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Research Article

Jacobi-Sobolev Orthogonal Polynomials: Asymptotics for N-Coherence of Measures

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Let us introduce the Sobolev-type inner product $\langle f,g\rangle = \langle f,g\rangle_1 + \lambda\langle f',g'\rangle_2$, where $\lambda > 0$ and $\langle f,g\rangle_1 = \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}dx$, $\langle f,g\rangle_2 = \int_{-1}^{1} f(x)g(x)((1-x)^{\alpha+1}(1+x)^{\beta+1})/(\prod_{k=1}^{M}|x-\xi_k|^{N_k+1})dx + \sum_{k=1}^{M}\sum_{i=0}^{N_k} M_{k,i}f^{(i)}(\xi_k)g^{(i)}(\xi_k)$, with $\alpha,\beta > -1, |\xi_k| > 1$, and $M_{k,i} > 0$, for all k,i. A Mehler-Heine-type formula and the inner strong asymptotics on $(-1,1)$ as well as some estimates for the polynomials orthogonal with respect to the above Sobolev inner product are obtained. Necessary conditions for the norm convergence of Fourier expansions in terms of such Sobolev orthogonal polynomials are given.

1. Introduction

For a nontrivial probability measure σ, supported on $[-1,1]$, we define the linear space $L^p(d\sigma)$ of all measurable functions f on $[-1,1]$ such that $\|f\|_{L^p(d\sigma)} < \infty$, where

$$\|f\|_{L^p(d\sigma)} = \begin{cases} \left(\int_{-1}^{1} |f(x)|^p d\sigma(x)\right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{-1<x<1} |f(x)|, & \text{if } p = \infty. \end{cases} \tag{1.1}$$

Let us now introduce the Sobolev-type spaces (see, e.g., [1, Chapter 3] in a more general framework)

$$W^{\mathcal{N},p} = \left\{ f : \|f\|^p_{W^{\mathcal{N},p}} = \|f\|^p_{L^p(d\mu_{\alpha,\beta})} + \lambda \|f'\|^p_{L^p(d\nu_{\alpha,\beta})} \right.$$
$$\left. + \sum_{k=1}^{M}\sum_{i=0}^{N_k} M_{k,i} \left| f^{(i+1)}(\xi_k) \right|^p < \infty \right\}, \quad 1 \le p < \infty, \tag{1.2}$$
$$W^{\mathcal{N},\infty} = \left\{ f : \|f\|_{W^{\mathcal{N},\infty}} = \max\left\{ \|f\|_{L^\infty(d\mu_{\alpha,\beta})}, \lambda \|f'\|_{L^\infty(d\nu_{\alpha,\beta})} \right\} < \infty \right\},$$

where $\lambda > 0$ and $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta dx$, $d\nu_{\alpha,\beta}(x) = ((1-x)^{\alpha+1}(1+x)^{\beta+1})/(\prod_{k=1}^{M}|x-\xi_k|^{N_k+1})dx$ with $\alpha, \beta > -1$, $|\xi_k| > 1$, and $M_{k,i} > 0$, for all k, i. We denote by $\mathcal{N}$ the vector of dimension M with components $(N_1, \ldots, N_M)$.

Let f and g in $W^{\mathcal{N},2}$. We can introduce the Sobolev-type inner product

$$\langle f, g \rangle = \langle f, g \rangle_1 + \lambda \langle f', g' \rangle_2, \tag{1.3}$$

where $\lambda > 0$ and

$$\langle f, g \rangle_1 = \int_{-1}^{1} f(x) g(x) (1-x)^\alpha (1+x)^\beta dx, \tag{1.4}$$

$$\langle f, g \rangle_2 = \int_{-1}^{1} f(x) g(x) \frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{\prod_{k=1}^{M}|x-\xi_k|^{N_k+1}} dx + \sum_{k=1}^{M}\sum_{i=0}^{N_k} M_{k,i} f^{(i)}(\xi_k) g^{(i)}(\xi_k), \tag{1.5}$$

where $\lambda > 0$, $\alpha, \beta > -1$, $|\xi_k| > 1$, and $M_{k,i} > 0$, for all k, i. In the sequel, we will assume that $\xi_k < -1$, and, therefore, $|x - \xi_k| = x - \xi_k$ for all $k = 1, 2, \ldots, M$, and $-1 \le x \le 1$.

Using the standard Gram-Schmidt method for the canonical basis $(x^n)_{n\ge0}$ in the linear space of polynomials, we obtain a unique sequence (up to a constant factor) of polynomials $(Q_n^{(\alpha,\beta,\mathcal{N})})_{n\ge0}$ orthogonal with respect to the above inner product. In the sequel, they will called Jacobi-Sobolev orthogonal polynomials.

For $M = 1$ and $N_1 = 0$, the pair of measures $(d\mu_{\alpha,\beta}, d\nu_{\alpha,\beta} + M_{1,0}\delta_\xi)$ is a 0-coherent pair, studied in [2–4] (see also [5] in a more general framework). In [6], the authors established the distribution of the zeros of the polynomials orthogonal with respect to the above Sobolev inner product (1.3) when $M = 1$ and $N_1 = 0$. Some results concerning interlacing and separation properties of their zeros with respect to the zeros of Jacobi polynomials are also obtained assuming we are working in a coherent case. More recently, for a noncoherent pair of measures, when $\alpha = \beta$, $M = 2$, $N_1 = N_2 = 0$, and $\xi_1 = -\xi_2$, the distribution of zeros of the corresponding Sobolev orthogonal polynomials as well as some asymptotic results (more precisely, inner strong asympttics, outer relative asymptotics, and Mehler-Heine formulas) for these sequences of polynomials are deduced in [7–9]. In the Jacobi case, some analog problems have been considered in [10, 11].

The aim of this contribution is to study necessary conditions for $W^{\mathcal{N},p}$-norm convergence of the Fourier expansion in terms of Jacobi-Sobolev orthogonal polynomials. In order to prove it, we need some estimates and strong asymptotics for the polynomials $Q_n^{(\alpha,\beta,\mathcal{N})}(x)$ as well as for their derivatives $Q'^{(\alpha,\beta,\mathcal{N})}_n(x)$. A Mehler-Heine-type formula, inner strong asymptotics, upper bounds in $(-1, 1)$, and $W^{\mathcal{N},p}$ norms of Jacobi-Sobolev orthonormal polynomials are obtained. Thus, we extend the results of [10] for generalized N-coherent pairs of measures.

The structure of the manuscript is as follows. In Section 2, we give some basic properties of Jacobi polynomials that we will use in the sequel. In Section 3, an algebraic relation between the sequences of polynomials $(Q_n^{(\alpha,\beta,\mathcal{N})})_{n\geq 0}$ and Jacobi orthonormal polynomials is stated. It involves $N+1$ (where $N = \sum_{k=1}^{M}(N_k+1)$) consecutive terms of such sequences in such a way that we obtain a generalization of the relations satisfied in the coherent case. Upper bounds for the polynomials $Q_n^{(\alpha,\beta,\mathcal{N})}(x)$ and their derivatives in $[-1,1]$ are deduced. The inner strong asymptotics as well as a Mehler-Heine-type formula are obtained. Finally, the asymptotic behavior of these polynomials with respect to the $W^{\mathcal{N},p}$ norm is studied. In Section 4, necessary conditions for the convergence of the Fourier expansions in terms of the sequence of Jacobi-Sobolev orthogonal polynomials are presented.

