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Research Article

Commutators of Littlewood-Paley Operators on the Generalized Morrey Space

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Let μ_Ω, μ_S^{ϖ}, and $\mu_\lambda^{*,\varpi}$ denote the Marcinkiewicz integral, the parameterized area integral, and the parameterized Littlewood-Paley g_λ^* function, respectively. In this paper, the authors give a characterization of BMO space by the boundedness of the commutators of μ_Ω, μ_S^{ϖ}, and $\mu_\lambda^{*,\varpi}$ on the generalized Morrey space $L^{p,\varphi}(\mathbb{R}^n)$.

1. Introduction

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(\mu x) = \Omega(x), \quad \text{for any } \mu > 0,\ x \in \mathbb{R}^n \setminus \{0\}. \tag{1.1}$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0. \tag{1.2}$$

(c) $\Omega \in \mathrm{Lip}(S^{n-1})$, that is,

$$\left|\Omega(x') - \Omega(y')\right| \le \left|x' - y'\right|, \quad \text{for any } x', y' \in S^{n-1}. \tag{1.3}$$

In 1958, Stein [1] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{1.4}$$

where

$$F_{\Omega,t}(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{\left|x-y\right|^{n-1}} f(y)dy. \tag{1.5}$$

We refer to see [1, 2] for the properties of μ_Ω.

Let $0 < \varpi < n$ and $\lambda > 1$. The parameterized area integral μ_S^ϖ and the parameterized Littlewood-Paley g_λ^* function $\mu_\lambda^{*,\varpi}$ are defined by

$$\mu_S^\varpi f(x) = \left(\iint_{\Gamma(x)} \left|\frac{1}{t^\varpi}\int_{|y-z|<t} \frac{\Omega(y-z)}{\left|y-z\right|^{n-\varpi}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \tag{1.6}$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}$, and

$$\mu_\lambda^{*,\varpi} f(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+\left|x-y\right|}\right)^{\lambda n} \left|\frac{1}{t^\varpi}\int_{|y-z|<t} \frac{\Omega(y-z)}{\left|y-z\right|^{n-\varpi}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \tag{1.7}$$

respectively. μ_S^ϖ and $\mu_\lambda^{*,\varpi}$ play very important roles in harmonic analysis and PDE (e.g., see [3–8]).

Before stating our result, let us recall some definitions. For $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$, the commutator $[b, \mu_\Omega]$ formed by b and the Marcinkiewicz integral μ_Ω are defined by

$$[b, \mu_\Omega] f(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{\left|x-y\right|^{n-1}} (b(x) - b(y)) f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}. \tag{1.8}$$

Let $0 < \varpi < n$ and $\lambda > 1$. The commutator $[b, \mu_S^{\varpi}]$ of μ_S^{ϖ} and the commutator $[b, \mu_\lambda^{*,\varpi}]$ of $\mu_\lambda^{*,\varpi}$ are defined, respectively, by

$$[b, \mu_S^{\varpi}] f(x) = \left(\iint_{\Gamma(x)} \left| \frac{1}{t^{\varpi}} \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} (b(x) - b(z)) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \tag{1.9}$$

$$[b, \mu_\lambda^{*,\varpi}] f(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \times \left| \frac{1}{t^{\varpi}} \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} (b(x) - b(z)) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}. \tag{1.10}$$

Let $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$. It is said that $b \in \mathrm{BMO}(\mathbb{R}^n)$ if

$$\|b\|_* := \sup_{B \subset \mathbb{R}^n} M(b, B) < \infty, \tag{1.11}$$

where $B = B(x, r)$ denotes the ball in $\mathbb{R}^n$ centered at x and with radius r,

$$M(b, B) = \frac{1}{|B|} \int_B |b(x) - b_B| dx, \tag{1.12}$$

and $b_B = (1/|B|) \int_B b(y) dy$.

There are some results about the boundedness of the commutators formed by BMO functions with μ_Ω, μ_S^{ϖ}, and $\mu_\lambda^{*,\varpi}$ (see [7, 9, 10]).

Many important operators gave a characterization of BMO space. In 1976, Coifman et al. [11] gave a characterization of BMO space by the commutator of Riesz transform; in 1982, Chanillo [12] studied the commutator formed by Riesz potential and BMO and gave another characterization of BMO space.

The purpose of this paper is to give a characterization of BMO space by the boundedness of the commutators of μ_Ω, μ_S^{ϖ}, and $\mu_\lambda^{*,\varpi}$ on the generalized Morrey space $L^{p,\varphi}(\mathbb{R}^n)$.

Definition 1.1. Let $1 < p < \infty$. Suppose that $\varphi : (0, \infty) \to (0, \infty)$ be such that $\varphi(t)$ is nonincreasing and $t^{1/p}\varphi(t)$ is nondecreasing. The generalized Morrey space $L^{p,\varphi}$ is defined by

$$L^{p,\varphi}(\mathbb{R}^n) = \left\{ f \in L_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\varphi}} < \infty \right\}, \tag{1.13}$$

where

$$\|f\|_{L^{p,\varphi}} = \sup_{\substack{x \in \mathbb{R}^n \\ r > 0}} \frac{1}{\varphi(|B(x,r)|)} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p}. \tag{1.14}$$

We refer to see [13, 14] for the known results of the generalized Morrey space $L^{p,\varphi}$ for some suitable φ. Noting that $\varphi(t) \equiv t^{-1/p}$, we get the Lebesque space $L^p(\mathbb{R}^n)$. For $\varphi(t) = t^{(\lambda/n-1)/p}$ $(0 < \lambda < n)$, $L^{p,\varphi}(\mathbb{R}^n)$ coincides with the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$.

The main result in this paper is as follows.

