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Research Article

Precise Asymptotics in the Law of Iterated Logarithm for Moving Average Process under Dependence

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Let $\{\xi_i,\ -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed and ϕ-mixing random variables, and let $\{a_i,\ -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers. In this paper, we get precise asymptotics in the law of the logarithm for linear process $\{X_k = \sum_{i=-\infty}^{+\infty} a_{i+k}\xi_i,\ k \geq 1\}$, which extend Liu and Lin's (2006) result to moving average process under dependence assumption.

1. Introduction and Main Results

Let $\{\xi_i,\ -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed random variables with zero means and finite variances, and let $\{a_i,\ -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers. Let

$$X_k = \sum_{i=-\infty}^{+\infty} a_{i+k}\xi_i, \quad k \geq 1, \tag{1.1}$$

be the moving average process based on $\{\xi_i,\ -\infty < i < \infty\}$. As usual, we denote $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$ as the sequence of partial sums.

Under the assumption that $\{\xi_i,\ -\infty < i < \infty\}$ is a sequence of independent identically distributed random variables, many limiting results have been obtained. Ibragimov [1] established the central limit theorem; Burton and Dehling [2] obtained a large deviation principle; Yang [3] established the central limit theorem and the law of the iterated logarithm;

Li et al. [4] obtained the complete convergence result for $\{X_k,\ k \geq 1\}$. As we know, $X_k\ (k \geq 1)$ are dependent even if $\{\xi_i,\ -\infty < i < \infty\}$ is a sequence of i.i.d. random variables. Therefore, we introduce the definition of ϕ-mixing,

$$\phi(m) := \sup_{k\geq 1}\left\{|P(B \mid A) - P(B)|,\ A \in \mathcal{F}_{-\infty}^{k},\ P(A) \neq 0,\ B \in \mathcal{F}_{k+m}^{\infty}\right\} \longrightarrow 0, \quad m \longrightarrow \infty, \tag{1.2}$$

where $\mathcal{F}_a^b = \sigma(\xi_i,\ a \leq i \leq b)$. Many limiting results of moving average for ϕ-mixing have been obtained. For example, Zhang [5] got complete convergence.

Theorem A. *Suppose that $\{\xi_i,\ -\infty < i < \infty\}$ is a sequence of identically distributed and ϕ-mixing random variables with $\sum_{m=1}^{\infty}\phi^{1/2}(m) < \infty$, and $\{X_k,\ k \geq 1\}$ is defined as (1.1). Let $h(x) > 0\ (x > 0)$ be a slowly varying function and $1 \leq t < 2$, $r \geq 1$, then $E\xi_1 = 0$ and $E|\xi_1|^{rt}h(|Y_t^t|) < \infty$ imply*

$$\sum_{n=1}^{\infty} n^{r-2}h(n)P\left(|S_n| \geq n^{1/t}\epsilon\right) < \infty, \quad \forall\epsilon > 0. \tag{1.3}$$

Li and Zhang [6] achieved precise asymptotics in the law of the iterated logarithm.

Theorem B. *Suppose that $\{\xi_i,\ -\infty < i < \infty\}$ is a sequence of identically distributed and ϕ-mixing random variables with mean zeros and finite variances, $\sum_{m=1}^{\infty}\phi^{1/2}(m) < \infty$, and $0 < \sigma^2 = E\xi_1^2 + 2\sum_{k=2}^{n}E\xi_1\xi_k < \infty$, $E\xi_1^2(\log^+|\xi_1|)^{\delta-1} < \infty$, for $\delta > 0$. Suppose that $\{X, X_k,\ k \geq 1\}$ is defined as in (1.1), where $\{a_i,\ -\infty < i < \infty\}$ is a sequence of real number with $\sum_{i=-\infty}^{\infty}|a_i| < \infty$, then one has*

$$\lim_{\epsilon\searrow 0}\epsilon^{2\delta+2}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\left(|S_n| \geq \sqrt{n\log n}\,\tau\epsilon\right) = \frac{\tau^{2\delta+2}}{\delta+1}E|N|^{2\delta+2}, \tag{1.4}$$

where $\tau =: \sigma\sum_{i=-\infty}^{\infty}a_i$, N is a standard normal random variable.

On the other hand, since Hsu and Robbins [7] introduced the concept of the complete convergence, there have been extensions in some directions. For the case of i.i.d. random variables, Davis [8] proved $\sum_{n=1}^{\infty}(\log n/n)P(|S_n| \geq \sqrt{n\log n}\epsilon) < \infty$, for $\epsilon > 0$ if and only if $EX_1 = 0$, $EX_1^2 < \infty$. Gut and Spătaru [9] gave the precise asymptotics of $\sum_{n=2}^{\infty}((\log n)^{\delta}/n)P(|S_n| \geq \sqrt{n\log n}\epsilon)$. We know that complete convergence can be derived from complete moment convergence. Liu and Lin [10] introduced a new kind of convergence of $\sum_{n=2}^{\infty}((\log n)^{\delta-1}/n^2)E|S_n|^2 I\{|S_n| \geq \sqrt{n\log n}\epsilon\}$. In this note, we show that the precise asymptotics for the moment convergence hold for moving-average process when $\{\xi_i,\ -\infty < i < \infty\}$ is a strictly stationary ϕ-mixing sequences. Now, we state the main results.

Theorem 1.1. *Suppose that $\{X, X_k, k \geq 1\}$ is defined as in (1.1), where $\{a_i,\ -\infty < i < \infty\}$ is a sequence of real number with $\sum_{i=-\infty}^{\infty}|a_i| < \infty$, and $\{\xi_i,\ -\infty < i < \infty\}$ is a sequence of identically* *distributed ϕ-mixing random variables with mean zeros and finite variances, $\sum_{m=1}^{\infty} \phi^{1/2}(m) < \infty$ and $0 < \sigma^2 = E\xi_1^2 + 2\sum_{k=2}^{n} E\xi_1\xi_k < \infty$, $E\xi_1^2(\log^+|\xi_1|)^{\delta} < \infty$, for $0 < \delta \leq 1$, then one has*

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} E|S_n|^2 I\left\{|S_n| \geq \sqrt{n \log n}\tau\epsilon\right\} = \frac{\tau^{2\delta+2}}{\delta} E|N|^{2\delta+2}, \tag{1.5}$$

where $\tau =: \sigma \sum_{i=-\infty}^{\infty} a_i$.

