Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 947518, 13 pages
doi:10.1155/2010/947518

Research Article

Decomposition of Polyharmonic Functions with Respect to the Complex Dunkl Laplacian

Guangbin Ren[1] and Helmuth R. Malonek[2]

[1] *Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China*
[2] *Departamento de Matemática, Universidade de Aveiro, 3810-193 Aveiro, Portugal*

Correspondence should be addressed to Helmuth R. Malonek, hrmalon@ua.pt

Received 28 December 2009; Accepted 26 March 2010

Academic Editor: Yuming Xing

Let Ω be a G-invariant convex domain in $\mathbb{C}^N$ including 0, where G is a complex Coxeter group associated with reduced root system $R \subset \mathbb{R}^N$. We consider holomorphic functions f defined in Ω which are Dunkl polyharmonic, that is, $(\Delta_h)^n f = 0$ for some integer n. Here $\Delta_h = \sum_{j=1}^N \mathfrak{D}_j^2$ is the complex Dunkl Laplacian, and $\mathfrak{D}_j$ is the complex Dunkl operator attached to the Coxeter group G, $\mathfrak{D}_j f(z) = (\partial f/\partial z_j)(z) + \sum_{v\in R_+} \kappa_v((f(z)-f(\sigma_v z))/\langle z, v\rangle)v_j$, where κ_v is a multiplicity function on R and σ_v is the reflection with respect to the root v. We prove that any complex Dunkl polyharmonic function f has a decomposition of the form $f(z) = f_0(z) + (\sum_{n=1}^N z_j^2) f_1(z) + \cdots + (\sum_{n=1}^N z_j^2)^{n-1} f_{n-1}(z)$, for all $z \in \Omega$, where f_j are complex Dunkl harmonic functions, that is, $\Delta_h f_j = 0$.

1. Introduction

A fundamental result in the theory of polyharmonic functions is the celebrated Almansi theorem [1–3], which shows that for any polyharmonic function f of degree n in a starlike domain D in $\mathbb{R}^N$ with center 0, that is,

$$(\Delta_{\mathbb{R}})^n f := \left(\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}\right)^n, \quad f = 0, \tag{1.1}$$

there exist uniquely harmonic functions $f_0, \ldots, f_{n-1}$ such that

$$f(x) = f_0(x) + |x|^2 f_1(x) + \cdots + |x|^{2(n-1)} f_{n-1}(x), \quad \forall x \in D. \tag{1.2}$$

The Almansi formula is a genuine analogy to the Taylor formula:

$$f(t) = f(0) + t\frac{f'(0)}{1!} + \cdots + t^n \frac{f^{(n)}(0)}{n!} + \cdots . \tag{1.3}$$

Compared with the Taylor formula, the Almansi formula is obtained by the scheme

$$\frac{d}{dt} \longmapsto \Delta_{\mathbb{R}}, \tag{1.4}$$

and since the constants $f^{(n)}(0)/n!$ are solutions of $(d/dt)(f^{(n)}(0)/n!) = 0$, they are replaced by the solutions of the Laplace equation $\Delta_{\mathbb{R}} f_k = 0$.

In [1], Aronszajn et al. indicated some applications of the Almansi formula in several complex variables. Its most eminent application is in spherical harmonic function theory [4, 5]. The polyharmonic functions have also applications in the theory of elasticity [6], in radar imaging [7], and in multivariate approximation [8, 9].

The purpose of this article is to extend Almansi's theorem to the theory of complex Dunkl harmonics. The theory of Dunkl harmonics developed by Dunkl [10–13] is an extension of the theory of ordinary harmonics. In 1989, Dunkl [10] constructed for each Coxeter group a family of commutative differential-difference operators $\mathfrak{D}_j$, called Dunkl operators, which can be considered as perturbations of the usual partial derivatives by reflection parts. These operators step from the analysis of quantum many body system of Calogero-Moser-Sutherland type [14] in mathematical physics. They also have roots in the theory of special functions of several variables. With Dunkl operators in place of the usual partial derivatives, one can define the Laplacian in the Dunkl setting, which is a parametrized operator and invariant under reflection groups. These parametrized Laplacian suggests the theory of Dunkl harmonics. In [15], we obtained the Almansi decomposition for the real Dunkl operator. Now we continue to consider the Almansi decomposition for the complex Dunkl operator.