Theorem 1.2. *Assume that $\varphi(t)$ is nonincreasing and $t^{1/p}\varphi(t)$ is nondecreasing. Suppose that $[b, \mu_\Omega]$ is defined as* (1.8), *Ω satisfies* (1.1), (1.2), *and*

$$\left|\Omega(x') - \Omega(y')\right| \le \frac{C_1}{\left(\log(2/\left|x' - y'\right|)\right)^\gamma}, \quad C_1 > 0,\ \gamma > 1,\ x', y' \in S^{n-1}. \tag{1.15}$$

If $[b, \mu_\Omega]$ is bounded on $L^{p,\varphi}(\mathbb{R}^n)$ for some p $(1 < p < \infty)$, then $b \in BMO(\mathbb{R}^n)$.

Theorem 1.3. *Let $0 < \varpi < n$ and $1 < p < \infty$. Assume that $\varphi(t)$ is nonincreasing and $t^{1/p}\varphi(t)$ is nondecreasing. Suppose that $[b, \mu_S^\varpi]$ is defined as* (1.9), *Ω satisfies* (1.1), (1.2), *and* (1.15). *If $[b, \mu_S^\varpi]$ is a bounded operator on $L^{p,\varphi}(\mathbb{R}^n)$ for some p $(1 < p < \infty)$, then $b \in BMO(\mathbb{R}^n)$.*

Theorem 1.4. *Let $0 < \varpi < n$, $\lambda > 1$, and $1 < p < \infty$. Assume that $\varphi(t)$ is nonincreasing and $t^{1/p}\varphi(t)$ is nondecreasing. Suppose that $[b, \mu_\lambda^{*,\varphi}]$ is defined as* (1.10), *Ω satisfies* (1.1), (1.2), *and* (1.15). *If $[b, \mu_\lambda^{*,\varpi}]$ is on $L^{p,\varphi}(\mathbb{R}^n)$ for some p $(1 < p < \infty)$, then $b \in BMO(\mathbb{R}^n)$.*

Remark 1.5. It is easy to check that $[b, \mu_S^\varpi](f)(x) \le 2^{\lambda n}[b, \mu_\lambda^{*,\varpi}](f)(x)$ (see, e.g., the proof of (19) in [15, page 89]), we therefore give only the proofs of Theorem 1.2 for $[b, \mu_\Omega]$ and Theorem 1.3 for $[b, \mu_S^\varpi]$.

Remark 1.6. It is easy to see that the condition (1.15) is weaker than $\mathrm{Lip}_\beta(S^{n-1})$ for $0 < \beta \le 1$. In the proof of Theorems 1.2 and 1.3, we will use some ideas in [16]. However, because Marcinkiewicz integral and the parameterized Littlewood-Paley operators are neither the convolution operator nor the linear operators, hence, we need new ideas and nontrivial estimates in the proof.

2. Proof of Theorem 1.2

Let us begin with recalling some known conclusion.

Similar to the proof of [17], we can easily get the following.

Lemma 2.1. *If Ω satisfies conditions* (1.1), (1.2), *and* (1.15), *let $\beta > 0$, then for $|x| > 2|y|$, we have*

$$\left|\frac{\Omega(x-y)}{\left|x-y\right|^\beta} - \frac{\Omega(x)}{|x|^\beta}\right| \le \frac{C}{|x|^\beta\left(\log(|x|/\left|y\right|)\right)^\gamma}. \tag{2.1}$$

Now let us return to the proof of Theorem 1.2. Suppose that $[b, \mu_\Omega]$ is a bounded operator on $L^{p,\varphi}(\mathbb{R}^n)$, we are going to prove that $b \in \mathrm{BMO}(\mathbb{R}^n)$.

We may assume that $\|[b,\mu_\Omega]\|_{L^{p,\varphi}\mapsto L^{p,\varphi}} = 1$. We want to prove that, for any $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, the inequality

$$N = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |b(y) - a_0|\,dy \le A(p,\Omega,n,\gamma) \tag{2.2}$$

holds, where $a_0 = |B(x_0,r)|^{-1}\int_{B(x_0,r)} b(y)dy$. Since $[b-a_0,\mu_\Omega] = [b,\mu_\Omega]$, we may assume that $a_0 = 0$. Let

$$f(y) = [\operatorname{sgn}(b(y)) - c_0]\,\chi_{B(x_0,r)}(y), \tag{2.3}$$

where $c_0 = (1/|B(x_0,r)|)\int_{B(x_0,r)} \operatorname{sgn}(b(y))dy$. Since $(1/|B(x_0,r)|)\int_{B(x_0,r)} b(y)dy = a_0 = 0$, we can easily get $|c_0| < 1$. Then, f has the following properties:

$$\|f\|_\infty \le 2, \tag{2.4}$$

$$\operatorname{supp} f \subset B(x_0,r), \tag{2.5}$$

$$\int_{\mathbb{R}^n} f(y)dy = 0, \tag{2.6}$$

$$f(y)b(y) > 0, \quad y \in B(x_0,r), \tag{2.7}$$

$$\frac{1}{|B(x_0,r)|}\int_{\mathbb{R}^n} f(y)b(y)dy = N. \tag{2.8}$$

In this proof for $j = 1,\ldots,15$, A_j is a positive constant depending only on Ω, p, n, γ, and A_i $(1 \le i < j)$. Since Ω satisfies (1.2), then there exists an A_1 such that $0 < A_1 < 1$ and

$$\sigma\left(\left\{x' \in S^{n-1} : \Omega(x') \ge \frac{2C_1}{(\log(2/A_1))^\gamma}\right\}\right) > 0, \tag{2.9}$$

where σ is the measure on S^{n-1} which is induced from the Lebesgue measure on $\mathbb{R}^n$. By the condition (1.15), it is easy to see that

$$\Lambda := \left\{x' \in S^{n-1} : \Omega(x') \ge \frac{2C_1}{(\log(2/A_1))^\gamma}\right\} \tag{2.10}$$

is a closed set. We claim that

$$\text{if } x' \in \Lambda \text{ and } y' \in S^{n-1}, \text{ satisfying } |x' - y'| \le A_1, \text{ then } \Omega(y') \ge \frac{C_1}{(\log(2/A_1))^\gamma}. \tag{2.11}$$

