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Research Article

A Hilbert-Type Integral Inequality in the Whole Plane with the Homogeneous Kernel of Degree −2

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By applying the way of real and complex analysis and estimating the weight functions, we build a new Hilbert-type integral inequality in the whole plane with the homogeneous kernel of degree −2 involving some parameters and the best constant factor. We also consider its reverse. The equivalent forms and some particular cases are obtained.

1. Introduction

If $f(x), g(x) \geq 0$, satisfying $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have (see [1])

$$\iint_0^\infty \frac{f(x)g(y)}{x+y} dx\, dy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well known as Hilbert's integral inequality, which is important in analysis and in its applications [1, 2]. In recent years, by using the way of weight functions, a number of extensions of (1.1) were given by Yang [3]. Noticing that inequality (1.1) is a Homogenous kernel of degree −1, in 2009, a survey of the study of Hilbert-type inequalities with the homogeneous kernels of degree negative numbers and some parameters is given by [4]. Recently, some inequalities with the homogenous kernels of degree 0 and nonhomogenous kernels have been studied (see [5–9]).

All of the above inequalities are built in the quarter plane. Yang [10] built a new Hilbert-type integral inequality in the whole plane as follows:

$$\iint_{-\infty}^{\infty} \frac{f(x)g(y)}{1+e^{x+y}} dx\, dy < \pi \left(\int_{-\infty}^{\infty} e^{-x} f^2(x) dx \int_{-\infty}^{\infty} e^{-x} g^2(x) \right)^{1/2}, \tag{1.2}$$

where the constant factor π is the best possible. Zeng and Xie [11] also give a new inequality in the whole plane.

By applying the method of [10, 11] and using the way of real and complex analysis, the main objective of this paper is to give a new Hilbert-type integral inequality in the whole plane with the homogeneous kernel of degree −2 involving some parameters and a best constant factor. The reverse form is considered. As applications, we also obtain the equivalent forms and some particular cases.

2. Some Lemmas

Lemma 2.1. *If $|\lambda| < 1$, $0 < \alpha_1 < \alpha_2 < \pi$, define the weight functions $\omega(x)$ and $\varpi(y)$ $(x, y \in (-\infty, \infty))$ as follow:*

$$\begin{aligned} \omega(x) &:= \int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{1}{x^2 + 2xy\cos\alpha_i + y^2} \right\} \frac{|x|^{1+\lambda}}{|y|^{\lambda}} dy, \\ \varpi(y) &:= \int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{1}{x^2 + 2xy\cos\alpha_i + y^2} \right\} \frac{|y|^{1-\lambda}}{|x|^{-\lambda}} dx. \end{aligned} \tag{2.1}$$

Then we have $\omega(x) = \varpi(y) = k(\lambda)$ $(x, y \neq 0)$, where

$$\begin{aligned} k(\lambda) &:= \frac{\pi}{\sin\lambda\pi} \left[\frac{\sin\lambda\alpha_1}{\sin\alpha_1} + \frac{\sin\lambda(\pi-\alpha_2)}{\sin\alpha_2} \right] \quad (0 < |\lambda| < 1); \\ k(0) &:= \lim_{\lambda\to 0^+} k(\lambda) = \left[\frac{\alpha_1}{\sin\alpha_1} + \frac{\pi-\alpha_2}{\sin\alpha_2} \right]. \end{aligned} \tag{2.2}$$

Proof. For $x \in (-\infty, 0)$, setting $u = y/x$, $u = -y/x$, respectively, in the following first and second integrals, we have

$$\begin{aligned} \omega(x) &= \int_{-\infty}^{0} \frac{1}{x^2 + 2xy\cos\alpha_1 + y^2} \cdot \frac{(-x)^{1+\lambda}}{(-y)^{\lambda}} dy \\ &\quad + \int_{0}^{\infty} \frac{1}{x^2 + 2xy\cos\alpha_2 + y^2} \cdot \frac{(-x)^{1+\lambda}}{y^{\lambda}} dy \\ &= \int_0^{\infty} \frac{u^{-\lambda}}{u^2 + 2u\cos\alpha_1 + 1} du + \int_0^{\infty} \frac{u^{-\lambda}}{u^2 - 2u\cos\alpha_2 + 1} du. \end{aligned} \tag{2.3}$$