Theorem 1.2. *Under the conditions in Theorem 1.1, one has*

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta} \sum_{n=3}^{\infty} \frac{(\log \log n)^{\delta-1}}{n^2 \log n} ES_n^2 I\left\{|S_n| \geq \sqrt{n \log \log n}\tau\epsilon\right\} = \frac{\tau^{2\delta+2} E|N|^{2(\delta+1)}}{\delta}. \tag{1.6}$$

Remark 1.3. In this paper, we generate the results of Liu and Lin [10] to linear process under dependence based on Theorem B by using the technique of dealing with the innovation process in Zhang [5].

We first proceed with some useful lemmas.

Lemma 1.4. *Let $\{X, X_k, k \geq 1\}$ be defined as in (1.1), and let $\{\xi_i, -\infty < i < \infty\}$ be a sequence of identically distributed ϕ-mixing random variables with $E\xi_1 = 0$, $E\xi_1^2 < \infty$, $0 < \sigma^2 = E\xi_1^2 + 2\sum_{k=2}^{\infty} E\xi_1\xi_k < \infty$, $\sum_{m=1}^{\infty} \phi^{1/2}(2^m) < \infty$, then*

$$\frac{S_n}{\tau\sqrt{n}} \xrightarrow{D} N(0,1). \tag{1.7}$$

The proof is similar to Theorem 1 in [11]. Set $\Delta_n = \sup_x |P(|S_n| \geq \sqrt{n}x) - P(|N| \geq x)|$. From Lemma 1.4, one can get $\Delta_n \to 0$ as $n \to \infty$.

Lemma 1.5 (see [2]). *Let $\sum_{i=-\infty}^{+\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{i=-\infty}^{+\infty} a_i$ and $k \geq 1$, then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=-\infty}^{+\infty} \left| \sum_{j=i+1}^{i+n} a_j \right|^k = |a|^k. \tag{1.8}$$

Lemma 1.6 (see [12]). *Let $\{X_i, i \geq 1\}$ be a sequence of ϕ-mixing random variables with zero means and finite second moments. Let $S_n = \sum_{i=1}^{n} X_i$. If exists C_n such that $\max_{1\leq i\leq n} ES_n^2 \leq C_n$, then for all $q \geq 2$, there exists $C = C(q, \phi(\cdot))$ such that*

$$E \max_{1\leq i\leq n} |S_i|^q \leq C\left(C_n^{q/2} + E \max_{1\leq i\leq n} |X_i|^q\right). \tag{1.9}$$

2. Proofs

Proof of Theorem 1.1. Without loss of generality, we assume that $\tau = 1$. We have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}ES_n^2 I\left\{|S_n| \geq \sqrt{n\log n}\epsilon\right\} \\
&\quad = \epsilon^2\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\left\{|S_n| \geq \sqrt{n\log n}\epsilon\right\} + \sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\int_{\sqrt{n\log n}\epsilon}^{\infty} 2xP(|S_n| \geq x)dx.
\end{aligned} \tag{2.1}$$

Set $d(\epsilon) = \exp(M\epsilon^{-2})$, where $M > 1$. By Theorem B, we need to show

$$\lim_{\epsilon\searrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\int_{\sqrt{n\log n}\epsilon}^{\infty} 2xP(|S_n| \geq x)dx = \frac{1}{\delta(\delta+1)}E|N|^{2\delta+2}. \tag{2.2}$$

By Proposition 5.1 in [10], we have

$$\lim_{\epsilon\searrow 0}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\int_{\sqrt{n\log n}\epsilon}^{\infty} 2xP\left(|N| \geq \frac{x}{\sqrt{n}}\right)dx = \frac{1}{\delta(\delta+1)}E|N|^{2\delta+2}. \tag{2.3}$$

Hence, Theorem 1.1 will be proved if we show the following two propositions. □

Proposition 2.1. *One has*

$$\lim_{\epsilon\searrow 0}\epsilon^{2\delta}\sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-1}}{n^2}\left|\int_{\sqrt{n\log n}\epsilon}^{\infty} 2xP(|S_n| \geq x)dx - \int_{\sqrt{n\log n}\epsilon}^{\infty} 2xP\left(|N| \geq \frac{x}{\sqrt{n}}\right)dx\right| = 0. \tag{2.4}$$

Proof. Write

$$\begin{aligned}
&\sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-1}}{n^2}\left|\int_{\sqrt{n\log n}\epsilon}^{\infty} 2xP(|S_n| \geq x)dx - \int_{\sqrt{n\log n}\epsilon}^{\infty} 2xP\left(|N| \geq \frac{x}{\sqrt{n}}\right)dx\right| \\
&\quad \leq \sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-1}}{n}\log n \\
&\qquad \times \int_0^{\infty} 2(x+\epsilon)\left|P\left(|S_n| \geq \sqrt{n\log n}(x+\epsilon)\right) - P\left(|N| \geq \sqrt{\log n}(x+\epsilon)\right)\right|dx \\
&\quad \leq \sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-1}}{n}(\Delta_{n1}+\Delta_{n2}+\Delta_{n3}),
\end{aligned} \tag{2.5}$$

where

$$\Delta_{n1} = \log n \int_0^{1/\sqrt{\log n}\Delta_n^{1/4}} 2(x+\epsilon)\left|P\left(|S_n| \geq \sqrt{n\log n}(x+\epsilon)\right) - P\left(|N| \geq \sqrt{\log n}(x+\epsilon)\right)\right| dx,$$
$$\Delta_{n2} = \log n \int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty} 2(x+\epsilon)P\left(|S_n| \geq \sqrt{n\log n}(x+\epsilon)\right)dx,$$
$$\Delta_{n3} = \log n \int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty} 2(x+\epsilon)P\left(|N| \geq \sqrt{\log n}(x+\epsilon)\right)dx. \tag{2.6}$$