In fact, since $|\Omega(x') - \Omega(y')| \leq C_1/(\log(2/|x'-y'|\ ))^{\gamma} \leq C_1/(\log(2/A_1))^{\gamma}$, note that $\Omega(x') \geq 2(C_1/(\log(2/A_1))^{\gamma})$, we can get $\Omega(y') \geq C_1/(\log(2/A_1))^{\gamma}$. Taking $A_2 > 3/A_1$, let

$$G = \left\{x \in \mathbb{R}^n : |x - x_0| \geq A_2 r,\ (x - x_0)' \in \Lambda\right\}. \tag{2.12}$$

For $x \in G$, we have

$$\begin{aligned} \left|[b, \mu_\Omega] f(x)\right| &\geq \left|\mu_\Omega(bf)(x)\right| - |b(x)|\left|\mu_\Omega f(x)\right| \\ &= \left\{\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega\left((x-y)'\right)}{|x-y|^{n-1}} b(y) f(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{1/2} \\ &\quad - |b(x)| \left\{\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega\left((x-y)'\right)}{|x-y|^{n-1}} f(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{1/2} \\ &:= I_1 - I_2. \end{aligned} \tag{2.13}$$

For I_1, noting that if $y \in B(x_0, r)$, then $|x - x_0| > A_2|y - x_0|$ for $x \in G$. Thus, we have

$$\left|(x-y)' - (x-x_0)'\right| \leq 2\frac{|y - x_0|}{|x - x_0|} \leq \frac{2}{A_2} < A_1. \tag{2.14}$$

Using (2.11), we get $\Omega((x-y)') \geq C_1/(\log(2/A_1))^{\gamma}$. Noting that $|x - x_0| \simeq |x - y|$, it follows from (2.5), (2.7), (2.8), and Hölder's inequality that

$$\begin{aligned} I_1 &\geq \left\{\int_{|x-x_0|}^\infty \left(\int_{B(x_0,r)} \frac{\Omega\left((x-y)'\right) b(y) f(y)}{|x-y|^{n-1}} \chi_{\{|x-y|\leq t\}}(y)\,dy\right)^2 \frac{dt}{t^3}\right\}^{1/2} \\ &\geq \left(\int_{|x-x_0|}^\infty \int_{B(x_0,r)} \frac{\Omega\left((x-y)'\right) b(y) f(y)}{|x-y|^{n-1}} \chi_{\{|x-y|\leq t\}}\,dy \frac{dt}{t^3}\right) \left(\int_{|x-x_0|}^\infty \frac{dt}{t^3}\right)^{-1/2} \\ &\geq \frac{C_1}{(\log(2/A_1))^{\gamma}} |x - x_0| \int_{B(x_0,r)} |x-y|^{-n+1} b(y) f(y) \int_{\substack{|x-x_0|\leq t \\ |x-y|\leq t}} \frac{dt}{t^3}\,dy \\ &\geq \frac{C}{(\log(2/A_1))^{\gamma}} |x - x_0|^{-n} \int_{B(x_0,r)} b(y) f(y)\,dy \\ &= A_3 N r^n |x - x_0|^{-n}. \end{aligned} \tag{2.15}$$

For $x \in G$, by $\Omega \in L^\infty(S^{n-1})$, (2.4), (2.5), (2.6), the Minkowski inequality, and Lemma 2.1, we obtain

$$
\begin{aligned}
I_2 &= |b(x)| \left\{ \int_0^\infty \left| \int_{\mathbb{R}^n} f(y) \left(\frac{\Omega(x-y)}{|x-y|^{n-1}} \chi_{\{|x-y|\le t\}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \chi_{\{|x-x_0|\le t\}} \right) dy \right|^2 \frac{dt}{t^3} \right\}^{1/2} \\
&\le |b(x)| \left\{ \left(\int_0^\infty \left(\int_{|x-y|\le t<|x-x_0|} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \right. \\
&\quad + \left(\int_0^\infty \left(\int_{|x-x_0|\le t<|x-y|} \frac{|\Omega(x-x_0)|}{|x-x_0|^{n-1}} |f(y)| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad \left. + \left(\int_0^\infty \left(\int_{\substack{|x-x_0|\le t \\ |x-y|\le t}} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| |f(y)| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \right\} \\
&\le |b(x)| \left\{ \int_{B(x_0,r)} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \left(\int_{|x-y|\le t<|x-x_0|} \frac{dt}{t^3} \right)^{1/2} dy \right. \\
&\quad + \int_{B(x_0,r)} \frac{|\Omega(x-x_0)|}{|x-x_0|^{n-1}} |f(y)| \left(\int_{|x-y|>t\ge|x-x_0|} \frac{dt}{t^3} \right)^{1/2} dy \\
&\quad \left. + \int_{B(x_0,r)} |f(y)| \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| \left(\int_{\substack{|x-y|\le t \\ |x-x_0|\le t}} \frac{dt}{t^3} \right)^{1/2} dy \right\} \\
&\le C|b(x)| \left(r^{1/2} \int_{B(x_0,r)} \frac{|f(y)|}{|x-x_0|^{n+1/2}} dy + \int_{B(x_0,r)} \frac{|f(y)|}{|x-x_0|^n (\log(|x-x_0|/r))^\gamma} dy \right) \\
&\le A_4 |b(x)| r^n |x-x_0|^{-n} \left(\log \frac{|x-x_0|}{r} \right)^{-\gamma}.
\end{aligned} \tag{2.16}
$$

Let

$$
F = \left\{ x \in G : |b(x)| > \frac{A_3 N}{2A_4} \left(\log \frac{|x-x_0|}{r} \right)^\gamma,\ |x-x_0| < N^{1/n} r \right\}. \tag{2.17}
$$