Throughout this paper, positive constants are denoted by $c, c_1, \ldots$ and they may vary at every occurrence. The notation $u_n \cong v_n$ means that the sequence u_n/v_n converges to 1 and notation $u_n \sim v_n$ means $c_1 u_n \leq v_n \leq c_2 u_n$ for sufficiently large n.

2. Preliminaries

For $\alpha,\ \beta > -1$, we denote by $(p_n^{(\alpha,\beta)})_{n\geq 0}$ the sequence of Jacobi polynomials which are orthonormal on $[-1,1]$ with respect to the inner product

$$\langle f, g\rangle_1 = \int_{-1}^{1} f g\, d\mu_{\alpha,\beta}. \tag{2.1}$$

We will denote by $k(\pi_n)$ the leading coefficient of any polynomial $\pi_n(x)$, and $\hat{\pi}_n(x) = (k(\pi_n))^{-1}\pi_n(x)$. Now, we list some properties of the Jacobi orthonormal polynomials which we will use in the sequel.

Proposition 2.1. *(a) The leading coefficient of $p_n^{(\alpha,\beta)}$ is (see [12, formulas (4.3.4) and (4.21.6)])*

$$k\left(p_n^{\alpha,\beta}\right) = \frac{1}{2^n}\binom{2n+\alpha+\beta}{n} \times \left(\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}\right)^{-1/2}. \tag{2.2}$$

(b) *The derivatives of Jacobi polynomials satisfy (see [12, formula (4.21.7)])*

$$\frac{d}{dx}p_n^{(\alpha,\beta)}(x) = \sqrt{n(n+\alpha+\beta+1)}\,p_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{2.3}$$

(c) *For $\alpha,\beta \geq -1/2$, and $q = \max\{\alpha,\beta\}$*

$$\max_{-1\leq x\leq 1}\left|p_n^{(\alpha,\beta)}(x)\right| = \left|p_n^{(\alpha,\beta)}(a)\right| \cong c n^{(q+1)/2}, \tag{2.4}$$

where $a = 1$ if $q = \alpha$ and $a = -1$ if $q = \beta$ (see [12, Theorem 7.32.1]).

(d) *For the polynomials* $p_n^{(\alpha,\beta)}$*, we get the following estimate (see [12, formula (7.32.6)], [13, Theorem 1]):*

$$\left|p_n^{(\alpha,\beta)}(x)\right| \leq c(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}, \tag{2.5}$$

where $x \in (-1,1)$ *and* $\alpha, \beta \geq -1/2$.

(e) *Mehler-Heine formula (see [12, Theorem 8.1.1])*

$$\lim_{n\to\infty} n^{-\alpha-1/2} p_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = 2^{(\alpha-\beta)/2} z^{-\alpha} J_\alpha(z), \tag{2.6}$$

where α, β *are real numbers and* $J_\alpha(z)$ *is the Bessel function of the first kind. This formula holds locally uniformly, that is, on every compact subset of the complex plane.*

(f) *Inner strong asymptotics. For* $p_n^{(\alpha,\beta)}(x)$*, when* $x \in [-1+\epsilon, 1-\epsilon]$ *and* $0 < \epsilon < 1$*, we get (see [12, Theorem 8.21.8])*

$$p_n^{(\alpha,\beta)}(x) = r_n^{\alpha,\beta}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}\cos(k\theta+\gamma) + O\left(n^{-1}\right), \tag{2.7}$$

where $x = \cos\theta$*,* $k = n + (\alpha+\beta+1)/2$*,* $\gamma = -(\alpha+(1/2))\pi/2$*, and* $r_n^{\alpha,\beta} \cong (2/\pi)^{1/2}$.

(g) *For* α*,* $\beta > -1$*,* $\tau = \max\{\alpha, \beta\}$*, and* $1 \leq p \leq \infty$ *(see [12, p.391. Exercise 91], [14, (2.2)], [15, Theorem 2]),*

$$\|p_n^{(\alpha,\beta)}\|_{L^p(d\mu_{\alpha,\beta})} \sim \begin{cases} c, & \text{if } 2\tau > p\tau - 2 + p/2, \\ (\log n)^{1/p}, & \text{if } 2\tau = p\tau - 2 + p/2, \\ n^{\tau+1/2-(2\tau+2)/p}, & \text{if } 2\tau < p\tau - 2 + p/2. \end{cases} \tag{2.8}$$

Let $\{s_n(x)\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to the inner product (1.5), and let

$$\Pi_N(x) = 1 + \sum_{k=1}^{N} b_k T_k(x) \tag{2.9}$$

be the N-th polynomial orthonormal with respect to $dx/\pi\sqrt{1-x^2}\prod_{k=1}^{M}(x-\xi_k)^{N_k+1}$, where $N = \sum_{k=1}^{M}(N_k+1)$ and $T_k(x) = \cos k\theta$, $x = \cos\theta$, are the Tchebychev polynomials of the first kind.

Proposition 2.2 ([16, Lemma 2.1]). *For* $n \geq N$*, there exist constants* $A_{n,i}$ *such that*

$$s_n(x) = \sum_{i=0}^{N} A_{n,i} p_{n-i}^{(\alpha+1,\beta+1)}(x) \tag{2.10}$$

and $\lim_{n\to\infty} A_{n,i} = A_i$, *where*

$$A_0 = \frac{1}{\sqrt{2^N b_N}}, \quad A_i = \frac{b_i}{\sqrt{2^N b_N}}, \quad 1 \le i \le N. \tag{2.11}$$

Next, we will consider the polynomials

$$v_{n+1}(x) = \sum_{i=0}^{N} a_{n,i} p_{n-i+1}^{(\alpha,\beta)}(x), \tag{2.12}$$

where $a_{n,i} = A_{n,i}\sqrt{(n+1)(n+\alpha+\beta+2)/(n-i+1)(n-i+\alpha+\beta+2)}$, $0 \le i \le N$. Notice that

$$\sum_{i=0}^{N} a_{n,i} \cong \frac{\Pi_N(1)}{\sqrt{2^N b_N}}. \tag{2.13}$$

Taking into account that the zeros of the polynomial $\Pi_N(x)$ orthogonal with respect to $dx/\pi\sqrt{1-x^2}\prod_{k=1}^{M}(x-\xi_k)^{N_k+1}$ on the interval $[-1,1]$ are real, simple, and located in $(-1,1)$, we have $\Pi_N(1) \ne 0 \ne \Pi_N(-1)$. Therefore, $\sum_{i=0}^{N} a_{n,i} \ne 0$ for n large enough.

On the other hand, using (b) in Proposition 2.1, we have

$$v'_{n+1}(x) = \sqrt{(n+1)(n+\alpha+\beta+2)}\, s_n(x). \tag{2.14}$$

From Proposition 2.1 and (2.12), we get the following.

