\prec\phi(z)$ and ϕ is the best dominant.

Theorem 4.3. *Let $q(z)\neq 0$ be univalent in U such that $zq'(z)/q(z)$ is starlike univalent in U and*

$$\operatorname{Re}\left\{1+\frac{\epsilon}{\gamma}q(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0,\quad(\epsilon,\gamma\in C,\ \gamma\neq 0). \tag{4.9}$$

If $f\in\Sigma_p$ satisfies the subordination

$$\epsilon\frac{z[D_\lambda^n f(z)]'}{[D_\lambda^n f(z)]}+\gamma\left[1+\frac{z[D_\lambda^n f(z)]''}{[D_\lambda^n f(z)]'}-\frac{z[D_\lambda^n f(z)]'}{[D_\lambda^n f(z)]}\right]\prec\epsilon q(z)+\frac{\gamma zq'(z)}{q(z)}, \tag{4.10}$$

then $z[D_\lambda^n f(z)]'/[D_\lambda^n f(z)]\prec q(z)$ and q is the best dominant.

Proof. Our aim is to apply Lemma 4.1. Setting

$$J(z)=\frac{z[D_\lambda^n f(z)]'}{[D_\lambda^n f(z)]}=\frac{-p+\sum_{k=p+1}^{\infty}k(k\lambda+p\lambda+1)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k z^{k+p}},\quad(n\in N_0;\ p\in N), \tag{4.11}$$

$\theta(w)=w$ and $\phi(w)=\gamma/w$, $\gamma\neq 0$. It can be easily observed that J is analytic in U, θ is analytic in C, ϕ is analytic in $C/\{0\}$ and $\phi(w)\neq 0$. By computation shows that

$$\frac{zJ'(z)}{J(z)}=1+\frac{z[D_\lambda^n f(z)]''}{[D_\lambda^n f(z)]'}-\frac{z[D_\lambda^n f(z)]'}{[D_\lambda^n f(z)]} \tag{4.12}$$

which yields, by (4.10), the following subordination:

$$\epsilon J(z)+\gamma\frac{zJ'(z)}{J(z)}\prec\epsilon q(z)+\frac{\gamma zq'(z)}{q(z)}, \tag{4.13}$$

that is,

$$\theta(J(z))+zJ'(z)\phi(J(z))\prec\theta\big(q(z)\big)+zq'(z)\phi\big(q(z)\big). \tag{4.14}$$

Now by letting

$$\begin{aligned} Q(z)&=zq'(z)\phi\big(q(z)\big)=\frac{\gamma zq'(z)}{q(z)},\\ h(z)&=\theta\big(q(z)\big)+Q(z)=\epsilon q(z)+\frac{\gamma zq'(z)}{q(z)}. \end{aligned} \tag{4.15}$$

We find Qi starlike univalent in U and that

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}=\operatorname{Re}\left\{1+\frac{\epsilon}{\gamma}q(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0. \tag{4.16}$$

Hence by Lemma 4.1, $z[D_\lambda^n f(z)]'/[D_\lambda^n f(z)]\prec q(z)$ and q is the best dominant. □

Corollary 4.4. *If $f\in\Sigma_p$ and assume that* (4.9) *holds, then*

$$1+\frac{z\left[D_\lambda^n f(z)\right]''}{\left[D_\lambda^n f(z)\right]'}\prec\frac{1+Az}{1+Bz}+\frac{(A-B)z}{(1+Az)(1+Bz)} \tag{4.17}$$

implies that $z[D_\lambda^n f(z)]'/[D_\lambda^n f(z)]\prec(1+Az)/(1+Bz)$, $-1\le B<A\le 1$ and $(1+Az)/(1+Bz)$ is the best dominant.

Proof. By setting $\epsilon=\gamma=1$ and $q(z)=(1+Az)/(1+Bz)$ in Theorem 4.3, then we can obtain the result. □

Corollary 4.5. *If $f\in\Sigma_p$ and assume that* (4.9) *holds, then*

$$1+\frac{z\left[D_\lambda^n f(z)\right]''}{\left[D_\lambda^n f(z)\right]'}\prec e^{\alpha z}+\alpha z \tag{4.18}$$

implies that $z[D_\lambda^n f(z)]'/[D_\lambda^n f(z)]\prec e^{\alpha z}$, $|\alpha|<\pi$ and $e^{\alpha z}$ is the best dominant.

Proof. By setting $\epsilon=\gamma=1$ and $q(z)=e^{\alpha z}$ in Theorem 4.3, where $|\alpha|<\pi$. □

Theorem 4.6. *Let $w, \gamma \in C$, and ϕ be convex and univalent in U with $\phi(0) = 1$ and $\mathrm{Re}\{w\phi(z)+\gamma\} > 0$ for all $z \in U$. If $f \in \Sigma_p$ satisfies the subordination*

$$\frac{1 + \gamma + \left(z[D_\lambda^n f(z)]''/[D_\lambda^n f(z)]'\right) - ((w/p) + 1)\left(z[D_\lambda^n f(z)]'/[D_\lambda^n f(z)]\right)}{w - \gamma\left(p[D_\lambda^n f(z)]/z[D_\lambda^n f(z)]'\right)} \prec \phi(z), \qquad (4.19)$$

then $-z[D_\lambda^n f(z)]'/p[D_\lambda^n f(z)] \prec \phi(z)$ and ϕ is the best dominant.

Proof. Our aim is to apply Lemma 4.2. Setting

$$q(z) = \frac{-z[D_\lambda^n f(z)]'}{p[D_\lambda^n f(z)]} = \frac{p + \sum_{k=p+1}^{\infty} k(k\lambda + p\lambda + 1)^n a_k z^{k+p}}{p + \sum_{k=p+1}^{\infty} p(k\lambda + p\lambda + 1)^n a_k z^{k+p}}, \quad (n \in N_0;\ p \in N). \qquad (4.20)$$

It can be easily observed that q is analytic in U and $q(0) = 1$. Computation shows that

$$\frac{zq'(z)}{q(z)} = 1 + \frac{z[D_\lambda^n f(z)]''}{[D_\lambda^n f(z)]'} - \frac{z[D_\lambda^n f(z)]'}{[D_\lambda^n f(z)]} \qquad (4.21)$$

which yields, by (4.19), the following subordination:

$$q(z) + \frac{zq'(z)}{wq(z) + \gamma} \prec \phi(z), \quad (z \in U). \qquad (4.22)$$

Hence by Lemma 4.2, $-z[D_\lambda^n f(z)]'/[pD_\lambda^n f(z)] \prec \phi(z)$ and ϕ is the best dominant. □

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