Without loss of generality, we may assume that $N > A_2 > 1$, otherwise, we get the desired result. Since $\varphi(t)$ is nonincreasing, it follows that $\varphi(|B(x_0, N^{1/n}r)|) \le \varphi(|B(x_0, r)|) = \varphi(r^n)$. By (2.13), (2.15), and (2.16), we have

$$\begin{aligned}
\|f\|_{L^{p,\varphi}}^p &\ge \|[b,\mu_\Omega]f\|_{L^{p,\varphi}}^p \\
&\ge \frac{1}{(\varphi(|B(x_0,N^{1/n}r)|))^p|B(x_0,N^{1/n}r)|}\int_{|x-x_0|<N^{1/n}r}|[b,\mu_\Omega]f(x)|^p dx \\
&\ge \frac{1}{(\varphi(r^n))^p Nr^n}\int_{(G\setminus F)\cap\{|x-x_0|<N^{1/n}r\}}\left(\frac{1}{2}A_3Nr^n|x-x_0|^{-n}\right)^p dx \\
&\ge \frac{1}{(\varphi(r^n))^p Nr^n}\int_{\{A_5(|F|+(A_2r)^n)^{1/n}<|x-x_0|<N^{1/n}r\}\cap G}\left(\frac{1}{2}A_3Nr^n|x-x_0|^{-n}\right)^p dx \\
&= \frac{\omega_{n-1}}{(\varphi(r^n))^p Nr^n}\left(\frac{A_3Nr^n}{2}\right)^p\int_{A_5(|F|+(A_2r)^n)^{1/n}}^{N^{1/n}r} t^{-pn+n-1}dt \\
&\ge \frac{\omega_{n-1}}{(\varphi(r^n))^p}(Nr^n)^{p-1}\frac{(A_3/2)^p}{n-np}\left(N^{1-p}r^{n(1-p)} - A_5^{(1-p)n}(|F|+(A_2r)^n)^{1-p}\right).
\end{aligned} \tag{2.18}$$

Thus,

$$\left(|F|+(A_2r)^n\right)^{1-p} \le A_6N^{1-p}r^{n(1-p)}\left(1+\varphi(r^n)^p\|f\|_{L^{p,\varphi}}^p\right). \tag{2.19}$$

Now, we claim that

$$\|f\|_{L^{p,\varphi}} \le \frac{C}{\varphi(r^n)}, \tag{2.20}$$

where C is independent of r. In fact,

$$\|f\|_{L^{p,\varphi}} = \sup_{\substack{x\in\mathbb{R}^n\\ t>0}}\frac{1}{\varphi(|B(x,t)|)}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}|f(y)|^p dy\right)^{1/p}. \tag{2.21}$$

Now, we consider the $L^{p,\varphi}$ norm of f in the following two cases.

Case 1 $(t > r)$. Since $s^{1/p}\varphi(s)$ is nondecreasing in s, then

$$\frac{1}{\varphi(|B(x,t)|)}\frac{1}{|B(x,t)|^{1/p}} \le \frac{1}{\varphi(r^n)}\frac{1}{r^{n/p}}. \tag{2.22}$$

Thus,

$$\begin{aligned}\|f\|_{L^{p,\varphi}} &\leq \sup_{\substack{x\in\mathbb{R}^n\\ t>0}} \frac{1}{\varphi(r^n)}\ \frac{1}{r^{n/p}}\left(\int_{B(x,t)} |f(y)|^p dy\right)^{1/p} \\ &= \sup_{\substack{x\in\mathbb{R}^n\\ t>0}} \frac{1}{\varphi(r^n)}\frac{1}{r^{n/p}}\left(\int_{B(x,t)\cap B(x_0,r)} |f(y)|^p dy\right)^{1/p} \\ &\leq \frac{C}{\varphi(r^n)}.\end{aligned} \tag{2.23}$$

Case 2 ($t \leq r$). Since $\varphi(s)$ is nonincreasing in s, then

$$\frac{1}{\varphi(|B(x,t)|)} \leq \frac{1}{\varphi(r^n)}. \tag{2.24}$$

Thus,

$$\begin{aligned}\|f\|_{L^{p,\varphi}} &\leq \sup_{\substack{x\in\mathbb{R}^n\\ t>0}} \frac{1}{\varphi(r^n)}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)} |f(y)|^p dy\right)^{1/p} \\ &\leq \frac{C}{\varphi(r^n)}.\end{aligned} \tag{2.25}$$

Now, (2.20) is established. Then, by (2.19) and (2.20), we get

$$|F| + (A_2 r)^n \geq A_7 N r^n. \tag{2.26}$$

If $N \leq 2A_7^{-1}A_2^n$, then Theorem 1.2 is proved. If $N > 2A_7^{-1}A_2^n$, then

$$|F| \geq \frac{A_7}{2} N r^n. \tag{2.27}$$

Let $g(y) = \chi_{B(x_0,r)}(y)$. For $x \in F$, we have

$$\begin{aligned}\left|[b,\mu_\Omega]g(x)\right| &\geq |b(x)|\left\{\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega\big((x-y)'\big)}{|x-y|^{n-1}} g(y)dy\right|^2 \frac{dt}{t^3}\right\}^{1/2} \\ &\quad - \left\{\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega\big((x-y)'\big)}{|x-y|^{n-1}} b(y)g(y)dy\right|^2 \frac{dt}{t^3}\right\}^{1/2} \\ &:= K_1 - K_2.\end{aligned} \tag{2.28}$$

Noting that if $y \in B(x_0, r)$ and $x \in F$, we get $|(x-y)' - (x-x_0)'| \le A_1$. Applying (2.11), we have $\Omega((x-y)') \ge C_1/(\log(2/A_1))^{\gamma}$. Since $|x-y| \simeq |x-x_0|$ when $y \in B(x_0, r)$ and $x \in F$, it follows that

$$\begin{aligned} K_1 &\ge |b(x)| \left\{ \int_{|x-x_0|}^{\infty} \left(\int_{B(x_0,r)} \frac{\Omega\left((x-y)'\right)}{|x-y|^{n-1}} \chi_{\{|x-y|\le t\}}(y)\,dy \right)^2 \frac{dt}{t^3} \right\}^{1/2} \\ &\ge |b(x)| \left(\int_{|x-x_0|}^{\infty} \int_{B(x_0,r)} \frac{\Omega\left((x-y)'\right)}{|x-y|^{n-1}} \chi_{\{|x-y|\le t\}}\,dy \frac{dt}{t^3} \right) \left(\int_{|x-x_0|}^{\infty} \frac{dt}{t^3} \right)^{-1/2} \\ &\ge \frac{C_1 |b(x)|}{(\log(2/A_1))^{\gamma}} |x-x_0| \int_{B(x_0,r)} |x-y|^{-n+1} \int_{\substack{|x-x_0|\le t \\ |x-y|\le t}} \frac{dt}{t^3}\,dy \\ &\ge A_8 |b(x)| |x-x_0|^{-n} \int_{B(x_0,r)} dy \\ &= A_8 r^n |b(x)| |x-x_0|^{-n}. \end{aligned} \tag{2.29}$$