Since $n \leq d(\epsilon)$ implies $\sqrt{\log n}\epsilon \leq \sqrt{M}$, we have

$$\Delta_{n1} \leq C\log n\Delta_n \left(\frac{1}{\sqrt{\log n}\Delta_n^{1/4}} + \epsilon\right)^2 \leq C\left(\Delta_n^{1/4} + \sqrt{M\Delta_n}\right)^2. \tag{2.7}$$

For Δ_{n3}, by Markov's inequality, we get

$$\Delta_{n3} \leq C\log n \int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty} \frac{1}{(\log n)^{3/2}(x+\epsilon)^2}dx \leq C\Delta_n^{1/4}. \tag{2.8}$$

From (2.7) and (2.8), we can get

$$\lim_{\epsilon\searrow 0} \epsilon^{2\delta} \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-1}}{n}(\Delta_{n1} + \Delta_{n3}) = 0. \tag{2.9}$$

Note that $\sum_{k=1}^n X_k = \sum_{i=-\infty}^{\infty}\sum_{k=1}^n a_{k+i}\xi_i = \sum_{i=-\infty}^{\infty} a_{ni}\xi_i$, where $a_{ni} = \sum_{k=1}^n a_{k+i}$. By Lemma 1.5, we can assume that

$$\sum_{i=-\infty}^{\infty} |a_{ni}|^t \leq n, \quad t \geq 1, \qquad \tilde{a} = \sum_{i=-\infty}^{\infty} |a_i| \leq 1. \tag{2.10}$$

Set $S_n' = \sum_{i=-\infty}^{\infty} a_{ni}\xi_i I\{|a_{ni}\xi_i| \leq \sqrt{n\log n}(x+\epsilon)\}$. As $E\xi_1 = 0$, by (2.10), we have

$$\begin{aligned}
\left|ES_n'\right| &\leq C\left|E\sum_{i=-\infty}^{\infty} a_{ni}\xi_i I\left\{|a_{ni}\xi_i| > \sqrt{n\log n(x+\epsilon)}\right\}\right| \\
&\leq C\sum_{i=-\infty}^{\infty} |a_{ni}|E|\xi_1| I\left\{|a_{ni}\xi_1| > \sqrt{n\log n(x+\epsilon)}\right\} \\
&\leq CnE|\xi_1| I\left\{\tilde{a}|\xi_1| > \sqrt{n\log n}(x+\epsilon)\right\}
\end{aligned}$$

$$\begin{aligned}&\le CnE|\xi_1|I\left\{|\xi_1|>\sqrt{n\log n}(x+\epsilon)\right\}\\&\le Cn\left(E\xi_1^2\right)^{1/2}\left(P\left(|\xi_1|>\sqrt{n\log n}(x+\epsilon)\right)\right)^{1/2}\\&\le C\frac{\sqrt{n}E\xi_1^2}{\sqrt{\log n}(x+\epsilon)}.\end{aligned}\tag{2.11}$$

So, when $x\in(1/\sqrt{\log n}\Delta_n^{1/4},\infty)$,

$$\frac{|ES'_n|}{\sqrt{n\log n}(x+\epsilon)}\le C\frac{E\xi_1^2}{\log n\left(1/\sqrt{\log n}\Delta_n^{1/4}+\epsilon\right)^2}<\epsilon,\quad \text{for } n \text{ large enough.}\tag{2.12}$$

By (2.12), we have

$$\begin{aligned}&\sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-1}}{n}\Delta_{n2}\\&\quad\le\sum_{n=2}^{d(\epsilon)}\int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty}\frac{(\log n)^{\delta}(x+\epsilon)}{n}\\&\qquad\times\left[P\left(\sup_i|a_{ni}\xi_i|>\sqrt{n\log n}(x+\epsilon)\right)+P\left(\left|S'_n-ES'_n\right|\ge\frac{\sqrt{n\log n}(x+\epsilon)}{2}\right)\right]dx\\&\quad=:H_1+H_2.\end{aligned}\tag{2.13}$$

Set $I_{nj}=\{j\in\mathcal{L},\ 1/(j+1)<|a_{nj}|\le 1/j,\ j=1,2,\ldots\}$, then $\bigcup_{j\ge1}I_{nj}=\mathcal{L}$ (referred by [4]). We can get

$$\sum_{j=1}^{k}\#I_{nj}\le n(k+1).\tag{2.14}$$

Then,

$$\begin{aligned}&P\left\{\sup_i|a_{ni}\xi_i|>\sqrt{n\log n}(x+\epsilon)\right\}\\&\quad\le\sum_{i=-\infty}^{\infty}P\left\{|a_{ni}\xi_i|>\sqrt{n\log n}(x+\epsilon)\right\}\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} P\left\{ |\xi_1| \geq \sqrt{n \log n} j (x+\epsilon) \right\} \leq \sum_{j=1}^{\infty} (\# I_{nj}) P\left\{ |\xi_1| \geq \sqrt{n \log n} j (x+\epsilon) \right\} \\
&\leq \sum_{j=1}^{\infty} \sum_{k \geq j} (\# I_{nj}) P\left\{ \sqrt{n \log n} k (x+\epsilon) \leq |\xi_1| < \sqrt{n \log n} (k+1)(x+\epsilon) \right\} \\
&\leq \sum_{k=1}^{\infty} \sum_{j=1}^{k} (\# I_{nj}) P\left\{ \sqrt{n \log n} k (x+\epsilon) \leq |\xi_1| < \sqrt{n \log n} (k+1)(x+\epsilon) \right\} \\
&\leq \sum_{k=1}^{\infty} n(k+1) P\left\{ \sqrt{n \log n} k (x+\epsilon) \leq |\xi_1| < \sqrt{n \log n} (k+1)(x+\epsilon) \right\} \\
&\leq \frac{\sqrt{n} E|\xi_1| I\left\{ |\xi_1| \geq \sqrt{n \log n} (x+\epsilon) \right\}}{\sqrt{\log n} (x+\epsilon)}.
\end{aligned}
\tag{2.15}
$$