Setting a complex function as $f(z) = 1/(z^2 + 2z\cos\alpha_1 + 1)$, where $z_1 = -e^{i\alpha_1}$ and $z_2 = -e^{-i\alpha_1}$ are the first-order poles of $f(z)$, and $z = \infty$ is the first-order zero point of $f(z)$, in view of the theorem of obtaining real integral by residue [12], it follows for $0 < |\lambda| < 1$ that

$$\begin{aligned}
\int_0^\infty \frac{u^{-\lambda}du}{u^2 + 2u\cos\alpha_1 + 1} &= \int_0^\infty \frac{u^{(1-\lambda)-1}du}{u^2 + 2u\cos\alpha_1 + 1} \\
&= \frac{2\pi i}{1 - e^{2\pi(1-\lambda)i}}\left[\operatorname{Re} s\left(z^{-\lambda}f(z), z_1\right) + \operatorname{Re} s\left(z^{-\lambda}f(z), z_2\right)\right] \\
&= \frac{2\pi i}{1 - e^{2\pi(1-\lambda)i}}\left[\frac{z_1^{-\lambda}}{z_1 - z_2} + \frac{z_2^{-\lambda}}{z_2 - z_1}\right] \\
&= \frac{-\pi \cdot (-1)^{-\lambda}}{\sin\pi(1-\lambda)\cdot(-1)^{1-\lambda}}\left[\frac{\cos(-\lambda)\alpha_1 + i\sin(-\lambda)\alpha_1}{-2i\sin\alpha_1} + \frac{\cos\lambda\alpha_1 + i\sin\lambda\alpha_1}{2i\sin\alpha_1}\right] \\
&= \frac{\pi\sin\lambda\alpha_1}{\sin\pi\lambda\sin\alpha_1}.
\end{aligned} \tag{2.4}$$

For $\lambda = 0$, we can find by the integral formula that

$$\int_0^\infty \frac{1}{u^2 + 2u\cos\alpha_1 + 1}du = \frac{\alpha_1}{\sin\alpha_1}. \tag{2.5}$$

Obviously, we find that for $0 < |\lambda| < 1$,

$$\begin{aligned}
\int_0^\infty \frac{u^{-\lambda}}{u^2 - 2u\cos\alpha_2 + 1}du &= \int_0^\infty \frac{u^{-\lambda}}{u^2 + 2u\cos(\pi - \alpha_2) + 1}du \\
&= \frac{\pi \cdot \sin\lambda(\pi - \alpha_2)}{\sin\lambda\pi \cdot \sin\alpha_2}; \quad \text{for } \lambda = 0, \\
\int_0^\infty \frac{1}{u^2 - 2u\cos\alpha_2 + 1}du &= \frac{\pi - \alpha_2}{\sin\alpha_2}.
\end{aligned} \tag{2.6}$$

Hence we find $\omega(x) = k(\lambda)$ $(x \in (-\infty, 0))$.

For $x \in (0,\infty)$, setting $u = -y/x$, $u = y/x$, respectively, in the following first and second integrals, we have

$$\begin{aligned}\omega(x) &= \int_{-\infty}^{0} \frac{1}{x^2+2xy\cos\alpha_2+y^2}\cdot\frac{x^{1+\lambda}}{(-y)^{\lambda}}dy \\ &\quad+\int_{0}^{\infty} \frac{1}{x^2+2xy\cos\alpha_1+y^2}\cdot\frac{x^{1+\lambda}}{y^{\lambda}}dy \\ &= \int_{0}^{\infty} \frac{u^{-\lambda}}{u^2-2u\cos\alpha_2+1}du+\int_{0}^{\infty} \frac{u^{-\lambda}}{u^2+2u\cos\alpha_1+1}du = k(\lambda).\end{aligned} \tag{2.7}$$

By the same way, we still can find that $\varpi(y) = \omega(x) = k(\lambda)$ $(y,x \neq 0;\ |\lambda| < 1)$. The lemma is proved. □

Note 1. (1) It is obvious that $\omega(0) = \varpi(0) = 0$; (2) If $\alpha_1 = \alpha_2 = \alpha \in (0,\pi)$, then it follows that

$$\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\} = \frac{1}{x^2+2xy\cos\alpha+y^2}, \tag{2.8}$$

and by Lemma 2.1, we can obtain

$$\varpi(y) = \omega(x) = \frac{\pi\cos\lambda(\alpha-\pi/2)}{\cos(\lambda\pi/2)\sin\alpha} \quad (y,x\neq 0). \tag{2.9}$$