By $\Omega \in L^{\infty}(S^{n-1})$, $|x-x_0| \simeq |x-y|$ when $y \in B(x_0, r)$ and $x \in F$ and the Minkowski inequality, we have

$$\begin{aligned} K_2 &\le C \int_{B(x_0,r)} \frac{|b(y)|}{|x-y|^n}\,dy \\ &\le A_9 |x-x_0|^{-n} \int_{B(x_0,r)} |b(y)|\,dy \\ &= A_9 N r^n |x-x_0|^{-n}. \end{aligned} \tag{2.30}$$

Thus, by (2.28), (2.29), and (2.30), we get, for $x \in F$,

$$\left| [b, \mu_\Omega] g(x) \right| \ge A_8 r^n |b(x)| |x-x_0|^{-n} - A_9 N r^n |x-x_0|^{-n}. \tag{2.31}$$

Similar to the proof of (2.20), we can easily get $\|g\|_{L^{p,\varphi}} \le C/\varphi(r^n)$. Thus, by (2.31),

$\varphi(Nr^n) \le \varphi(r^n)$, and $|b(x)| > (NA_3/2A_4)(\log(|x-x_0|/r))^\gamma$ when $x \in F$, we have

$$\begin{aligned}
\frac{A_{10}}{\varphi(r^n)} &\ge \|g\|_{L^{p,\varphi}} \ge \|[b,\mu_\Omega]g\|_{L^{p,\varphi}} \\
&\ge \frac{1}{\varphi(Nr^n)(Nr^n)^{1/p}}\left(\int_{|x-x_0|<N^{1/n}r} \left|[b,\mu_\Omega]g(x)\right|^p dx\right)^{1/p} \\
&\ge \frac{1}{\varphi(r^n)(Nr^n)^{1/p}}\int_{|x-x_0|<N^{1/n}r} \left|[b,\mu_\Omega]g(x)\right| dx \left(\int_{|x-x_0|<N^{1/n}r} dx\right)^{-1/p'} \\
&\ge \frac{1}{\varphi(r^n)Nr^n}\int_F \left|[b,\mu_\Omega]g(x)\right| dx \\
&\ge \frac{A_8 r^n}{\varphi(r^n)Nr^n}\int_F |b(x)||x-x_0|^{-n}dx - \frac{A_9 Nr^n}{\varphi(r^n)Nr^n}\int_F |x-x_0|^{-n}dx \\
&\ge \frac{A_{11}}{\varphi(r^n)}\int_F \left(\log\frac{|x-x_0|}{r}\right)^\gamma |x-x_0|^{-n}dx \\
&\quad - \frac{A_9}{\varphi(r^n)}\int_F |x-x_0|^{-n}dx \\
&:= L_1 - L_2.
\end{aligned} \tag{2.32}$$

We first estimate L_2. Since $A_2 r < |x-x_0| < N^{1/n}r$ for $x \in F$, we have

$$L_2 \le \frac{A_9\omega_{n-1}}{\varphi(r^n)}\int_{A_2 r}^{N^{1/n}r} \rho^{-1}d\rho \le \frac{A_{12}}{\varphi(r^n)}\log N. \tag{2.33}$$

Now, the estimate of L_1 is divided into two cases, namely, 1: $\gamma \ge n$; 2: $1 < \gamma < n$.

Case 1 ($\gamma \ge n$). Since the function $\log s/s$ is decreasing for $s \ge 3$ and $3r < A_2 r < |x-x_0| < N^{1/n}r$ for $x \in F$, by (2.27), we get

$$\begin{aligned}
L_1 &= \frac{A_{11}r^{-n}}{\varphi(r^n)}\int_F \left(\frac{\log(|x-x_0|/r)}{|x-x_0|/r}\right)^n \left(\log\frac{|x-x_0|}{r}\right)^{\gamma-n} dx \\
&\ge \frac{A_7 A_{11}}{2\varphi(r^n)}(\log A_2)^{\gamma-n} N \left(\frac{\log N^{1/n}}{N^{1/n}}\right)^n \\
&\ge \frac{A_{13}}{\varphi(r^n)}(\log N)^n.
\end{aligned} \tag{2.34}$$

Case 2 ($1 < \gamma < n$). Since the function $(\log s)^{\gamma}/s^n$ is decreasing for $s \geq 3$ and $3r < A_2 r < |x - x_0| < N^{1/n} r$ for $x \in F$, by (2.27), we have

$$L_1 = \frac{A_{11} r^{-n}}{\varphi(r^n)} \int_F \frac{(\log(|x - x_0|/r))^{\gamma}}{(|x - x_0|/r)^n} dx \geq \frac{A_7 A_{11}}{2\varphi(r^n)} N \frac{(\log N^{1/n})^{\gamma}}{N} \geq \frac{A_{14}}{\varphi(r^n)} (\log N)^{\gamma}. \tag{2.35}$$

From Cases 1 and 2, we know that there exists a constant $\tau > 1$ such that

$$L_1 \geq \frac{A_{15}}{\varphi(r^n)} (\log N)^{\tau}. \tag{2.36}$$

So by (2.32), (2.33), and (2.36), we get

$$A_{10} \geq A_{15} (\log N)^{\tau} - A_{12} \log N. \tag{2.37}$$

Then, $N \leq A(\Omega, p, n, \gamma)$. Theorem 1.2 is proved.