Proposition 2.3. (a) *For* α, $\beta \ge -1/2$, *and* $q = \max\{\alpha, \beta\}$,

$$\max_{-1\le x\le 1} |v_n(x)| = |v_n(a)| \cong c n^{(q+1)/2}, \tag{2.15}$$

where $a = 1$ *if* $q = \alpha$ *and* $a = -1$ *if* $q = \beta$.

(b) *When* $x \in (-1,1)$ *and* α, $\beta \ge -1/2$, *we get the following estimate for the polynomials* v_n:

$$|v_n(x)| \le c(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}. \tag{2.16}$$

(c) *Mehler-Heine type formula. We get*

$$\lim_{n\to\infty} n^{-\alpha-1/2} v_n\left(\cos\frac{z}{n}\right) = 2^{(\alpha-\beta)/2}\frac{\Pi_N(1)}{\sqrt{2^N b_N}} z^{-\alpha} J_\alpha(z), \tag{2.17}$$

where α, β *are real numbers, and* $J_\alpha(z)$ *is the Bessel function of the first kind. This formula holds locally uniformly, that is, on every compact subset of the complex plane.*

(d) *Inner strong asymptotics. When* $x \in [-1+\epsilon, 1-\epsilon]$ *and* $0 < \epsilon < 1$, *we get*

$$v_n(x) = (1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}\sum_{i=0}^{N} a_{n-1,i} r_{n-i}^{\alpha,\beta} \cos\left((k-i)\theta + \gamma\right) + O\left(n^{-1}\right), \tag{2.18}$$

where $x = \cos\theta$, $k = n + (\alpha+\beta+1)/2$, $\gamma = -(\alpha + (1/2))\pi/2$, *and* $r_n^{\alpha,\beta} \cong (2/\pi)^{1/2}$.

(e) *For* $\alpha,\ \beta > -1$, $1 \le p \le \infty$, *and* $1 \le j \le n$,

$$\|v_j\|_{L^p(d\mu_{\alpha,\beta})} \le c\|p_j^{(\alpha,\beta)}\|_{L^p(d\mu_{\alpha,\beta})} \le c\|p_n^{(\alpha,\beta)}\|_{L^p(d\mu_{\alpha,\beta})}. \tag{2.19}$$

3. Asymptotics of Jacobi-Sobolev Orthogonal Polynomials

Let $\{Q_n^{(\alpha,\beta,\mathcal{N})}(x)\}_{n=0}^{\infty}$ denote the sequence of polynomials orthogonal with respect to (1.3) normalized by the condition that they have the same leading coefficient as $v_n(x)$, that is, $k(Q_n^{(\alpha,\beta,\mathcal{N})}) = a_{n-1,0}k(p_n^{(\alpha,\beta)})$.

The following relation between $Q_n^{(\alpha,\beta,\mathcal{N})}$ and $v_n(x)$ holds.

Proposition 3.1. *For* $\alpha, \beta > -1$,

$$v_n(x) = Q_n^{(\alpha,\beta,\mathcal{N})}(x) + \sum_{k=1}^{N} \alpha_{n-k}^{(n)} Q_{n-k}^{(\alpha,\beta,\mathcal{N})}(x), \quad n \ge 1, \tag{3.1}$$

where, for $1 \le k \le N$,

$$\alpha_{n-k}^{(n)} = \frac{\sum_{i=k}^{N} a_{n-1,i}\left\langle p_{n-i}^{(\alpha,\beta)}, Q_{n-k}^{(\alpha,\beta,\mathcal{N})}\right\rangle_1}{\|Q_{n-k}^{(\alpha,\beta,\mathcal{N})}\|_{W^{\mathcal{N},2}}^2}. \tag{3.2}$$

Moreover, $|\alpha_{n-N}^{(n)}| = O(1/n^2)$ *and* $|\alpha_{n-k}^{(n)}| = O(1/n)$ *for* $1 \le k < N$.

Proof. Expanding $v_n(x)$ with respect to the basis $\{Q_k^{(\alpha,\beta,\mathcal{N})}\}_{k=0}^{n}$ of the linear space of polynomials with degree at most n, we get

$$v_n(x) = Q_n^{(\alpha,\beta,\mathcal{N})}(x) + \sum_{k=1}^{n} \alpha_{n-k}^{(n)} Q_{n-k}^{(\alpha,\beta,\mathcal{N})}(x), \tag{3.3}$$

where, for $k = 1, \ldots, n$,

$$\alpha_{n-k}^{(n)} = \frac{\left\langle v_n, Q_{n-k}^{(\alpha,\beta,\mathcal{N})}\right\rangle}{\left\langle Q_{n-k}^{(\alpha,\beta,\mathcal{N})}, Q_{n-k}^{(\alpha,\beta,\mathcal{N})}\right\rangle}. \tag{3.4}$$

For $k = 1, \dots, n$,

$$\left\langle v_n, Q_{n-k}^{(\alpha,\beta,\mathcal{N})} \right\rangle = \left\langle v_n, Q_{n-k}^{(\alpha,\beta,\mathcal{N})} \right\rangle_1 + \lambda\sqrt{n(n+\alpha+\beta+1)}\left\langle s_{n-1}, \left(Q_{n-k}^{(\alpha,\beta,\mathcal{N})}\right)' \right\rangle_2$$
$$= \int_{-1}^{1} v_n(x) Q_{n-k}^{(\alpha,\beta,\mathcal{N})}(x)\, d\mu_{\alpha,\beta}(x) \tag{3.5}$$
$$= \sum_{i=0}^{N} a_{n-1,i} \int_{-1}^{1} p_{n-i}^{(\alpha,\beta)}(x) Q_{n-k}^{(\alpha,\beta,\mathcal{N})}(x)\, d\mu_{\alpha,\beta}(x).$$

Therefore,

$$\left\langle v_n, Q_{n-k}^{(\alpha,\beta,\mathcal{N})} \right\rangle = 0, \quad k > N,$$
$$\left\langle v_n, Q_{n-k}^{(\alpha,\beta,\mathcal{N})} \right\rangle = \sum_{i=k}^{N} a_{n-1,i} \int_{-1}^{1} p_{n-i}^{(\alpha,\beta)}(x) Q_{n-k}^{(\alpha,\beta,\mathcal{N})}(x)\, d\mu_{\alpha,\beta}(x), \quad 1 \le k \le N. \tag{3.6}$$

As a conclusion,

$$\alpha_{n-k}^{(n)} = 0, \quad k > N, \tag{3.7}$$

$$\alpha_{n-k}^{(n)} = \frac{\sum_{i=k}^{N} a_{n-1,i} \left\langle p_{n-i}^{(\alpha,\beta)}, Q_{n-k}^{(\alpha,\beta,\mathcal{N})} \right\rangle_1}{\| Q_{n-k}^{(\alpha,\beta,\mathcal{N})} \|_{W^{\mathcal{N},2}}^2}, \quad 1 \le k \le N. \tag{3.8}$$