As a direct consequence, we will show that the Almansi Theorem implies the Gauss decomposition of the homogeneous polynomials into complex Dunkl harmonics.

We need some notations before stating our main result. Let R be a root system in $\mathbb{R}^N$ and G the associated Coxeter group. Let $\kappa : R \to \mathbb{C}$ be a fixed multiplicity function $v \mapsto \kappa_v$ on R. Fix a positive subsystem R_+ of R, and denote $\gamma = \gamma_\kappa := \sum_{v\in R_+} \kappa_v$. We will always assume that

$$\operatorname{Re} \gamma_\kappa > -\frac{N}{2}. \tag{1.5}$$

Let $\mathfrak{D}_j$ be the Dunkl operator attached to the Coxeter group G and the multiplicity function κ, defined by (see [16])

$$\mathfrak{D}_j f(z) = \frac{\partial f}{\partial z_j}(z) + \sum_{v\in R_+} \kappa_v \frac{f(z) - f(\sigma_v z)}{\langle z, v\rangle} v_j, \tag{1.6}$$

where σ_v denotes the reflection in the hyperplane orthogonal to v.

The Dunkl operators enjoy the regularity property: if $f \in H(\Omega)$, the space of holomorphic functions in Ω, then $\mathfrak{D}_i f \in H(\Omega)$. This follows immediately from the formula

$$\frac{f(z)-f(\sigma_v z)}{\langle z,v\rangle} = \int_0^1 \left\langle \nabla f(t\sigma_v z + (1-t)z), \frac{2v}{|v|^2} \right\rangle dt \tag{1.7}$$

for any $f \in H(\Omega)$ and $v \in R$.

The Dunkl Laplacian is defined as

$$\Delta_h = \mathfrak{D}_1^2 + \cdots + \mathfrak{D}_N^2, \tag{1.8}$$

more precisely,

$$\Delta_h f(z) = \Delta f(z) + 2\sum_{v\in R_+} \kappa_v \frac{\langle \nabla f(z), v\rangle}{\langle z,v\rangle} - 2\sum_{v\in R_+} \kappa_v \frac{f(z)-f(\sigma_v z)}{\langle z,v\rangle^2}|v|^2. \tag{1.9}$$

Here Δ and ∇ are the complex Laplacian and gradient operator:

$$\begin{gathered} \Delta := \Delta_{\mathbb{C}} = \frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}, \\ \nabla = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right). \end{gathered} \tag{1.10}$$

Throughout this paper we let Ω be a G-invariant convex domain in $\mathbb{C}^N$ including 0, that is, $G(\Omega) \subset \Omega$, $0 \in \Omega$, and $tx + (1-t)y \in \Omega$ for all $t \in [0,1]$ and $x, y \in \Omega$. This class of domain turns out to be natural for the Almansi decomposition. It is known that Δ_h is a regular operator in such a domain. Namely, if $f \in H(\Omega)$, then $\Delta_h f \in H(\Omega)$.

Definition 1.1. A holomorphic function $f : \Omega \to \mathbb{C}$ is *Dunkl polyharmonic* of degree n if $(\Delta_h)^n f = 0$. If $n = 1$, it is called *Dunkl harmonic* function.

Let I be the identity operator. For any $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$ we define the operator $I_s : H(\Omega) \to H(\Omega)$ by

$$I_s f(x) = \int_0^1 f(tx)t^{s-1} dt. \tag{1.11}$$

If f is Dunkl harmonic in Ω, then so is $I_s f$. For any $j \in \mathbb{N}$, by assumption (1.5) we can introduce the operator:

$$Q_j = \frac{1}{4^j j!} I_{(N+2(j-1))/2+\gamma_\kappa} I_{(N+2(j-2))/2+\gamma_\kappa} \cdots I_{N/2+\gamma_\kappa}. \tag{1.12}$$

For any $z \in \mathbb{C}^N$ and $j \in \mathbb{N}$, we denote

$$z^2 = z_1^2 + \cdots + z_N^2, \qquad z^{2j} = \left(z^2\right)^j = \left(z_1^2 + \cdots + z_N^2\right)^j. \tag{1.13}$$

Our main result is the following theorem.