3. Proof of Theorem 1.3

Similar to the proof of Theorem 1.2, we only give the outline.

Suppose that $[b, \mu_S^{\varpi}]$ is a bounded operator on $L^{p,\varphi}(\mathbb{R}^n)$, we are going to prove that $b \in \mathrm{BMO}(\mathbb{R}^n)$.

We may assume that $\|[b, \mu_S^{\varpi}]\|_{L^{p,\varphi} \mapsto L^{p,\varphi}} = 1$. We want to prove that, for any $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, the inequality

$$N = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |b(y) - a_0| dy \leq B(\Omega, p, n, \varpi) \tag{3.1}$$

holds, where $a_0 = |B(x_0, r)|^{-1} \int_{B(x_0, r)} b(y) dy$. Since $[b - a_0, \mu_S^{\varpi}] = [b, \mu_S^{\varpi}]$, we may assume that $a_0 = 0$. Let $f(y)$ be as (2.3), then (2.4)–(2.8) hold. In this proof for $j = 1, \ldots, 13$, B_j is a positive constant depending only on Ω, p, n, ϖ, and B_i $(1 \leq i < j)$. Since Ω satisfies (1.2), then there exists a B_1 such that $0 < B_1 < 1$ and

$$\sigma\left(\left\{x' \in S^{n-1} : \Omega(x') \geq \frac{2C_1}{(\log(2/B_1))^{\gamma}}\right\}\right) > 0, \tag{3.2}$$

where σ is the measure on S^{n-1} which is induced from the Lebesgue measure on $\mathbb{R}^n$. By the condition (1.15), it is easy to see that

$$\Lambda := \left\{ x' \in S^{n-1} : \Omega(x') \geq \frac{2C_1}{(\log(2/B_1))^{\gamma}} \right\} \tag{3.3}$$

is a closed set. As the proof of (2.11), we can get the following:

$$\text{if } x' \in \Lambda \text{ and } y' \in S^{n-1}, \text{ satisfying } |x' - y'| \leq B_1, \text{ then } \Omega(y') \geq \frac{C_1}{(\log(2/B_1))^{\gamma}}. \tag{3.4}$$

Taking $B_2 > 3/B_1 + 1$, let

$$G = \{ x \in \mathbb{R}^n : |x - x_0| \geq B_2 r, \ (x - x_0)' \in \Lambda \}. \tag{3.5}$$

For $x \in G$, we have

$$\begin{aligned}
\left|[b, \mu_S^{\varpi}] f(x)\right| &= \left(\int_0^{\infty} \int_{|x-y|<t} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} (b(x) - b(z)) f(z) dz \right|^2 \frac{dy dt}{t^{n+1+2\varpi}} \right)^{1/2} \\
&\geq \Bigg(\int_{4|x-x_0|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \\
&\quad \times \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} (b(x) - b(z)) f(z) dz \right|^2 \frac{dy dt}{t^{n+1+2\varpi}} \Bigg)^{1/2} \\
&\geq \Bigg(\int_{4|x-x_0|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|, (y-x_0)' \in \Lambda}} \\
&\quad \times \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} b(z) f(z) dz \right|^2 \frac{dy dt}{t^{n+1+2\varpi}} \Bigg)^{1/2} \\
&\quad - |b(x)| \left(\int_{4|x-x_0|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} f(z) dz \right|^2 \frac{dy dt}{t^{n+1+2\varpi}} \right)^{1/2} \\
&:= I_1 - I_2.
\end{aligned} \tag{3.6}$$

For I_1, noting that if $|z-x_0|<r$, $|x-x_0|>B_2|z-x_0|$, and $|y-x_0|>2B_2|z-x_0|$, then we get

$$\left|(y-z)'-(y-x_0)'\right|\leq 2\frac{|z-x_0|}{|y-x_0|}\leq \frac{1}{B_2}<B_1. \tag{3.7}$$

Then by (3.4), we get $\Omega((y-z)')\geq C_1/(\log(2/B_1))^{\gamma}$. Since $4|x-x_0|>|y-x_0|+|z-x_0|\geq|y-z|\geq|y-x_0|-|z-x_0|>2|x-x_0|-|x-x_0|/2=3|x-x_0|/2$ and $4|x-x_0|>|x-y|\geq|y-x_0|-|x-x_0|>|x-x_0|$, we get $4|x-x_0|\geq|y-z|\geq 3|x-x_0|/2$ and $4|x-x_0|>|x-y|>|x-x_0|$. Thus, by (2.5), (2.7), (2.8), and the Hölder inequality, we get

$$\begin{aligned}
I_1 &\geq C\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t,\,(y-x_0)'\in\Lambda\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\int_{B(x_0,r)}\frac{\Omega(y-z)}{|y-z|^{n-\varpi}}b(z)f(z)\chi_{\{(|y-z|)<t\}}dz\frac{dydt}{t^{n+1+2\varpi}}\\
&\quad\times\left(\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t,\,(y-x_0)'\in\Lambda\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\frac{dydt}{t^{n+1+2\varpi}}\right)^{-1/2}\\
&\geq C|x-x_0|^{2\varpi-n}\int_{B(x_0,r)}b(z)f(z)\int_{\substack{(y-x_0)'\in\Lambda\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\int_{\substack{4|x-x_0|<t,\,|x-y|<t\\ |y-z|<t}}\frac{dtdy}{t^{n+1+2\varpi}}dz\\
&= C|x-x_0|^{2\varpi-n}\int_{B(x_0,r)}b(z)f(z)\int_{\substack{(y-x_0)'\in\Lambda\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\int_{4|x-x_0|<t}\frac{dtdy}{t^{n+1+2\varpi}}dz\\
&\geq C|x-x_0|^{-n}\int_{B(x_0,r)}b(z)f(z)dz\\
&= B_3Nr^n|x-x_0|^{-n}.
\end{aligned} \tag{3.8}$$