Using the extremal property for monic orthogonal polynomials with respect to the corresponding norm (see [12, Theorem 3.1.2]),

$$\| \widehat{p}_n^{(\alpha,\beta)} \|_{L^2(d\mu_{\alpha,\beta})}^2 = \inf\left\{ \|p\|_{L^2(d\mu_{\alpha,\beta})}^2 : \deg p = n, p \text{ monic} \right\}, \tag{3.9}$$

we get

$$\| \widehat{Q}_n^{(\alpha,\beta,\mathcal{N})} \|_{W^{\mathcal{N},2}}^2 \ge \frac{\lambda}{\prod_{k=1}^{M} (1-\xi_k)^{N_k+1}} \| \widehat{Q}'^{(\alpha,\beta,\mathcal{N})}_n \|_{L^2(d\mu_{\alpha+1,\beta+1})}^2$$
$$\ge \frac{\lambda n^2}{\prod_{k=1}^{M} (1-\xi_k)^{N_k+1}} \| \widehat{p}_{n-1}^{(\alpha+1,\beta+1)} \|_{L^2(d\mu_{\alpha+1,\beta+1})}^2. \tag{3.10}$$

Thus,

$$\|Q_n^{(\alpha,\beta,\mathcal{N})}\|_{W^{\mathcal{N},2}}^2 \geq cn^2\left(k\left(Q_n^{(\alpha,\beta,\mathcal{N})}\right)\right)^2\left(k\left(p_{n-1}^{(\alpha+1,\beta+1)}\right)\right)^{-2} \geq cn^2. \tag{3.11}$$

Finally, from (3.8), we find that

$$\left|\alpha_{n-N}^{(n)}\right| = \frac{|a_{n-1,N}\,a_{n-N-1,0}|}{\|Q_{n-N}^{(\alpha,\beta,\mathcal{N})}\|_{W^{\mathcal{N},2}}^2} = O\left(\frac{1}{n^2}\right), \tag{3.12}$$

and from Schwarz inequality,

$$\left|\left\langle p_{n-i}^{(\alpha,\beta)}, Q_{n-k}^{(\alpha,\beta,\mathcal{N})}\right\rangle_1\right| \leq \sqrt{\left\langle Q_{n-k}^{(\alpha,\beta,\mathcal{N})}, Q_{n-k}^{(\alpha,\beta,\mathcal{N})}\right\rangle_1} \leq \|Q_{n-k}^{(\alpha,\beta,\mathcal{N})}\|_{W^{\mathcal{N},2}}. \tag{3.13}$$

Thus,

$$\left|\alpha_{n-k}^{(n)}\right| = O\left(\frac{1}{n}\right), \quad 1 \leq k < N. \tag{3.14}$$

□

Using (3.1) in a recursive way, we get the representation of the polynomial $Q_n^{(\alpha,\beta,\mathcal{N})}$ in terms of the elements of the sequence $\{v_n(x)\}_{n=0}^{\infty}$. More precisely we get the following.

Proposition 3.2. *For α, $\beta > -1$, it holds that*

$$Q_n^{(\alpha,\beta,\mathcal{N})}(x) = \sum_{m=0}^{n} b_{1,n}^{(m)} v_{n-m}(x), \tag{3.15}$$

where $b_{1,n}^{(0)} = 1$, $b_{k,n}^{(1)} = -\alpha_{n-k}^{(n)}$, and $b_{k,n}^{(m)} = -b_{1,n}^{(m-1)}\alpha_{n-k-m+1}^{(n-m+1)} + b_{k+1,n}^{(m-1)}$, $k = 1, 2, \ldots, N$, $m = 2, 3, \ldots, n$. Moreover, $|b_{k,n}^{(m)}| = O(1/n)$ for $m = 1, 2, \ldots, N-1$, and $|b_{k,n}^{(m)}| = O(1/n^2)$ for $m = N$, $N+1, \ldots, n$.

Proof. Let denote by $b_{1,n}^{(0)} = 1$, $b_{k,n}^{(1)} = -\alpha_{n-k}^{(n)}$, and $b_{k,n}^{(m)} = -b_{1,n}^{(m-1)}\alpha_{n-k-m+1}^{(n-m+1)} + b_{k+1,n}^{(m-1)}$, $k = 1, 2, \ldots, N$, $m = 2, 3, \ldots, n$. First, we prove that

$$Q_n^{(\alpha,\beta,\mathcal{N})}(x) = \sum_{m=0}^{l} b_{1,n}^{(m)} v_{n-m}(x) + \sum_{k=1}^{N} b_{k,n}^{(l+1)} Q_{n-k-l}^{(\alpha,\beta,\mathcal{N})}(x), \tag{3.16}$$

where $l = 0, 1, \ldots$, and, by convention, $Q_{-s}^{(\alpha,\beta,\mathcal{N})}(x) = 0$, $s = 1, 2, \ldots$.

We will prove (3.16) by induction. When $l = 0$, it is a trivial result. On the other hand, applying (3.1) in a recursive way, we get

$$\begin{aligned} Q_n^{(\alpha,\beta,\mathcal{N})}(x) &= v_n(x) - \sum_{k=1}^{N} \alpha_{n-k}^{(n)} Q_{n-k}^{(\alpha,\beta,\mathcal{N})}(x) \\ &= v_n(x) - \alpha_{n-1}^{(n)} \left[v_{n-1}(x) - \sum_{k=1}^{N} \alpha_{n-k-1}^{(n-1)} Q_{n-k-1}^{(\alpha,\beta,\mathcal{N})}(x) \right] - \sum_{k=2}^{N} \alpha_{n-k}^{(n)} Q_{n-k}^{(\alpha,\beta,\mathcal{N})}(x) \\ &= \sum_{m=0}^{1} b_{1,n}^{(m)} v_{n-m}(x) - b_{1,n}^{(1)} \sum_{k=1}^{N} \alpha_{n-k-1}^{(n-1)} Q_{n-k-1}^{(\alpha,\beta,\mathcal{N})}(x) + \sum_{k=1}^{N-1} b_{k+1,n}^{(1)} Q_{n-k-1}^{(\alpha,\beta,\mathcal{N})}(x) \\ &= \sum_{m=0}^{1} b_{1,n}^{(m)} v_{n-m}(x) + \sum_{k=1}^{N} b_{k,n}^{(2)} Q_{n-k-1}^{(\alpha,\beta,\mathcal{N})}(x) - b_{N+1,n}^{(1)} Q_{n-N-1}^{(\alpha,\beta,\mathcal{N})}(x). \end{aligned} \tag{3.17}$$

Taking into account (3.7), we have $b_{N+1,n}^{(1)} = 0$. Thus, (3.16) follows for $l = 1$. Now, we assume (3.16) holds for $l \geq 1$. Again, from (3.1),

$$\begin{aligned} \sum_{k=1}^{N} b_{k,n}^{(l+1)} Q_{n-k-l}^{(\alpha,\beta,\mathcal{N})}(x) &= b_{1,n}^{(l+1)} Q_{n-l-1}^{(\alpha,\beta,\mathcal{N})}(x) + \sum_{k=2}^{N} b_{k,n}^{(l+1)} Q_{n-k-l}^{(\alpha,\beta,\mathcal{N})}(x) \\ &= b_{1,n}^{(l+1)} \left[v_{n-l-1}(x) - \sum_{k=1}^{N} \alpha_{n-k-l-1}^{(n-l-1)} Q_{n-k-l-1}^{(\alpha,\beta,\mathcal{N})}(x) \right] + \sum_{k=1}^{N-1} b_{k+1,n}^{(l+1)} Q_{n-k-l-1}^{(\alpha,\beta,\mathcal{N})}(x) \\ &= b_{1,n}^{(l+1)} v_{n-l-1}(x) + \sum_{k=1}^{N} b_{k,n}^{(l+2)} Q_{n-k-l-1}^{(\alpha,\beta,\mathcal{N})}(x) - b_{N+1,n}^{(l+1)} Q_{n-N-l-1}^{(\alpha,\beta,\mathcal{N})}(x). \end{aligned} \tag{3.18}$$