So, we get

$$
\begin{aligned}
H_1 &\leq CE|\xi_1| \int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty} \sum_{n=2}^{d(\epsilon)} \frac{(\log n)^{\delta-1/2}}{n^{1/2}} \\
&\qquad \times \sum_{k=n}^{\infty} I\left\{ \sqrt{k \log k}(x+\epsilon) < |\xi_1| \leq \sqrt{(k+1)\log(k+1)}(x+\epsilon) \right\} dx \\
&\leq C \int_0^{\infty} \sum_{k=2}^{\infty} E|\xi_1| I\left\{ \sqrt{k \log k}(x+\epsilon) < |\xi_1| \leq \sqrt{(k+1)\log(k+1)}(x+\epsilon) \right\} dx \sum_{n=2}^{k} \frac{(\log n)^{\delta-1/2}}{n^{1/2}} \\
&\leq CE\xi_1^2 \int_0^{\infty} (x+\epsilon)^{-1} \sum_{k=2}^{\infty} (\log k)^{\delta-1} I\left\{ \sqrt{k \log k}(x+\epsilon) < |\xi_1| \leq \sqrt{(k+1)\log(k+1)}(x+\epsilon) \right\} dx \\
&\leq CE\xi_1^2 \int_0^{\infty} \left| \log^+ |\xi_1| - \log(x+\epsilon) \right|^{(\delta-1)} (x+\epsilon)^{-1} I\{ |\xi_1| > (x+\epsilon) \} dx \\
&\leq CE\xi_1^2 \left| \log^+ |\xi_1| - \log \epsilon \right|^{\delta} \leq CE\xi_1^2 \left(\log^+ |\xi_1| \right)^{\delta} + CE\xi_1^2 (-\log \epsilon)^{\delta}.
\end{aligned}
\tag{2.16}
$$

Therefore,

$$
\lim_{\epsilon \searrow 0} \epsilon^{2\delta} H_1 = 0. \tag{2.17}
$$

By Lemma 1.6, noting that $\sum_{m=1}^{\infty}\phi^{1/2}(m)<\infty$, for $q>2$,

$$\begin{aligned}
H_2 \le{}& C\sum_{n=2}^{d(\epsilon)}\int_0^{\infty}\frac{(\log n)^{\delta-q/2}}{n^{1+q/2}}(x+\epsilon)^{1-q}\\
&\times\Biggl\{\Biggl(\sum_{i=-\infty}^{\infty}E(a_{ni}\xi_1)^2 I\Bigl\{|a_{ni}\xi_1|\le\sqrt{n\log n}(x+\epsilon)\Bigr\}\Biggr)^{q/2}\\
&\quad+\sum_{i=-\infty}^{\infty}E|a_{ni}\xi_1|^q I\Bigl\{|a_{ni}\xi_1|\le\sqrt{n\log n}\epsilon\Bigr\}\\
&\quad+\sum_{i=-\infty}^{\infty}E|a_{ni}\xi_1|^q I\Bigl\{\sqrt{n\log n}\epsilon<|a_{ni}\xi_1|\le\sqrt{n\log n}(x+\epsilon)\Bigr\}\Biggr\}dx\\
=:{}& H_{21}+H_{22}+H_{23}.
\end{aligned}\tag{2.18}$$

For H_{21}, we have

$$\begin{aligned}
H_{21}&\le\sum_{n=2}^{d(\epsilon)}\int_0^{\infty}\frac{(\log n)^{\delta-q/2}}{n}(x+\epsilon)^{1-q}\Bigl(E\xi_1^2 I\Bigl\{|a_{ni}\xi_1|\le\sqrt{n\log n}(x+\epsilon)\Bigr\}\Bigr)^{q/2}dx\\
&\le C\epsilon^{-2\delta}M^{\delta+1-q/2}.
\end{aligned}\tag{2.19}$$

Then, for $0<\delta\le 1$, $q>2$, we have

$$\lim_{M\to\infty}\limsup_{\epsilon\searrow 0}\epsilon^{2\delta}H_{21}=0.\tag{2.20}$$

For H_{22}, we decompose it into two parts,

$$\begin{aligned}
H_{22}&\le\sum_{n=2}^{d(\epsilon)}\int_0^{\infty}\frac{(\log n)^{\delta-q/2}}{n^{1+q/2}}(x+\epsilon)^{1-q}\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}|a_{ni}|^q E|\xi_1|^q I\Bigl\{|a_{ni}\xi_1|\le\sqrt{n\log n}\epsilon\Bigr\}dx\\
&\le\epsilon^{2-q}\sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-q/2}}{n^{1+q/2}}\\
&\quad\times\sum_{j=1}^{\infty}(\#I_{nj})j^{-q}\Biggl\{\sum_{k=0}^{2n}E|\xi_1|^q I\Bigl\{\sqrt{k\log k}\epsilon\le|\xi_1|<\sqrt{(k+1)\log(k+1)}\epsilon\Bigr\}\\
&\qquad+\sum_{k=2n+1}^{(j+1)n}E|\xi_1|^q I\Bigl\{\sqrt{k\log k}\epsilon\le|\xi_1|<\sqrt{(k+1)\log(k+1)}\epsilon\Bigr\}\Biggr\}\\
&=:H_{221}+H_{222}.
\end{aligned}\tag{2.21}$$

It is easy to see that

$$\sum_{j=m}^{\infty}(\#I_{nj})(j+1)^{-q}(m+1)^{q-1} \le \sum_{j=1}^{\infty}(\#I_{nj})(j+1)^{-1} \le \sum_{i=-\infty}^{\infty}|a_{ni}| = \sum_{j=1}^{\infty}\sum_{i\in I_{nj}}|a_{ni}| \le n. \tag{2.22}$$