Lemma 2.2. *If $p > 1$, $1/p+1/q = 1$, $|\lambda| < 1$, $0 < \alpha_1 < \alpha_2 < \pi$, and $f(x)$ is a nonnegative measurable function in $(-\infty,\infty)$, then we have*

$$\begin{aligned}J &:= \int_{-\infty}^{\infty} |y|^{p(1-\lambda)-1}\left(\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}f(x)dx\right)^{p}dy \\ &\leq k^{p}(\lambda)\int_{-\infty}^{\infty}|x|^{-p\lambda-1}f^{p}(x)dx.\end{aligned} \tag{2.10}$$

Proof. By Lemma 2.1 and Hölder's inequality [13], we have

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}f(x)\right)^p \\
&=\left[\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}\left(\frac{|x|^{(-\lambda)/q}}{|y|^{\lambda/p}}f(x)\right)\left(\frac{|y|^{\lambda/p}}{|x|^{(-\lambda)/q}}\right)dx\right]^p \\
&\le\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}\frac{|x|^{(1-p)\lambda}}{|y|^{\lambda}}f^p(x)dx \\
&\quad\times\left(\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}\frac{|y|^{(q-1)\lambda}}{|x|^{(-\lambda)}}dx\right)^{p-1} \\
&=k^{p-1}(\lambda)|y|^{p(\lambda-1)+1}\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}\frac{|x|^{(1-p)\lambda}}{|y|^{\lambda}}f^p(x)dx.
\end{aligned}\tag{2.11}$$

Then by Fubini theorem, it follows that

$$\begin{aligned}
J&\le k^{p-1}(\lambda)\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}\frac{|x|^{(1-p)\lambda}}{|y|^{\lambda}}f^p(x)dx\right]dy \\
&=k^{p-1}(\lambda)\int_{-\infty}^{\infty}\omega(x)|x|^{-p\lambda-1}f^p(x)dx \\
&=k^{p}(\lambda)\int_{-\infty}^{\infty}|x|^{-p\lambda-1}f^p(x)dx.
\end{aligned}\tag{2.12}$$

The lemma is proved. □

3. Main Results and Applications

Theorem 3.1. *If $p>1$, $1/p+1/q=1$, $|\lambda|<1$, $0<\alpha_1<\alpha_2<\pi$, $f,g\ge 0$, satisfying $0<\int_{-\infty}^{\infty}|x|^{-p\lambda-1}f^p(x)dx<\infty$ and $0<\int_{-\infty}^{\infty}|y|^{q\lambda-1}g^q(y)dy<\infty$, then we have*

$$\begin{aligned}
I&:=\iint_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}f(x)g(y)dx\,dy \\
&<k(\lambda)\left(\int_{-\infty}^{\infty}|x|^{-p\lambda-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|y|^{q\lambda-1}g^q(y)dy\right)^{1/q},
\end{aligned}\tag{3.1}$$

$$J = \int_{-\infty}^{\infty} |y|^{p(1-\lambda)-1} \left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{1}{x^2 + 2xy\cos\alpha_i + y^2} \right\} f(x)dx \right)^p dy$$
$$< k^p(\lambda) \int_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)dx, \tag{3.2}$$

where the constant factor $k(\lambda)$ and $k^p(\lambda)$ are the best possible and $k(\lambda)$ is defined by Lemma 2.1. Inequality (3.1) *and* (3.2) *are equivalent.*

Proof. If (2.11) takes the form of equality for a $y \in (-\infty, 0) \cup (0, \infty)$, then there exist constants A and B, such that they are not all zero, and $A(|x|^{(1-p)\lambda}/|y|^{\lambda}) f^p(x) = B(|y|^{(q-1)\lambda}/|x|^{(-\lambda)}) g^q(y)$ a.e. in $(-\infty, 0) \cup (0, \infty)$. Hence, there exists a constant C, such that $A\cdot|x|^{-p\lambda} f^p(x) = B\cdot|y|^{q\lambda} g^q(y) = C$ a.e. in $(0, \infty)$. We suppose $A \neq 0$ (otherwise $B = A = 0$). Then $|x|^{-p\lambda-1} f^p(x) = C/A|x|$ a. e. in $(-\infty, \infty)$, which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)dx < \infty$. Hence (2.11) takes the form of strict inequality, so does (2.10), and we have (3.2).