Now, we prove that $b_{k,n}^{(l+1)} = 0$ for $k > N$. For $l = 0$, this follows from (3.7). Since $b_{k,n}^{(l+1)} = -b_{1,n}^{(l)} \alpha_{n-k-l}^{(n-l)} + b_{k+1,n}^{(l)}$ and $\alpha_{n-k-l}^{(n-l)} = 0$, for $k > N$ the statement follows by induction. Thus, (3.16) holds for $l + 1$. Now taking $l = n$ in (3.16), we get (3.15).

Finally, we prove that $|b_{k,n}^{(l)}| = O(1/n)$ for $l = 1, 2, \ldots, N-1$, and $|b_{k,n}^{(l)}| = O(1/n^2)$ for $l = N, N+1, \ldots, n$. First, the following inequality holds:

$$\left| b_{k,n}^{(l)} \right| = \begin{cases} O\left(\dfrac{1}{n}\right), & \text{if } 1 \leq k \leq N - 1, \\ O\left(\dfrac{1}{n(n-l+1)}\right), & \text{if } N - 1 + 1 \leq k \leq \text{N}, \\ 0, & \text{if } k > N, \end{cases} \tag{3.19}$$

$l = 1, 2, \ldots, N$. Indeed, for $l = 1$, (3.19) follows from Proposition 3.1 and (3.7). Now, we assume that the relation (3.19) holds for $l \geq 1$. Thus, for $1 \leq k \leq N - l - 1$,

$$\left|b_{1,n}^{(l)}\right| = O\left(\frac{1}{n}\right), \qquad \left|b_{k+1,n}^{(l)}\right| = O\left(\frac{1}{n}\right), \qquad \left|\alpha_{n-l-k}^{(n-l)}\right| = O\left(\frac{1}{n-l}\right), \tag{3.20}$$

for $N - l \leq k \leq N - 1$

$$\left|b_{1,n}^{(l)}\right| = O\left(\frac{1}{n}\right), \qquad \left|b_{k+1,n}^{(l)}\right| = O\left(\frac{1}{n(n-l+1)}\right), \qquad \left|\alpha_{n-l-k}^{(n-l)}\right| = O\left(\frac{1}{n-l}\right), \tag{3.21}$$

for $k = N$

$$\left|b_{1,n}^{(l)}\right| = O\left(\frac{1}{n}\right), \qquad \left|b_{k+1,n}^{(l)}\right| = 0, \qquad \left|\alpha_{n-l-k}^{(n-l)}\right| = O\left(\frac{1}{(n-l)^2}\right), \tag{3.22}$$

and for $k > N$

$$\left|b_{k+1,n}^{(l)}\right| = 0, \qquad \left|\alpha_{n-l-k}^{(n-l)}\right| = 0. \tag{3.23}$$

Therefore, from

$$b_{k,n}^{(l+1)} = -b_{1,n}^{(l)} \alpha_{n-k-l}^{(n-l)} + b_{k+1,n}^{(l)}, \tag{3.24}$$

the relation (3.19) holds for $l + 1$. As consequence, $|b_{k,n}^{(l)}| = O(1/n)$ for $l = 1, 2, \ldots, N - 1$ and $1 \leq k \leq N$.

Now, we will prove by induction that $|b_{k,n}^{(l)}| = O(1/n^2)$ for $l \geq N$ and $1 \leq k \leq N$.

The case $l = N$ follows from (3.19). We assume that $|b_{k,n}^{(l)}| = O(1/n^2)$ for $l \geq N$ and $1 \leq k \leq N$. For $1 \leq k \leq N - 1$,

$$\left|b_{1,n}^{(l)}\right| = O\left(\frac{1}{n^2}\right), \qquad \left|b_{k+1,n}^{(l)}\right| = O\left(\frac{1}{n^2}\right), \qquad \left|\alpha_{n-l-k}^{(n-l)}\right| = O\left(\frac{1}{n-l}\right), \tag{3.25}$$

and for $k = N$

$$\left|b_{1,n}^{(l)}\right| = O\left(\frac{1}{n^2}\right), \qquad \left|b_{k+1,n}^{(l)}\right| = 0, \qquad \left|\alpha_{n-l-k}^{(n-l)}\right| = O\left(\frac{1}{n-l}\right). \tag{3.26}$$

Therefore, from

$$b_{k,n}^{(l+1)} = -b_{1,n}^{(l)} \alpha_{n-k-l}^{(n-l)} + b_{k+1,n}^{(l)}, \tag{3.27}$$

the statement holds for $l + 1$. □

Next, we will give some properties of the Jacobi-Sobolev orthogonal polynomials.

Proposition 3.3. (a) *For the polynomials* $Q_n^{(\alpha,\beta,\mathcal{N})}$*, we get*

$$\left|Q_n^{(\alpha,\beta,\mathcal{N})}(x)\right| \leq c(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}, \tag{3.28}$$

where $x \in (-1,1)$*, and* $\alpha, \beta \geq -1/2$*.*

(b) For the polynomials $Q'^{(\alpha,\beta,\mathcal{N})}_n$*, we get*

$$\left|Q'^{(\alpha,\beta,\mathcal{N})}_n(x)\right| \leq cn(1-x)^{-\alpha/2-3/4}(1+x)^{-\beta/2-3/4}, \tag{3.29}$$

where $x \in (-1,1)$*, and* $\alpha, \beta > -1$*.*

Proof. (a) Using Proposition 3.2, we have

$$\left|Q_n^{(\alpha,\beta,\mathcal{N})}(x)\right| \leq |v_n(x)| + O\left(\frac{1}{n}\right)\sum_{m=1}^{N-1}|v_{n-m}(x)| + O\left(\frac{1}{n^2}\right)\sum_{m=N}^{n}|v_{n-m}(x)|. \tag{3.30}$$

Therefore, from Proposition 2.3(b), the statement follows immediately.