So,

$$\sum_{j=m}^{\infty}(\#I_{nj})j^{-q} \le Cnm^{-(q-1)}. \tag{2.23}$$

Now, we estimate H_{221}, by (2.23),

$$\begin{aligned}
H_{221} &\le \epsilon^{2-q}\sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-q/2}}{n^{q/2}}\sum_{k=0}^{2n}E|\xi_1|^q I\left\{\sqrt{k\log k}\epsilon \le |\xi_1| < \sqrt{(k+1)\log(k+1)}\epsilon\right\} \\
&\le \epsilon^{2-q}\sum_{k=2}^{d(\epsilon)}E|\xi_1|^q I\left\{\sqrt{k\log k}\epsilon \le |\xi_1| < \sqrt{(k+1)\log(k+1)}\epsilon\right\}\sum_{n=[k/2]}^{d(\epsilon)}\frac{(\log n)^{\delta-q/2}}{n^{q/2}} \\
&\le \epsilon^{2-q}\sum_{k=2}^{d(\epsilon)}\frac{(\log k)^{\delta-q/2}}{k^{q/2-1}}E|\xi_1|^q I\left\{\sqrt{k\log k}\epsilon \le |\xi_1| < \sqrt{(k+1)\log(k+1)}\epsilon\right\} \\
&\le \sum_{k=2}^{d(\epsilon)}(\log k)^{\delta-1}E\xi_1^2 I\left\{\sqrt{k\log k}\epsilon \le |\xi_1| < \sqrt{(k+1)\log(k+1)}\epsilon\right\} \\
&\le \sum_{k=2}^{d(\epsilon)}(\log k)^{-1}\left|\log^+|\xi_1| - \log\epsilon\right|^{\delta}E\xi_1^2 I\left\{\sqrt{k\log k}\epsilon \le |\xi_1| < \sqrt{(k+1)\log(k+1)}\epsilon\right\} \\
&\le CE\xi_1^2(\log^+|\xi_1|)^{\delta} + CE\xi_1^2(-\log\epsilon)^{\delta}.
\end{aligned} \tag{2.24}$$

For H_{222}, we have

$$\begin{aligned}
H_{222} &\le \epsilon^{2-q}\sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} \\
&\quad\times\sum_{k=2n+1}^{\infty}\sum_{j\ge k/n-1}(\#I_{nj})j^{-q}E|\xi_1|^q I\left\{\sqrt{k\log k}\epsilon \le |\xi_1| < \sqrt{(k+1)\log(k+1)}\epsilon\right\} \\
&\le \epsilon^{2-q}\sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-q/2}}{n^{1-q/2}}\sum_{k=2n+1}^{\infty}k^{1-q}E|\xi_1|^q I\left\{\sqrt{k\log k}\epsilon \le |\xi_1| < \sqrt{(k+1)\log(k+1)}\epsilon\right\} \\
&\le \sum_{k=2}^{d(\epsilon)}(\log k)^{\delta-1}E\xi_1^2 I\left\{\sqrt{k\log k}\epsilon \le |\xi_1| < \sqrt{(k+1)\log(k+1)}\epsilon\right\} \\
&\le \sum_{k=2}^{d(\epsilon)}(\log k)^{-1}\left|\log^+|\xi_1| - \log\epsilon\right|^{\delta}E\xi_1^2 I\left\{\sqrt{k\log k}\epsilon \le |\xi_1| < \sqrt{(k+1)\log(k+1)}\epsilon\right\} \\
&\le CE\xi_1^2(\log^+|\xi_1|)^{\delta} + CE\xi_1^2(-\log\epsilon)^{\delta}.
\end{aligned} \tag{2.25}$$

From (2.24) and (2.25), we can get

$$\lim_{\epsilon\searrow 0}\epsilon^{2\delta}H_{22}=\lim_{\epsilon\searrow 0}\epsilon^{2\delta}H_{221}+\lim_{\epsilon\searrow 0}\epsilon^{2\delta}H_{222}=0. \tag{2.26}$$

Finally, $q>2$, and we will get

$$\begin{aligned}
H_{23} &\le C\sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-q/2}}{n^{1+q/2}}E|\xi_1|^q I\left\{|\tilde{a}\xi_1|>\sqrt{n\log n}\epsilon\right\}\\
&\quad\times\int_0^\infty (x+\epsilon)^{1-q}\sum_{j=1}^\infty(\#I_{nj})j^{-q}\left\{\sum_{k=0}^{2n}+\sum_{k=2n+1}^{(j+1)n}\right\}\\
&\qquad\times I\left\{\sqrt{k\log k}(x+\epsilon)<|\xi_1|\le\sqrt{(k+1)\log(k+1)}(x+\epsilon)\right\}dx\\
&\le C\sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-q/2}}{n^{q/2}}E|\xi_1|^q I\left\{|\tilde{a}\xi_1|>\sqrt{n\log n}\epsilon\right\}\\
&\quad\times\int_0^\infty (x+\epsilon)^{1-q}I\left\{|\xi_1|\le\sqrt{2n\log(2n)}(x+\epsilon)\right\}dx+C\sum_{n=2}^{d(\epsilon)}\frac{(\log n)^{\delta-q/2}}{n^{1-q/2}}E|\xi_1|^q\\
&\quad\times\int_0^\infty\sum_{k=2n+1}^\infty\frac{k^{1-q}}{(x+\epsilon)^{q-1}}I\left\{\sqrt{k\log k}(x+\epsilon)<|\xi_1|\le\sqrt{(k+1)\log(k+1)}(x+\epsilon)\right\}dx\\
&\le C\sum_{k=2}^\infty E\xi_1^2 I\left\{\sqrt{k\log k}\epsilon<|\xi_1|\le\sqrt{(k+1)\log(k+1)}\epsilon\right\}\sum_{n=2}^k\frac{(\log n)^{\delta-1}}{n}\\
&\quad+CE\xi_1^2\int_0^\infty(x+\epsilon)^{-1}\\
&\qquad\times\sum_{k=4}^\infty(\log k)^{\delta-1}I\left\{\sqrt{k\log k}(x+\epsilon)<|\xi_1|\le\sqrt{(k+1)\log(k+1)}(x+\epsilon)\right\}dx\\
&\le C\sum_{k=2}^\infty(\log k)^\delta E\xi_1^2 I\left\{\sqrt{k\log k}\epsilon<|\xi_1|\le\sqrt{(k+1)\log(k+1)}\epsilon\right\}\\
&\quad+CE\xi_1^2\int_0^\infty\left|\log^+|\xi_1|-\log(x+\epsilon)\right|^{(\delta-1)}(x+\epsilon)^{-1}I\{|\xi_1|>(x+\epsilon)\}dx\\
&\le CE\xi_1^2\left|\log^+|\xi_1|-\log\epsilon\right|^\delta\le CE\xi_1^2\left(\log^+|\xi_1|\right)^\delta+CE\xi_1^2(-\log\epsilon)^\delta,
\end{aligned}\tag{2.27}$$