By the Hölder's inequality [13], we have

$$I = \int_{-\infty}^{\infty} \left(|y|^{1/q-\lambda} \int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{1}{x^2 + 2xy\cos\alpha_i + y^2} \right\} f(x)dx \right) \left(|y|^{\lambda-1/q} g(y)dy \right)$$
$$\leq J^{1/p} \left(\int_{-\infty}^{\infty} |y|^{q\lambda-1} g^q(y)dy \right)^{1/q}. \tag{3.3}$$

By (3.2), we have (3.1). On the other hand, suppose that (3.1) is valid. Setting

$$g(y) = |y|^{p(1-\lambda)-1} \left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{1}{x^2 + 2xy\cos\alpha_i + y^2} \right\} f(x)dx \right)^{p-1}, \tag{3.4}$$

then it follows $J = \int_{-\infty}^{\infty} |y|^{q\lambda-1} g^q(y)dy$. By (2.10), we have $J < \infty$. If $J = 0$, then (3.2) is obvious value; if $0 < J < \infty$, then by (3.1), we obtain

$$0 < \int_{-\infty}^{\infty} |y|^{q\lambda-1} g^q(y)dy = J = I$$
$$< k(\lambda) \left(\int_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |y|^{q\lambda-1} g^q(y)dy \right)^{1/q}, \tag{3.5}$$
$$J^{1/p} = \left(\int_{-\infty}^{\infty} |y|^{q\lambda-1} g^q(y)dy \right)^{1/p} < k(\lambda) \left(\int_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)dx \right)^{1/p}.$$

Hence we have (3.2), which is equivalent to (3.1).

For $\varepsilon > 0$, define functions $\widetilde{f}(x)$, $\widetilde{g}(x)$ as follows:

$$\widetilde{f}(x) := \begin{cases} x^{\lambda-2\varepsilon/p}, & x \in (1,\infty), \\ 0, & x \in [-1,1], \\ (-x)^{\lambda-2\varepsilon/p}, & x \in (-\infty,-1), \end{cases}$$
$$\widetilde{g}(x) := \begin{cases} x^{-\lambda-2\varepsilon/q}, & x \in (1,\infty), \\ 0, & x \in [-1,1], \\ (-x)^{-\lambda-2\varepsilon/q}, & x \in (-\infty,-1). \end{cases} \tag{3.6}$$

Then $\widetilde{L} := \{\int_{-\infty}^{\infty} |x|^{-p\lambda-1} \widetilde{f}^p(x)dx\}^{1/p} \{\int_{-\infty}^{\infty} |y|^{q\lambda-1} \widetilde{g}^q(y)dy\}^{1/q} = 1/\varepsilon$ and

$$\widetilde{I} := \iint_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{1}{x^2 + 2xy\cos\alpha_i + y^2} \right\} \widetilde{f}(x)\widetilde{g}(y)\,dx\,dy = I_1 + I_2 + I_3 + I_4, \tag{3.7}$$

where

$$I_1 := \int_{-\infty}^{-1} (-x)^{\lambda-2\varepsilon/p} \left[\int_{-\infty}^{-1} \frac{(-y)^{-\lambda-2\varepsilon/q}}{x^2 + 2xy\cos\alpha_1 + y^2} dy \right] dx,$$
$$I_2 := \int_{-\infty}^{-1} (-x)^{\lambda-2\varepsilon/p} \left[\int_{1}^{\infty} \frac{y^{-\lambda-2\varepsilon/q}}{x^2 + 2xy\cos\alpha_2 + y^2} dy \right] dx,$$
$$I_3 := \int_{1}^{\infty} x^{\lambda-2\varepsilon/p} \left[\int_{-\infty}^{-1} \frac{(-y)^{-\lambda-2\varepsilon/q}}{x^2 + 2xy\cos\alpha_2 + y^2} dy \right] dx,$$
$$I_4 := \int_{1}^{\infty} x^{\lambda-2\varepsilon/p} \left[\int_{1}^{\infty} \frac{y^{-\lambda-2\varepsilon/q}}{x^2 + 2xy\cos\alpha_1 + y^2} dy \right] dx. \tag{3.8}$$