On the other hand, taking into account Proposition 2.1(d), Proposition 2.2, (2.14), and (3.15), the proof of (b) can be done in a similar way. □

Now, we show that, like for the classical Jacobi polynomials, the polynomial $Q_n^{(\alpha,\beta,\mathcal{N})}(x)$ attains its maximum in $[-1,1]$ at the end-points. More precisely,

Proposition 3.4. (a) *For* α, $\beta \geq -1/2$*, and* $q = \max\{\alpha,\beta\}$

$$\max_{-1\leq x\leq 1}\left|Q_n^{(\alpha,\beta,\mathcal{N})}(x)\right| = \left|Q_n^{(\alpha,\beta,\mathcal{N})}(a)\right| \sim n^{(q+1)/2}, \tag{3.31}$$

where $a = 1$ *if* $q = \alpha$ *and* $a = -1$ *if* $q = \beta$*.*

(b) For α, $\beta > -1$ *and* $q = \max\{\alpha,\beta\}$

$$\max_{-1\leq x\leq 1}\left|Q'^{(\alpha,\beta,\mathcal{N})}_n(x)\right| = \left|Q'^{(\alpha,\beta,\mathcal{N})}_n(b)\right| \sim n^{(q+5)/2}, \tag{3.32}$$

where $b = 1$ *if* $q = \alpha$ *and* $b = -1$ *if* $q = \beta$*.*

Proof. Here, we will prove only the case when $\alpha \geq \beta$. The case when $\beta \geq \alpha$ can be done in a similar way.

(a) From Proposition 2.3(a),

$$|v_{n-m}(x)| \leq cn^{(\alpha+1)/2}, \quad m = 0,1,\ldots,n, \tag{3.33}$$

for $x \in [-1,1]$ and $\alpha \geq \beta \geq -1/2$. Therefore, according to (3.30),

$$\left|Q_n^{(\alpha,\beta,\mathcal{N}\mathcal{N})}(x)\right| \leq cn^{(\alpha+1)/2}, \tag{3.34}$$

for $x \in [-1,1]$ and $\alpha \geq \beta \geq -1/2$. From Proposition 3.1, we get

$$\left|Q_n^{(\alpha,\beta,\mathcal{N})}(x)\right| = |v_n(x)| - O\left(n^{(\alpha-1)/2}\right). \tag{3.35}$$

Finally, from Proposition 2.3(a), the statement follows.

(b) Taking into account Proposition 2.1(c), Proposition 2.2, (2.14), (3.1), and (3.15), we can conclude the proof in the same way as we did in (a). □

Corollary 3.5. *For α, $\beta \geq -1/2$,*

$$\left|Q_n^{(\alpha,\beta,\mathcal{N})}(\cos\theta)\right| \leq cA(n,\alpha,\beta,\theta), \tag{3.36}$$

and for α, $\beta > -1$,

$$\left|Q'^{(\alpha,\beta,\mathcal{N})}_n(\cos\theta)\right| \leq cnA(n,\alpha+1,\beta+1,\theta), \tag{3.37}$$

where

$$A(n,\alpha,\beta,\theta) = \begin{cases} \left(\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-1/2}\right), & \text{if } \dfrac{c}{n} \leq \theta \leq \dfrac{\pi-c}{n}, \\ n^{(\alpha+1)/2}, & \text{if } 0 \leq \theta \leq \dfrac{c}{n}, \\ n^{(\beta+1)/2}, & \text{if } \dfrac{\pi-c}{n} \leq \theta \leq \pi. \end{cases} \tag{3.38}$$

Proof. The inequality

$$n^{(\alpha+1)/2} \leq c\theta^{-\alpha-1/2} \tag{3.39}$$

holds for $\theta \in (0, c/n]$, as well as

$$n^{(\beta+1)/2} \leq c(\pi-\theta)^{-\beta-1/2}, \tag{3.40}$$

for $\theta \in [\pi - c/n, \pi)$. Therefore, from Propositions 3.3 and 3.4, the statement follows immediately. □

Next, we deduce a Mehler-Heine-type formula for $Q_n^{(\alpha,\beta,\mathcal{N})}$ and $Q'^{(\alpha,\beta,\mathcal{N})}_n$ (see Theorem 4.1 in [10]).

Proposition 3.6. *Uniformly on compact subsets of* $\mathbf{C}$,

(a)

$$\lim_{n\to\infty} n^{-\alpha-1/2} Q_n^{(\alpha,\beta,\mathcal{N})}\left(\cos\frac{z}{n}\right) = 2^{(\alpha-\beta)/2}\frac{\Pi_N(1)}{\sqrt{2^N b_N}} z^{-\alpha} J_\alpha(z), \tag{3.41}$$

(b)

$$\lim_{n\to\infty} n^{-\alpha-5/2} Q_n'^{(\alpha,\beta,\mathcal{N})}\left(\cos\frac{z}{n}\right) = 2^{(\alpha-\beta)/2}\frac{\Pi_N(1)}{\sqrt{2^N b_N}} z^{-\alpha-1} J_{\alpha+1}(z), \tag{3.42}$$

where α, β are real numbers, and $J_\alpha(z)$ is the Bessel function of the first kind.

Proof. To prove the proposition, we use the same technique as in [17].

(a) Multiplying in (3.1) by $(n+1)^{-\alpha-1/2}$, we obtain

$$V_n(z) = Y_n(z) + \sum_{k=1}^{N} A_{n-k}^{(n)} Y_{n-k}(z), \quad n \geq 1, \tag{3.43}$$

where $Y_n(z) = (n+1)^{-\alpha-1/2} Q_n^{(\alpha,\beta,\mathcal{N})}(\cos z/n)$, $V_n(z) = (n+1)^{-\alpha-1/2} v_n(\cos z/n)$ and $A_{n-k}^{(n)} = \alpha_{n-k}^{(n)}((n-k)/(n+1))^{\alpha+1/2}$, $k = 1,\ldots,N$. Moreover, $|A_{n-N}^{(n)}| = O(1/n^2)$ and $|A_{n-k}^{(n)}| = O(1/n)$ for $1 \leq k < N$.

Using the above relation in a recursive way as well as the same argument of Proposition 3.2, we have

$$Y_n(z) = \sum_{m=0}^{n} B_{1,n}^{(m)} V_{n-m}(z), \tag{3.44}$$

where $B_{1,n}^{(0)} = 1$, $|B_{1,n}^{(m)}| = O(1/n)$ for $m = 1,2,\ldots,N-1$, and $|B_{1,n}^{(m)}| = O(1/n^2)$ for $m = N, N+1,\ldots,n$. Thus,

$$|Y_n(z)| \leq \sum_{m=0}^{n} \left|B_{1,n}^{(m)}\right| |V_{n-m}(z)|. \tag{3.45}$$

On the other hand, from Proposition 2.3(c), $(V_n)_{n\geq 0}$ is uniformly bounded on compact subsets of $\mathbf{C}$. Thus, for a fixed compact set $K \subset \mathbf{C}$, there exists a constant C, depending only on K, such that when $z \in K$,

$$|V_n(z)| < C, \quad n \geq 0. \tag{3.46}$$

Thus, the sequence $(Y_n)_{n\geq 0}$ is uniformly bounded on $K \subset \mathbf{C}$. As a conclusion,

$$Y_n(z) = V_n(z) + O\left(n^{-1}\right), \quad z \in K, \tag{3.47}$$

and from Proposition 2.3(c), we obtain the result.

(b) Since we have uniform convergence in (3.41), taking derivatives and using a well known property of Bessel functions of the first kind (see [12, formula 1.71.5]), we obtain (3.42). □

Now, we give the inner strong asymptotics of $Q_n^{(\alpha,\beta,\mathcal{N})}$ on $(-1,1)$.