then

$$\lim_{\epsilon\searrow 0}\epsilon^{2\delta}H_{23}=0. \tag{2.28}$$

Hence, (2.4) can be referred from (2.9), (2.17), (2.20), (2.26), and (2.28). □

Proposition 2.2. *One has*

$$\lim_{\epsilon\searrow 0}\epsilon^{2\delta}\sum_{n=d(\epsilon)+1}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\left|\int_{\sqrt{n\log n}\epsilon}^{\infty}2xP(|S_n|\geq x)dx-\int_{\sqrt{n\log n}\epsilon}^{\infty}2xP\left(|N|\geq\frac{x}{\sqrt{n}}\right)dx\right|=0. \tag{2.29}$$

Proof. Consider the following:

$$\begin{aligned}
&\sum_{n=d(\epsilon)+1}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\left|\int_{\sqrt{n\log n}\epsilon}^{\infty}2xP(|S_n|\geq x)dx-\int_{\sqrt{n\log n}\epsilon}^{\infty}2xP\left(|N|\geq\frac{x}{\sqrt{n}}\right)dx\right|\\
&\quad\leq\sum_{n=d(\epsilon)+1}^{\infty}\frac{(\log n)^{\delta}}{n}\int_0^{\infty}2(x+\epsilon)P\left(|N|\geq\sqrt{\log n}(x+\epsilon)\right)\\
&\qquad+\sum_{n=d(\epsilon)+1}^{\infty}\frac{(\log n)^{\delta}}{n}\int_0^{\infty}2(x+\epsilon)P\left(|S_n|\geq\sqrt{n\log n}(x+\epsilon)\right)dx\\
&\quad=: G_1+G_2.
\end{aligned} \tag{2.30}$$

We first estimate G_1, for $\theta>2\delta$, by Markov's inequality,

$$\begin{aligned}
G_1&\leq\sum_{n=d(\epsilon)+1}^{\infty}\frac{(\log n)^{\delta}}{n}\int_0^{\infty}\frac{1}{(\log n)^{(\theta+2)/2}(x+\epsilon)^{\theta+1}}dx\\
&\leq CM^{\delta-\theta/2}\epsilon^{-2\delta}.
\end{aligned} \tag{2.31}$$

Hence,

$$\lim_{M\to\infty}\limsup_{\epsilon\searrow 0}\epsilon^{2\delta}G_1=0. \tag{2.32}$$

Now, we estimate G_2. Here, $n>M\epsilon^{-2}$, so

$$\frac{|ES'_n|}{\sqrt{n\log n}(x+\epsilon)}<\frac{E\xi_1^2}{(\log n)(x+\epsilon)^2}<\frac{E\xi_1^2}{M}<\epsilon,\quad\text{for } M\longrightarrow\infty. \tag{2.33}$$

We have

$$
\begin{aligned}
G_2 &\leq \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta}}{n} \int_0^{\infty} (x+\epsilon) \left[P\left(\sup_i |a_{ni}\xi_i| > \sqrt{n \log n}(x+\epsilon) \right) \right. \\
&\qquad \left. + P\left(\left| S_n' - ES_n' \right| \geq \frac{\sqrt{n \log n}(x+\epsilon)}{2} \right) \right] dx \\
&=: G_{21} + G_{22}.
\end{aligned} \tag{2.34}
$$

We estimate G_{21} first. Similar to the proof of (2.16), we have

$$
\begin{aligned}
G_{21} &\leq CE|\xi_1| \int_0^{\infty} \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-1/2}}{n^{1/2}} \\
&\qquad \times \sum_{k=n}^{\infty} I\left\{ \sqrt{k \log k}(x+\epsilon) < |\xi_1| \leq \sqrt{(k+1)\log(k+1)}(x+\epsilon) \right\} dx \\
&\leq CE|\xi_1| \int_0^{\infty} \sum_{k=d(\epsilon)+1}^{\infty} I\left\{ \sqrt{k \log k}(x+\epsilon) < |\xi_1| \leq \sqrt{(k+1)\log(k+1)}(x+\epsilon) \right\} dx \\
&\quad \times \sum_{n=d(\epsilon)+1}^{k} \frac{(\log n)^{\delta-1/2}}{n^{1/2}} \\
&\leq CE\xi_1^2 \int_0^{\infty} \sum_{k=d(\epsilon)+1}^{\infty} (\log n)^{\delta-1} (x+\epsilon)^{-1} \\
&\qquad \times I\left\{ \sqrt{k \log k}(x+\epsilon) < |\xi_1| \leq \sqrt{(k+1)\log(k+1)}(x+\epsilon) \right\} dx \\
&\leq CE\xi_1^2 \int_0^{\infty} \left| \log|\xi_1| - \log(x+\epsilon) \right|^{(\delta-1)} (x+\epsilon)^{-1} I\{ |\xi_1| > (x+\epsilon) \} dx \\
&\leq CE\xi_1^2 \left| \log^+ |\xi_1| - \log \epsilon \right|^{\delta} \leq CE\xi_1^2 \left(\log^+ |\xi_1| \right)^{\delta} + CE\xi_1^2 (-\log \epsilon)^{\delta},
\end{aligned} \tag{2.35}
$$

then

$$
\lim_{M \to \infty} \limsup_{\epsilon \searrow 0} \epsilon^{2\delta} G_{21} = 0. \tag{2.36}
$$