By Fubini theorem [14], we obtain

$$I_1 = I_4 = \int_1^\infty x^{-1-2\varepsilon} \int_{1/x}^\infty \frac{u^{-\lambda-2\varepsilon/q}}{u^2 + 2u\cos\alpha_1 + 1} du \quad \left(u = \frac{y}{x}\right)$$
$$= \int_1^\infty x^{-1-2\varepsilon} \left(\int_{1/x}^1 \frac{u^{-\lambda-2\varepsilon/q}du}{u^2 + 2u\cos\alpha_1 + 1} + \int_1^\infty \frac{u^{-\lambda-2\varepsilon/q}du}{u^2 + 2u\cos\alpha_1 + 1} \right) dx$$
$$= \int_0^1 \left(\int_{1/u}^\infty x^{-1-2\varepsilon} dy \right) \frac{u^{-\lambda-2\varepsilon/q}du}{u^2 + 2u\cos\alpha_1 + 1} + \frac{1}{2\varepsilon} \int_1^\infty \frac{u^{-\lambda-2\varepsilon/q}du}{u^2 + 2u\cos\alpha_1 + 1}$$

$$= \frac{1}{2\varepsilon}\left(\int_0^1 \frac{u^{-\lambda+2\varepsilon/p}}{u^2+2u\cos\alpha_1+1}du + \int_1^\infty \frac{u^{-\lambda-2\varepsilon/q}}{u^2+2u\cos\alpha_1+1}du\right),$$

$$I_2 = I_3 = \frac{1}{2\varepsilon}\left(\int_0^1 \frac{u^{-\lambda+2\varepsilon/p}}{u^2-2u\cos\alpha_2+1}du + \int_1^\infty \frac{u^{-\lambda-2\varepsilon/q}}{u^2-2u\cos\alpha_2+1}du\right). \tag{3.9}$$

In view of the above results, if the constant factor $k(\lambda)$ in (3.1) is not the best possible, then exists a positive number K with $K < k(\lambda)$, such that

$$\begin{aligned}&\int_0^1 \frac{u^{-\lambda+2\varepsilon/p}}{u^2+2u\cos\alpha_1+1}du + \int_1^\infty \frac{u^{-\lambda-2\varepsilon/q}}{u^2+2u\cos\alpha_1+1}du\\ &\quad+\int_0^1 \frac{u^{-\lambda+2\varepsilon/p}du}{u^2-2u\cos\alpha_2+1} + \int_1^\infty \frac{u^{-\lambda-2\varepsilon/q}du}{u^2-2u\cos\alpha_2+1} = \varepsilon\widetilde{I} < \varepsilon K\widetilde{L} = K.\end{aligned} \tag{3.10}$$

By Fatou lemma [14] and (3.10), we have

$$\begin{aligned}k(\lambda) &= \int_0^\infty \frac{u^{-\lambda}}{u^2+2u\cos\alpha_1+1}du + \int_0^\infty \frac{u^{-\lambda}}{u^2-2u\cos\alpha_2+1}du\\ &= \int_0^1 \lim_{\varepsilon\to 0^+} \frac{u^{-\lambda+2\varepsilon/p}}{u^2+2u\cos\alpha_1+1}du + \int_1^\infty \lim_{\varepsilon\to 0^+} \frac{u^{-\lambda-2\varepsilon/q}}{u^2+2u\cos\alpha_1+1}du\\ &\quad+\int_0^1 \lim_{\varepsilon\to 0^+} \frac{u^{-\lambda+2\varepsilon/p}}{u^2-2u\cos\alpha_2+1}du + \int_1^\infty \lim_{\varepsilon\to 0^+} \frac{u^{-\lambda-2\varepsilon/q}}{u^2-2u\cos\alpha_2+1}du\\ &\le \underline{\lim}_{\varepsilon\to 0^+}\left[\int_0^1 \frac{u^{-\lambda+2\varepsilon/p}}{u^2+2u\cos\alpha_1+1}du + \int_1^\infty \frac{u^{-\lambda-2\varepsilon/q}}{u^2+2u\cos\alpha_1+1}du\right.\\ &\quad\left.+\int_0^1 \frac{u^{-\lambda+2\varepsilon/p}}{u^2-2u\cos\alpha_2+1}du + \int_1^\infty \frac{u^{-\lambda-2\varepsilon/q}}{u^2-2u\cos\alpha_2+1}du\right] \le K,\end{aligned} \tag{3.11}$$

which contradicts the fact that $K < k(\lambda)$. Hence the constant factor $k(\lambda)$ in (3.1) is the best possible.