Proposition 3.7. *For $x \in [-1+\epsilon, 1-\epsilon]$ and $0<\epsilon<1$,*

$$Q_n^{(\alpha,\beta,\mathcal{N})}(x) = (1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4} \times \sum_{i=0}^{N} a_{n-1,i} r_{n-i}^{\alpha,\beta} \cos\big((k-i)\theta+\gamma\big) + O\Big(n^{-1}\Big), \tag{3.48}$$

$$\begin{aligned} Q_n'^{(\alpha,\beta,\mathcal{N})}(\cos\theta) &= \sqrt{n\big(n+\alpha+\beta+1\big)}(1-x)^{-\alpha/2-3/4}(1+x)^{-\beta/2-3/4} \\ &\quad \times \sum_{i=0}^{N} A_{n-1,i} r_{n-i-1}^{\alpha+1,\beta+1} \cos\big((k-i)\theta+\gamma_1\big) + O(1), \end{aligned} \tag{3.49}$$

where $x = \cos\theta$, $k = n + (\alpha+\beta+1)/2$, $\gamma = -(\alpha+(1/2))\pi/2$, $\gamma_1 = -(\alpha+(3/2))\pi/2$, and $r_n^{\alpha,\beta} \cong (2/\pi)^{1/2}$.

Proof. From Proposition 3.3(a), the sequence $(Q_n^{(\alpha,\beta,\mathcal{N})})_{n\geq 0}$ is uniformly bounded on compact subsets of $(-1,1)$; thus, from Proposition 3.1,

$$Q_n^{(\alpha,\beta,\mathcal{N})}(x) = v_n(x) + O\Big(\frac{1}{n}\Big). \tag{3.50}$$

Now, using Proposition 2.3(d), the relation (3.48) follows.

Concerning (3.49), it can be obtained in a similar way by using Propositions 2.1(f) and 2.2, (2.14), Propositions 3.1 and 3.3(b). □

Now, we can give the sharp estimate for the Sobolev norms of the Jacobi-Sobolev polynomials.

Proposition 3.8. *For $\alpha \geq \beta \geq -1/2$ and $1 \leq p \leq \infty$,*

$$\|Q_n^{(\alpha,\beta,\mathcal{N})}\|_{W^{\mathcal{N},p}} \sim \begin{cases} n, & \text{if } \dfrac{4(\alpha+2)}{(2\alpha+3)} > p, \\[2ex] n\big(\log n\big)^{1/p}, & \text{if } \dfrac{4(\alpha+2)}{(2\alpha+3)} = p, \\[2ex] n^{(\alpha+5)/2-(2\alpha+4)/p}, & \text{if } \dfrac{4(\alpha+2)}{(2\alpha+3)} < p. \end{cases} \tag{3.51}$$

Proof. Clearly, if $p=\infty$, then we get Proposition 3.4(b). Thus, in the proof, we will assume $1 \leq p < \infty$. Since by Proposition 3.2 and(2.14)

$$\left|\frac{d^{i+1}}{dx^{i+1}} Q_n^{(\alpha,\beta,\mathcal{N})}(\xi_k)\right| \leq cn\left|\frac{d^i}{dx^i} s_{n-1}(\xi_k)\right| + c\sum_{m=1}^{N-1}\left|\frac{d^i}{dx^i} s_{n-m-1}(\xi_k)\right| + O\Big(n^{-1}\Big)\sum_{m=N}^{n-1}\left|\frac{d^i}{dx^i} s_{n-m-1}^{(i)}(\xi_k)\right|, \tag{3.52}$$

where $k = 1,\dots,M$, $i = 0,1,\dots,N_k$, and $(d^i/dx^i)s_n(\xi_k)$ are bounded because of the orthonormality condition, we obtain

$$\left|\frac{d^{i+1}}{dx^{i+1}}Q_n^{(\alpha,\beta,\mathcal{N})}(\xi_k)\right| \le cn, \tag{3.53}$$

where $k = 1,\dots,M$, and $i = 0,1,\dots,N_k$.

On the other hand, using (3.30), Minkowski's inequality, and Proposition 2.3(e), we deduce

$$\|Q_n^{(\alpha,\beta,\mathcal{N})}\|_{L^p(d\mu_{\alpha,\beta})} \le c\|p_n^{(\alpha,\beta)}\|_{L^p(d\mu_{\alpha,\beta})} \le cn\|p_n^{(\alpha+1,\beta+1)}\|_{L^p(d\mu_{\alpha+1,\beta+1})}. \tag{3.54}$$

In the same way as above, we get

$$\|Q'^{(\alpha,\beta,\mathcal{N})}_n\|_{L^p(d\mu_{\alpha+1,\beta+1})} \le cn\|p_n^{(\alpha+1,\beta+1)}\|_{L^p(d\mu_{\alpha+1,\beta+1})}. \tag{3.55}$$

Thus, from (3.53), (3.54), and (3.55), we have

$$\begin{aligned}
&\|Q_n^{(\alpha,\beta,\mathcal{N})}\|^p_{W^{\mathcal{N},p}} \\
&\le \|Q_n^{(\alpha,\beta,\mathcal{N})}\|^p_{L^p(d\mu_{\alpha,\beta})} + \frac{\lambda}{\prod_{k=1}^M(-1-\xi_k)^{N_k+1}}\|Q'^{(\alpha,\beta,\mathcal{N})}_n\|^p_{L^p(d\mu_{\alpha+1,\beta+1})} + \sum_{k=1}^M\sum_{i=0}^{N_k}M_{k,i}\left|\frac{d^{i+1}}{dx^{i+1}}Q_n^{(\alpha,\beta,\mathcal{N})}(\xi_k)\right|^p \\
&\le cn^p\|p_n^{(\alpha+1,\beta+1)}\|_{L^p(d\mu_{\alpha+1,\beta+1})}.
\end{aligned} \tag{3.56}$$

Notice that the upper estimate in (3.54) and (3.55) can also be proved using the bounds for Jacobi-Sobolev polynomials given in Corollary 3.5.

In order to prove the lower bound in (3.51) we will need the following.

Proposition 3.9. *For $\alpha > -1$ and $1 \le p < \infty$,*

$$\|Q'^{(\alpha,\beta,\mathcal{N})}_n\|_{L^p(d\mu_{\alpha+1,\beta+1})} \ge c\begin{cases} cn, & \text{if } \dfrac{4(\alpha+2)}{(2\alpha+3)} > p, \\ cn(\log n)^{1/p}, & \text{if } \dfrac{4(\alpha+2)}{(2\alpha+3)} = p, \\ n^{(\alpha+5)/2-(2\alpha+4)/p}, & \text{if } \dfrac{4(\alpha+2)}{(2\alpha+3)} < p. \end{cases} \tag{3.57}$$