By (2.5) and (2.6), we have

$$\begin{aligned}
I_2 = |b(x)|\Bigg(\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\\
\times\left|\int_{\mathbb{R}^n}\left(\frac{\Omega(y-z)}{|y-z|^{n-\varpi}}\chi_{\{|y-z|<t\}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\varpi}}\chi_{\{|y-x_0|<t\}}\right)f(z)dz\right|^2\frac{dydt}{t^{n+1+2\varpi}}\Bigg)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq |b(x)|\Bigg(\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\\
&\qquad\times\left|\int_{\substack{|y-z|<t\\ |y-x_0|<t}}\left(\frac{\Omega(y-z)}{|y-z|^{n-\varpi}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\varpi}}\right)f(z)dz\right|^2\frac{dydt}{t^{n+1+2\varpi}}\Bigg)^{1/2}\\
&\quad+|b(x)|\left(\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\left|\int_{\substack{|y-z|<t\\ |y-x_0|\geq t}}\frac{\Omega(y-z)}{|y-z|^{n-\varpi}}f(z)dz\right|^2\frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2}\\
&\quad+|b(x)|\left(\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\left|\int_{\substack{|y-z|\geq t\\ |y-x_0|<t}}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\varpi}}f(z)dz\right|^2\frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2}\\
&:= I_2^1+I_2^2+I_2^3.
\end{aligned}
\tag{3.9}
$$

In I_2^2, we have $t \leq |y-x_0| < 3|x-x_0|$ and $t \geq 4|x-x_0|$. In I_2^3, we get $t \leq |y-z| < 4|x-x_0|$ and $t \geq 4|x-x_0|$. It is easy to see that $I_2^2 = I_2^3 = 0$. Now, we estimate I_2^1, by $\Omega \in L^\infty(S^{n-1})$, the Minkowski inequality, Lemma 2.1 for $|y-x_0| > 2|z-x_0|$, and (2.4), we get

$$
\begin{aligned}
I_2^1 &\leq C|b(x)|\int_{B(x_0,r)}|f(z)|dz\Bigg(\int_{2|x-x_0|<|y-x_0|<3|x-x_0|}\int_{\substack{4|x-x_0|\leq t,\,|y-z|<t\\ |y-x_0|<t,\,|x-y|\leq t}}\\
&\qquad\times\frac{1}{|y-x_0|^{2(n-\varpi)}(\log(|y-x_0|/r))^{2\gamma}}\frac{dtdy}{t^{n+1+2\varpi}}\Bigg)^{1/2}\\
&\leq B_4|b(x)|r^n|x-x_0|^{-n}\left(\log\frac{|x-x_0|}{r}\right)^{-\gamma}.
\end{aligned}
\tag{3.10}
$$

From (3.9) and (3.10), we get

$$
I_2 \leq B_4|b(x)|r^n|x-x_0|^{-n}\left(\log\frac{|x-x_0|}{r}\right)^{-\gamma}. \tag{3.11}
$$

Let

$$
F = \left\{x \in G : |b(x)| > \frac{B_3N}{2B_4}\left(\log\frac{|x-x_0|}{r}\right)^{\gamma},\ |x-x_0| < N^{1/n}r\right\}. \tag{3.12}
$$

Without loss of generality, we may assume that $N > B_2 > 1$, otherwise, we get the desired result. Since $\varphi(t)$ is nonincreasing, we have $\varphi(|B(x_0, N^{1/n}r)|) \le \varphi(|B(x_0, r)|) = \varphi(r^n)$. Then by, (3.6), (3.8), and (3.11), we get

$$\begin{aligned}
\|f\|_{L^{p,\varphi}}^p &\ge \left\|[b,\mu_S^{\varpi}]f\right\|_{L^{p,\varphi}}^p \\
&\ge \frac{1}{(\varphi(|B(x_0,N^{1/n}r)|))^p|B(x_0,N^{1/n}r)|}\int_{|x-x_0|<N^{1/n}r}\left|[b,\mu_S^{\varpi}]f(x)\right|^p dx \\
&\ge \frac{1}{(\varphi(r^n))^p Nr^n}\int_{(G\setminus F)\cap\{|x-x_0|<N^{1/n}r\}}\left(\frac{1}{2}B_3Nr^n|x-x_0|^{-n}\right)^p dx \\
&\ge \frac{1}{(\varphi(r^n))^p Nr^n}\int_{\{B_5(|F|+(B_2r)^n)^{1/n}<|x-x_0|<N^{1/n}r\}\cap G}\left(\frac{1}{2}B_3Nr^n|x-x_0|^{-n}\right)^p dx \\
&= \frac{1}{(\varphi(r^n))^p Nr^n}\left(\frac{B_3Nr^n}{2}\right)^p\int_{B_5\left(|F|+(B_2r)^n\right)^{1/n}}^{N^{1/n}r} t^{-pn+n-1}dt\int_{\Lambda}J(x')d\sigma(x') \\
&\ge \frac{1}{(\varphi(r^n))^p}\sigma(\Lambda)(Nr^n)^{p-1}\frac{(B_3/2)^p}{n-np}\left(N^{1-p}r^{n(1-p)}-B_5^{(1-p)n}\left(|F|+(B_2r)^n\right)^{1-p}\right).
\end{aligned} \tag{3.13}$$

Thus,

$$\left(|F|+(B_2r)^n\right)^{1-p} \le B_6N^{1-p}r^{n(1-p)}\left(1+(\varphi(r^n))^p\|f\|_{L^{p,\lambda}}^p\right). \tag{3.14}$$