If the constant factor in (3.2) is not the best possible, then by (3.3), we may get a contradiction that the constant factor in (3.1) is not the best possible. Thus the theorem is proved. □

In view of Note (2) and Theorem 3.1, we still have the following theorem.

Theorem 3.2. *If* $p > 1$, $1/p + 1/q = 1$, $|\lambda| < 1$, $0 < \alpha < \pi$, *and* $f, g \geq 0$, *satisfying* $0 < \int_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)dx < \infty$ *and* $0 < \int_{-\infty}^{\infty} |y|^{q\lambda-1} g^q(y)dy < \infty$, *then we have*

$$\begin{gathered}
\iint_{-\infty}^{\infty} \frac{1}{x^2 + 2xy\cos\alpha + y^2} f(x)g(y)dx\,dy \\
< \frac{\pi\cos\lambda(\alpha - \pi/2)}{\cos(\lambda\pi/2)\sin\alpha} \left(\int_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)dx\right)^{1/p} \left(\int_{-\infty}^{\infty} |y|^{q\lambda-1} g^q(y)dy\right)^{1/q}, \\
\int_{-\infty}^{\infty} |y|^{p(1-\lambda)-1} \left(\int_{-\infty}^{\infty} \frac{1}{x^2 + 2xy\cos\alpha + y^2} f(x)dx\right)^p dy \\
< \left[\frac{\pi\cos\lambda(\alpha - \pi/2)}{\cos(\lambda\pi/2)\sin\alpha}\right]^p \iint_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)dx,
\end{gathered} \tag{3.12}$$

where the constant factors $\pi\cos\lambda(\alpha - \pi/2)/\cos(\lambda\pi/2)\sin\alpha$ *and* $[\pi\cos\lambda(\alpha - \pi/2)/\cos(\lambda\pi/2)\sin\alpha]^p$ *are the best possible. Inequality* (3.12) *is equivalent.*

In particular, for $\alpha = \pi/3$, *we have the following equivalent inequalities:*

$$\begin{gathered}
\iint_{-\infty}^{\infty} \frac{1}{x^2 + xy + y^2} f(x)g(y)dx\,dy \\
< \frac{2\pi\cos(\lambda\pi/6)}{\sqrt{3}\cos(\lambda\pi/2)} \left(\int_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)dx\right)^{1/p} \left(\int_{-\infty}^{\infty} |y|^{q\lambda-1} g^q(y)dy\right)^{1/q}, \\
\int_{-\infty}^{\infty} |y|^{p(1-\lambda)-1} \left(\int_{-\infty}^{\infty} \frac{1}{x^2 + xy + y^2} f(x)dx\right)^p dy \\
< \left[\frac{2\pi\cos(\lambda\pi/6)}{\sqrt{3}\cos(\lambda\pi/2)}\right]^p \iint_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)dx.
\end{gathered} \tag{3.13}$$

Theorem 3.3. *As the assumptions of Theorem 3.1, replacing* $p > 1$ *by* $0 < p < 1$, *we have the equivalent reverses of* (3.1) *and* (3.2) *with the best constant factors.*

Proof. By the reverse Hölder's inequality [13], we have the reverse of (2.10) and (3.3). It is easy to obtain the reverse of (3.2). In view of the reverses of (3.2) and (3.3), we obtain the reverse of (3.1). On the other hand, suppose that the reverse of (3.1) is valid. Setting the same $g(y)$ as Theorem 3.1, by the reverse of (2.10), we have $J > 0$. If $J = \infty$, then the reverse of (3.2) is obvious value; if $J < \infty$, then by the reverse of (3.1), we obtain the reverses of (3.5). Hence we have the reverse of (3.2), which is equivalent to the reverse of (3.1).