By Lemma 1.6, for $q > 2$, we have

$$
\begin{aligned}
G_{22} &= \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta}}{n} \int_0^{\infty} (x+\epsilon) P\left\{ \left| S'_n - ES'_n \right| \geq \frac{\sqrt{n \log n}(x+\epsilon)}{2} \right\} dx \\
&\leq \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} \\
&\quad \times \int_0^{\infty} (x+\epsilon)^{1-q} \left\{ \left(\sum_{i=-\infty}^{\infty} E(a_{ni}\xi_1)^2 I\left\{ |a_{ni}\xi_i| \leq \sqrt{n \log n}(x+\epsilon) \right\} \right)^{q/2} \right. \\
&\qquad + \sum_{i=-\infty}^{\infty} E|a_{ni}\xi_1|^q I\left\{ |a_{ni}\xi_i| \leq \sqrt{n \log n}\epsilon \right\} \\
&\qquad \left. + \sum_{i=-\infty}^{\infty} E|a_{ni}\xi_1|^q I\left\{ \sqrt{n \log n}\epsilon < |a_{ni}\xi_i| \leq \sqrt{n \log n}(x+\epsilon) \right\} \right\} dx \\
&=: G_{221} + G_{222} + G_{223}.
\end{aligned}
\tag{2.37}
$$

For G_{221}, we have

$$
\begin{aligned}
G_{221} &\leq \sum_{n=d(\epsilon)+1}^{\infty} \int_0^{\infty} \frac{(\log n)^{\delta-q/2}}{n} (x+\epsilon)^{1-q} \left(E\xi_1^2 I\{|a_{ni}\xi_1| \leq n(x+\epsilon)\} \right)^{q/2} dx \\
&\leq C\epsilon^{2-q} \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n} \leq CM^{\delta+1-q/2}\epsilon^{-2\delta}.
\end{aligned}
\tag{2.38}
$$

Next, turning to G_{222}, it follows that

$$
\begin{aligned}
G_{222} &\leq \epsilon^{2-q} \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} \\
&\quad \times \sum_{j=1}^{\infty} (\#I_{nj}) j^{-q} \left\{ \sum_{k=2}^{2n} E|\xi_1|^q I\left\{ \sqrt{k \log k}\epsilon \leq |\xi_1| < \sqrt{(k+1)\log(k+1)}\epsilon \right\} \right. \\
&\qquad \left. + \sum_{k=2n+1}^{(j+1)n} E|\xi_1|^q I\left\{ \sqrt{k \log k}\epsilon \leq |\xi_1| < \sqrt{(k+1)\log(k+1)}\epsilon \right\} \right\} \\
&=: G_{2221} + G_{2222},
\end{aligned}
\tag{2.39}
$$

then

$$\begin{aligned}
G_{2221} &\le C\epsilon^{2-q}\sum_{n=d(\epsilon)+1}^{\infty}\frac{(\log n)^{\delta-q/2}}{n^{q/2}}\sum_{k=2}^{2n}E|\xi_1|^q I\left\{\sqrt{k\log k}\epsilon\le|\xi_1|<\sqrt{(k+1)\log(k+1)}\epsilon\right\}\\
&\le C\epsilon^{2-q}\sum_{k=2}^{\infty}E|\xi_1|^q I\left\{\sqrt{k\log k}\epsilon\le|\xi_1|<\sqrt{(k+1)\log(k+1)}\epsilon\right\}\sum_{n=[k/2]}^{\infty}\frac{(\log n)^{\delta-q/2}}{n^{q/2}}\\
&\le C\epsilon^{2-q}\sum_{k=2}^{\infty}\frac{(\log k)^{\delta-q/2}}{k^{q/2-1}}E|\xi_1|^q I\left\{\sqrt{k\log k}\epsilon\le|\xi_1|<\sqrt{(k+1)\log(k+1)}\epsilon\right\}\\
&\le C\sum_{k=2}^{\infty}(\log k)^{\delta-1}E\xi_1^2 I\left\{\sqrt{k\log k}\epsilon\le|\xi_1|<\sqrt{(k+1)\log(k+1)}\epsilon\right\}\\
&\le C\sum_{k=2}^{\infty}(\log k)^{-1}\left|\log^+|\xi_1|-\log\epsilon\right|^{\delta}E\xi_1^2 I\left\{\sqrt{k\log k}\epsilon\le|\xi_1|<\sqrt{(k+1)\log(k+1)}\epsilon\right\}\\
&\le CE\xi_1^2(\log^+|\xi_1|)^{\delta}+CE\xi_1^2(-\log\epsilon)^{\delta}.
\end{aligned}\tag{2.40}$$

For G_{2222}, it follows that

$$\begin{aligned}
G_{2222} &\le C\epsilon^{2-q}\sum_{n=d(\epsilon)+1}^{\infty}\frac{(\log n)^{\delta-q/2}}{n^{1+q/2}}\\
&\quad\times\sum_{k=2n+1}^{\infty}\sum_{j\ge k/n-1}(\#I_{nj})j^{-q}E|\xi_1|^q I\left\{\sqrt{k\log k}\epsilon\le|\xi_1|<\sqrt{(k+1)\log(k+1)}\epsilon\right\}\\
&\le C\epsilon^{2-q}\sum_{k=d(\epsilon)+1}^{\infty}k^{1-q}E|\xi_1|^q I\left\{\sqrt{k\log k}\epsilon\le|\xi_1|<\sqrt{(k+1)\log(k+1)}\epsilon\right\}\sum_{n=d(\epsilon)+1}^{[k/2]}\frac{(\log n)^{\delta-q/2}}{n^{1-q/2}}\\
&\le C\sum_{k=d(\epsilon)+1}^{\infty}(\log k)^{\delta-1}E\xi_1^2 I\left\{\sqrt{k\log k}\epsilon\le|\xi_1|<\sqrt{(k+1)\log(k+1)}\epsilon\right\}\\
&\le C\sum_{k=d(\epsilon)+1}^{\infty}(\log k)^{-1}\left|\log^+|\xi_1|-\log\epsilon\right|^{\delta}E\xi_1^2 I\left\{\sqrt{k\log k}\epsilon\le|\xi_1|<\sqrt{(k+1)\log(k+1)}\epsilon\right\}\\
&\le CE\xi_1^2(\log^+|\xi_1|)^{\delta}+CE\xi_1^2(-\log\epsilon)^{\delta}.
\end{aligned}\tag{2.41}$$