Proof. We will use a technique similar to [12, Theorem 7.34]. According to (3.42),

$$
\begin{aligned}
\int_0^{\pi/2} & \theta^{2\alpha+3}\left|{Q'}_n^{(\alpha,\beta,\mathcal{N})}(\cos\theta)\right|^p d\theta \\
&> \int_0^{\omega/n} \theta^{2\alpha+3}\left|{Q'}_n^{(\alpha,\beta,\mathcal{N})}(\cos\theta)\right|^p d\theta \\
&= cn^{-2\alpha-4}\int_0^{\omega} t^{2\alpha+3}\left|{Q'}_n^{(\alpha,\beta,\mathcal{N})}\left(\cos\frac{t}{n}\right)\right|^p dt \cong cn^{p(\alpha+5/2)-2\alpha-4} \\
&\quad\times \int_0^{\omega} t^{2\alpha+3}\left|t^{-\alpha-1}J_{\alpha+1}(t)\right|^p dt = cn^{p(\alpha+5/2)-2\alpha-4} \\
&\quad\times \int_0^{\omega} t^{2\alpha+3-p\alpha-p}|J_{\alpha+1}(t)|^p dt.
\end{aligned}
\tag{3.58}
$$

On the other hand, from (see [18, Lemma 2.1]), if $\gamma > -1 - p\alpha$ and $1 \le p < \infty$, we have

$$
\int_0^{\omega} t^{\gamma}|J_{\alpha}(t)|^p dt \sim \begin{cases} c, & \text{if } \gamma < \dfrac{p}{2-1}, \\[2ex] c\log\omega, & \text{if } \gamma = \dfrac{p}{2-1}. \end{cases}
\tag{3.59}
$$

Thus, for $4(\alpha+2)/(2\alpha+3) \le p$ and ω large enough, (3.57) follows.

Finally, from (3.49), we obtain

$$
\int_0^{\pi/2} \theta^{2\alpha+3}\left|{Q'}_n^{(\alpha,\beta,\mathcal{N})}(\cos\theta)\right|^p d\theta > \int_{\pi/4}^{\pi/2} \theta^{2\alpha+3}\left|{Q'}_n^{(\alpha,\beta,\mathcal{N})}(\cos\theta)\right|^p d\theta \ge cn^p.
\tag{3.60}
$$

The proof of Proposition 3.9 is complete. □

From (3.57), for $\alpha > -1$ and $1 \le p < \infty$,

$$
\|Q_n^{(\alpha,\beta,\mathcal{N})}\|_{W^{\mathcal{N},p}} \ge c\begin{cases} cn, & \text{if } \dfrac{4(\alpha+2)}{(2\alpha+3)} > p, \\[2ex] n(\log n)^{1/p}, & \text{if } \dfrac{4(\alpha+2)}{(2\alpha+3)} = p, \\[2ex] n^{(\alpha+5)/2-(2\alpha+4)/p}, & \text{if } \dfrac{4(\alpha+2)}{(2\alpha+3)} < p. \end{cases}
\tag{3.61}
$$

Thus, using (3.56) and (3.61), the statement follows. □

4. Necessary Conditions for the Norm Convergence

The analysis of the norm convergence of partial sums of the Fourier expansions in terms of Jacobi polynomials has been done by many authors. See, for instance, [19–21], and the references therein.

Let $q_n^{(\alpha,\beta,\mathcal{N})}$ be the Jacobi-Sobolev orthonormal polynomials, that is,

$$q_n^{(\alpha,\beta,\mathcal{N})}(x) = \left(\|Q_n^{(\alpha,\beta,\mathcal{N})}\|_{W^{\mathcal{N},2}}\right)^{-1} Q_n^{(\alpha,\beta,\mathcal{N})}(x). \tag{4.1}$$

For $f \in W^{\mathcal{N},1}$, its Fourier expansion in terms of Jacobi-Sobolev orthonormal polynomials is

$$\sum_{k=0}^{\infty} \widehat{f}(k) q_k^{(\alpha,\beta,\mathcal{N})}(x), \tag{4.2}$$

where

$$\widehat{f}(k) = \left\langle f, q_k^{(\alpha,\beta,\mathcal{N})} \right\rangle, \quad k = 0, 1, \ldots. \tag{4.3}$$

Let $S_n f$ be the n-th partial sum of the expansion (4.2)

$$S_n(f,x) = \sum_{k=0}^{n} \widehat{f}(k) q_k^{(\alpha,\beta,\mathcal{N})}(x). \tag{4.4}$$

Theorem 4.1. *Let $\alpha \geq \beta \geq -1/2$, and $1 \leq p \leq \infty$. If there exists a constant $c > 0$ such that*

$$\|S_n f\|_{W^{\mathcal{N},p}} \leq c\|f\|_{W^{\mathcal{N},p}}, \tag{4.5}$$

for every $f \in W^{\mathcal{N},p}$, then $p \in (p_0, q_0)$ with

$$q_0 = \frac{4(\alpha+2)}{2\alpha+3}, \qquad p_0 = \frac{4(\alpha+2)}{2\alpha+5}. \tag{4.6}$$

Proof. For the proof, we apply the same argument as in [20]. Assume that (4.5) holds. Then,

$$\left\| \left\langle f, q_n^{(\alpha,\beta,\mathcal{N})} \right\rangle q_n^{(\alpha,\beta,\mathcal{N})}(x) \right\|_{W^{\mathcal{N},p}} = \|S_n f - S_{n-1} f\|_{W^{\mathcal{N},p}} \leq c\|f\|_{W^{\mathcal{N},p}}. \tag{4.7}$$

Consider the linear functionals

$$T_n(f) = \left\langle f, q_n^{(\alpha,\beta,\mathcal{N})} \right\rangle \|q_n^{(\alpha,\beta,\mathcal{N})}\|_{W^{\mathcal{N},p}} \tag{4.8}$$

on $W^{\mathcal{N},p}$. Hence, for every f in $W^{\mathcal{N},p}\sup_n|T_n(f)| < \infty$ holds. From the Banach-Steinhaus theorem, this yields $\sup_n\|T_n\| < \infty$. On the other hand, by duality (see, for instance, [1, Theorem 3.8]), we have

$$\|T_n\| = \|q_n^{(\alpha,\beta,\mathcal{N})}\|_{W^{\mathcal{N},p}}\|q_n^{(\alpha,\beta,\mathcal{N})}\|_{W^{\mathcal{N},q}}, \tag{4.9}$$

where p is the conjugate of q. Therefore,

$$\sup_n\|q_n^{(\alpha,\beta,\mathcal{N})}(x)\|_{W^{\mathcal{N},p}}\|q_n^{(\alpha,\beta,\mathcal{N})}(x)\|_{W^{\mathcal{N},q}} < \infty. \tag{4.10}$$

On the other hand, from (3.51), we obtain the Sobolev norms of Jacobi-Sobolev orthonormal polynomials

$$\|q_n^{(\alpha,\beta,\mathcal{N})}\|_{W^{\mathcal{N},p}} \sim \begin{cases} c, & \text{if } p < q_0, \\ (\log n)^{1/p}, & \text{if } p = q_0, \\ n^{(\alpha+3)/2-(2\alpha+4)/p}, & \text{if } p > q_0, \end{cases} \tag{4.11}$$

for $\alpha \geq \beta \geq -1/2$ and $1 \leq p \leq \infty$. Now, from (4.11), it follows that the inequality (4.10) holds if and only if $p \in (p_0, q_0)$.

The proof of Theorem 4.1 is complete. □

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