If the constant factor $k(\lambda)$ in the reverse of (3.1) is not the best possible, then there exists a positive constant K (with $K > k(\lambda)$), such that the reverse of (3.1) is still valid as we replace $k(\lambda)$ by K. By the reverse of (3.10), we have

$$\begin{aligned} &\int_0^1 \left[\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right]u^{-\lambda+2\varepsilon/p}\,du \\ &+\int_1^\infty \left[\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right]u^{-\lambda-2\varepsilon/q}du > K. \end{aligned} \tag{3.14}$$

For $\varepsilon \to 0^+$, by the Levi's theorem [14], we find

$$\begin{aligned} &\int_0^1 \left[\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right]u^{-\lambda+2\varepsilon/p}du \\ &\longrightarrow \int_0^1 \left[\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right]u^{-\lambda}du. \end{aligned} \tag{3.15}$$

For $0 < \varepsilon < \varepsilon_0,\ q < 0$, such that $|\lambda + 2\varepsilon_0/q| < 1$, since

$$\begin{gathered} u^{-\lambda-2\varepsilon/q} \le u^{-\lambda-2\varepsilon_0/q}, \quad u \in [1,\infty), \\ \int_1^\infty \left[\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right]u^{-\lambda-2\varepsilon_0/q}du \le k\left(\lambda+\frac{2\varepsilon_0}{q}\right) < \infty, \end{gathered} \tag{3.16}$$

then by Lebesgue control convergence theorem [14], for $\varepsilon \to 0^+$, we have

$$\begin{aligned} &\int_1^\infty \left[\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right]u^{-\lambda-2\varepsilon/q}du \\ &\longrightarrow \int_1^\infty \left[\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right]u^{-\lambda}du. \end{aligned} \tag{3.17}$$

By (3.14), (3.15), and (3.17), for $\varepsilon \to 0^+$, we have $k(\lambda) \ge K$, which contradicts the fact that $k(\lambda) < K$. Hence the constant factor $k(\lambda)$ in the reverse of (3.1) is the best possible.

If the constant factor in reverse of (3.2) is not the best possible, then by the reverse of (3.3), we may get a contradiction that the constant factor in the reverse of (3.1) is not the best possible. Thus the theorem is proved. □

By the same way of Theorem 3.3, we still have the following theorem.

Theorem 3.4. *By the assumptions of Theorem 3.2, replacing $p > 1$ by $0 < p < 1$, we have the equivalent reverses of* (3.12) *with the best constant factors.*

References

[1] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, The University Press, Cambridge, UK, 2nd edition, 1952.

[2] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, vol. 53 of *Mathematics and Its Applications (East European Series)*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

[3] B. Yang, *The Norm of Operator and Hilbert-Type Inequalities*, Science Press, Beijing, China, 2009.

[4] B. C. Yang, "A survey of the study of Hilbert-type inequalities with parameters," *Advances in Mathematics*, vol. 38, no. 3, pp. 257–268, 2009.

[5] B. C. Yang, "On the norm of an integral operator and applications," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 1, pp. 182–192, 2006.

[6] J. Xu, "Hardy-Hilbert's inequalities with two parameters," *Advances in Mathematics*, vol. 36, no. 2, pp. 189–202, 2007.

[7] B. C. Yang, "On the norm of a Hilbert's type linear operator and applications," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 529–541, 2007.

[8] D. M. Xin, "A Hilbert-type integral inequality with a homogeneous kernel of zero degree," *Mathematical Theory and Applications*, vol. 30, no. 2, pp. 70–74, 2010.

[9] B. C. Yang, "A Hilbert-type integral inequality with a homogeneous kernel of degree zero," *Journal of Shandong University. Natural Science*, vol. 45, no. 2, pp. 103–106, 2010.

[10] B. C. Yang, "A new Hilbert-type inequality," *Bulletin of the Belgian Mathematical Society*, vol. 13, no. 3, pp. 479–487, 2006.

[11] Z. Zeng and Z. Xie, "On a new Hilbert-type integral inequality with the integral in whole plane," *Journal of Inequalities and Applications*, vol. 2010, Article ID 256796, 8 pages, 2010.

[12] Y. Ping, H. Wang, and L. Song Jr., *Complex Function*, Science Press, Beijing, China, 2004.

[13] J. Kuang, *Applied Inequalities*, Shangdong Science and Technology Press, Jinan, China, 2004.

[14] J. Kuang, *Introudction to Real Analysis*, Hunan Educiton Press, Changsha, China, 1996.