Finally, $q > 2$, we have

$$\begin{aligned}
G_{223} &\leq C \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{1+q/2}} E|\xi_1|^q I\left\{|\tilde{a}\xi_1| > \sqrt{n\log n}\epsilon\right\} \\
&\quad \times \int_0^{\infty} (x+\epsilon)^{1-q} \sum_{j=1}^{\infty} (\# I_{nj}) j^{-q} \left\{ \sum_{k=0}^{2n} + \sum_{k=2n+1}^{(j+1)n} \right\} \\
&\qquad \times I\left\{ \sqrt{k\log k}(x+\epsilon) < |\xi_1| \leq \sqrt{(k+1)\log(k+1)}(x+\epsilon) \right\} dx \\
&\leq C \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{q/2}} E|\xi_1|^q I\left\{|\tilde{a}\xi_1| > \sqrt{n\log n}\epsilon\right\} \\
&\quad \times \int_0^{\infty} (x+\epsilon)^{1-q} I\left\{ |\xi_1| \leq \sqrt{2n\log(2n)}(x+\epsilon) \right\} dx + C \sum_{n=d(\epsilon)+1}^{\infty} \frac{(\log n)^{\delta-q/2}}{n^{1-q/2}} E|\xi_1|^q \\
&\quad \times \int_0^{\infty} \sum_{k=2n+1}^{\infty} \frac{k^{1-q}}{(x+\epsilon)^{q-1}} I\left\{ \sqrt{k\log k}(x+\epsilon) < |\xi_1| \leq \sqrt{(k+1)\log(k+1)}(x+\epsilon) \right\} dx \\
&\leq C \sum_{k=d(\epsilon)+1}^{\infty} E\xi_1^2 I\left\{ \sqrt{k\log k}\epsilon < |\xi_1| \leq \sqrt{(k+1)\log(k+1)}\epsilon \right\} \sum_{n=d(\epsilon)+1}^{k} \frac{(\log n)^{\delta-1}}{n} \\
&\quad + CE\xi_1^2 \int_0^{\infty} (x+\epsilon)^{-1} \\
&\qquad \times \sum_{k=d(\epsilon)+1}^{\infty} (\log k)^{\delta-1} I\left\{ \sqrt{k\log k}(x+\epsilon) < |\xi_1| \leq \sqrt{(k+1)\log(k+1)}(x+\epsilon) \right\} dx \\
&\leq C \sum_{k=2}^{\infty} (\log k)^{\delta} E\xi_1^2 I\left\{ \sqrt{k\log k}\epsilon < |\xi_1| \leq \sqrt{(k+1)\log(k+1)}\epsilon \right\} \\
&\quad + CE\xi_1^2 \int_0^{\infty} \left|\log^+|\xi_1| - \log(x+\epsilon)\right|^{(\delta-1)} (x+\epsilon)^{-1} I\{|\xi_1| > (x+\epsilon)\} dx \\
&\leq CE\xi_1^2 \left|\log^+|\xi_1| - \log\epsilon\right|^{\delta} \leq CE\xi_1^2 (\log^+|\xi_1|)^{\delta} + CE\xi_1^2(-\log\epsilon)^{\delta}.
\end{aligned} \tag{2.42}$$

From (2.38) to (2.42), we can get

$$\lim_{M\to\infty} \lim_{\epsilon \searrow 0} \epsilon^{2\delta} G_{22} = 0. \tag{2.43}$$

(2.29) can be derived by (2.32), (2.36), and (2.43). □

Proof of Theorem 1.2. Without loss of generality, we set $\tau = 1$. It is easy to see that

$$\sum_{n=3}^{\infty} \frac{(\log\log n)^{\delta-1}}{n^2 \log n} ES_n^2 I\left\{|S_n| \geq \sqrt{n \log\log n}\epsilon\right\}$$
$$= \epsilon^2 \sum_{n=3}^{\infty} \frac{(\log\log n)^{\delta}}{n \log n} P\left\{|S_n| \geq \sqrt{n \log\log n}\epsilon\right\} \tag{2.44}$$
$$+ \sum_{n=3}^{\infty} \frac{(\log\log n)^{\delta-1}}{n^2 \log n} \int_{\sqrt{n\log\log n}\epsilon}^{\infty} 2xP(|S_n| \geq x)dx.$$

So, we only prove the following two propositions:

$$\epsilon^{2\delta+2} \sum_{n=3}^{\infty} \frac{(\log\log n)^{\delta}}{n \log n} P\left\{|S_n| \geq \sqrt{n \log\log n}\epsilon\right\} = \frac{E|N|^{2(\delta+1)}}{\delta+1}, \tag{2.45}$$

$$\epsilon^{2\delta} \sum_{n=3}^{\infty} \frac{(\log\log n)^{\delta-1}}{n^2 \log n} \int_{\sqrt{n\log\log n}\epsilon}^{\infty} 2xP(|S_n| \geq x)dx = \frac{E|N|^{2(\delta+1)}}{\delta(\delta+1)}. \tag{2.46}$$

The proof of (2.45) can be referred to [6], and the proof of (2.46) is similar to Propositions 2.1 and 2.2. □

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