Then, by (2.20) and (3.14), we get

$$|F|+(B_2r)^n \ge B_7Nr^n. \tag{3.15}$$

If $N \le 2B_7^{-1}B_2^n$, then Theorem 1.3 is proved. If $N > 2B_7^{-1}B_2^n$, then

$$|F| \ge \frac{B_7}{2}Nr^n. \tag{3.16}$$

Let $g(y) = \chi_{B(x_0,r)}(y)$. For $x \in F$, we have

$$\begin{aligned}
\left|[b,\mu_S^{\varpi}]g(x)\right| &= \left(\int_0^\infty \int_{|x-y|<t} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}(b(x)-b(z))g(z)dz\right|^2 \frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2} \\
&\geq \left(\int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}(b(x)-b(z))g(z)dz\right|^2 \frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2} \\
&\geq |b(x)|\left(\int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|,\,(y-x_0)'\in\Lambda}} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}g(z)dz\right|^2 \frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2} \\
&\quad - \left(\int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}b(z)g(z)dz\right|^2 \frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2} \\
&:= K_1 - K_2.
\end{aligned} \tag{3.17}$$

For K_1, as above mentioned, we have $\Omega((y-z)') \geq C_1/(\log(2/B_1))^{\gamma}$. Since $4|x-x_0| \geq |y-z| \geq 3|x-x_0|/2$ and $4|x-x_0| > |x-y| > |x-x_0|$, it follows the Hölder inequality that

$$\begin{aligned}
K_1 &= |b(x)|\left\{\int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t,\,(y-x_0)'\in\Lambda \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \left(\int_{B(x_0,r)} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}\chi_{\{(|y-z|)<t\}}dz\right)^2 \frac{dydt}{t^{n+1+2\varpi}}\right\}^{1/2} \\
&\geq |b(x)|\int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t,\,(y-x_0)'\in\Lambda \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \int_{B(x_0,r)} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}\chi_{\{(|y-z|)<t\}}dz\frac{dydt}{t^{n+1+2\varpi}} \\
&\quad \times \left(\int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t,\,(y-x_0)'\in\Lambda \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \frac{dydt}{t^{n+1+2\varpi}}\right)^{-1/2} \\
&\geq C|b(x)||x-x_0|^{2\varpi-n}\int_{B(x_0,r)} \int_{\substack{(y-x_0)'\in\Lambda \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \int_{4|x-x_0|<t} \frac{dtdy}{t^{n+1+2\varpi}}dz \\
&\geq B_8 N|x-x_0|^{-n}\int_{B(x_0,r)} dz \\
&= B_8 N r^n |x-x_0|^{-n}.
\end{aligned} \tag{3.18}$$

By $\Omega \in L^\infty(S^{n-1})$, the Minkowski inequality, and $|x-x_0| \simeq |y-z|$ for $2|x-x_0| < |y-x_0| < 3|x-x_0|$ and $z \in B(x_0, r)$, we get

$$\begin{aligned}
K_2 &= \left(\int_{4|x-x_0|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \left| \int_{B(x_0,r)} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} b(z) \chi_{\{|y-z|<t\}} dz \right|^2 \frac{dy\,dt}{t^{n+1+2\varpi}} \right)^{1/2} \\
&\leq \frac{C}{|x-x_0|^{n-\varpi}} \int_{B(x_0,r)} |b(z)| dz \left(\int_{2|x-x_0|<|y-x_0|<3|x-x_0|} \int_{4|x-x_0|}^{\infty} \frac{dt\,dy}{t^{n+1+2\varpi}} \right)^{1/2} \\
&\leq B_9 |x-x_0|^{-n} \int_{B(x_0,r)} |b(z)| dz \\
&= B_9 N r^n |x-x_0|^{-n}.
\end{aligned} \tag{3.19}$$

Thus, by (3.17), (3.18), and (3.19), we get, for $x \in F$,

$$\left| [b, \mu_S^{\varpi}] g(x) \right| \geq B_8 r^n |b(x)| |x-x_0|^{-n} - B_9 N r^n |x-x_0|^{-n}. \tag{3.20}$$

Thus, by (3.20), $\varphi(Nr^n) \leq \varphi(r^n)$, $|b(x)| > (NB_3/2B_4)(\log(|x-x_0|/r))^\gamma$ when $x \in F$ and the Hölder inequality, we have

$$\begin{aligned}
\frac{B_{10}}{\varphi(r^n)} &\geq \|g\|_{L^{p,\varphi}} \geq \left\| [b, \mu_S^{\varpi}] g \right\|_{L^{p,\varphi}} \\
&\geq \frac{1}{\varphi(Nr^n)(Nr^n)^{1/p}} \left(\int_{|x-x_0|<N^{1/n}r} \left| [b, \mu_S^{\varpi}] g(x) \right|^p dx \right)^{1/p} \\
&\geq \frac{1}{\varphi(r^n)(Nr^n)^{1/p}} \int_{|x-x_0|<N^{1/n}r} \left| [b, \mu_S^{\varpi}] g(x) \right| dx \left(\int_{|x-x_0|<N^{1/n}r} dx \right)^{-1/p'} \\
&\geq \frac{1}{\varphi(r^n) N r^n} \int_F \left| [b, \mu_S^{\varpi}] g(x) \right| dx \\
&\geq \frac{B_8 r^n}{\varphi(r^n) N r^n} \int_F |b(x)| |x-x_0|^{-n} dx - \frac{B_9 N r^n}{\varphi(r^n) N r^n} \int_F |x-x_0|^{-n} dx \\
&\geq \frac{B_{11}}{\varphi(r^n)} \int_F \left(\log \frac{|x-x_0|}{r} \right)^\gamma |x-x_0|^{-n} dx \\
&\quad - \frac{B_9}{\varphi(r^n)} \int_F |x-x_0|^{-n} dx \\
&:= L_1 - L_2.
\end{aligned} \tag{3.21}$$

As the proof of (2.33) and (2.36), we can get that there exists a constant $\tau > 1$ such that

$$L_1 \geq \frac{B_{12}}{\varphi(r^n)} (\log N)^{\tau},$$
$$L_2 \leq \frac{B_{13}}{\varphi(r^n)} \log N. \tag{3.22}$$

So, by (3.21) and (3.22), we get

$$B_{10} \geq B_{12} (\log N)^{\tau} - B_{13} \log N. \tag{3.23}$$

Then, $N \leq B(\Omega, p, n, \varpi)$. Theorem 1.3 is proved.

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