Theorem 1.2. *Assume that R is a root system in $\mathbb{R}^N$ and G its associated complex Coxter group. Let Ω be a G-invariant convex domain in $\mathbb{C}^N$ including 0. If f is a Dunkl polyharmonic function in Ω of degree n, then there exist uniquely Dunkl harmonic functions $f_0, \ldots, f_{n-1}$ such that*

$$f(z) = f_0(z) + z^2 f_1(z) + \cdots + z^{2(n-1)} f_{n-1}(z), \quad \forall x \in \Omega. \tag{1.14}$$

Moreover the Dunkl harmonic functions $f_0, \ldots, f_{n-1}$ are given by the following formulae:

$$\begin{aligned} f_0 &= \left(I - z^2 Q_1 \Delta_h\right)\left(I - z^4 Q_2 \Delta_h^2\right) \cdots \left(I - z^{2(n-1)} Q_{n-1} \Delta_h^{n-1}\right) f(z) \\ f_1 &= Q_1 \Delta_h \left(I - z^4 Q_2 \Delta_h^2\right) \cdots \left(I - z^{2(n-1)} Q_{n-1} \Delta_h^{n-1}\right) f(z) \\ &\vdots \\ f_{n-2} &= Q_{n-2} \Delta_h^{n-2} \left(I - z^{2(n-1)} Q_{n-1} \Delta_h^{n-1}\right) f(z) \\ f_{n-1} &= Q_{n-1} \Delta_h^{n-1} f(z). \end{aligned} \tag{1.15}$$

Conversely, the sum in (1.14)*, with $f_0, \ldots, f_{n-1}$ Dunkl harmonic in Ω, defines a Dunkl polyharmonic function in Ω of degree n.*

Remark 1.3. By the Scheme in (1.4), we know that the formulae of f_j above play the role of Taylor coefficient formulae. These formulae are new even in the classical case $\kappa = 0$.

2. Preliminaries

Let us recall some notation in the theory of Dunkl harmonics; see [16, 17]. Concerning root system and reflection groups, see [18].

A root system R is a finite set of nonzero vectors in $\mathbb{R}^N$ such that $\sigma_v R = R$ and $R \cap \mathbb{R}v = \{\pm v\}$ for all $v \in R$.

The positive subsystem R_+ is a subset of R such that $R = R_+ \cup (-R_+)$, where R_+ and $-R_+$ are separated by a hyperplane through the origin.

For a nonzero vector $v \in \mathbb{C}^N$, the reflection σ_v in the hyperplane orthogonal to v is defined by

$$\sigma_v z := z - 2\frac{\langle z, v\rangle}{|v|^2} v, \quad z \in \mathbb{C}^N, \tag{2.1}$$

where the symbol $\langle z, v\rangle = \sum_{j=1}^N z_j \overline{v}_j$ and $|z|^2 = \langle z, z\rangle$.

The Coxeter group G (or the finite reflection group) generated by the root system R is the subgroup of the unitary group $U(N)$ generated by $\{\sigma_u : u \in R\}$.

A multiplicity function κ_v is a G-invariant complex valued function defined on R, that is, $\kappa_v = \kappa_{gv}$ for all $g \in G$.

Notice that Dunkl operators were studied in literature for $\operatorname{Re}\kappa_v \geq 0$.

The Dunkl operator $\mathfrak{D}_j$, associated with the Coxeter group G and the multiplicity function κ, is the first-order differential-difference operator. The remarkable property of Dunkl operators is that they are commutative:

$$\mathfrak{D}_i\mathfrak{D}_j = \mathfrak{D}_j\mathfrak{D}_i. \tag{2.2}$$

The Dunkl Laplacian $\Delta_h = \sum_{j=1}^{N} \mathfrak{D}_j^2$ can be split into three parts

$$\Delta_h = \Delta + G_h + D_h \tag{2.3}$$

with

$$\begin{aligned} G_h f(z) &= 2\sum_{v\in R_+} \kappa_v \frac{\langle \nabla f(z), v\rangle}{\langle z, v\rangle}; \\ D_h f(z) &= -2\sum_{v\in R_+} \kappa_v \frac{f(z) - f(\sigma_v z)}{\langle z, v\rangle^2}|v|^2. \end{aligned} \tag{2.4}$$

When $\kappa = 0$, the Dunkl Laplacian Δ_h is just the ordinary complex Laplacian Δ.

Consider the natural action of $U(N)$ on functions $f : \mathbb{C}^N \to \mathbb{C}$, given by $gf(z) = f(g^{-1}z)$. The Dunkl Laplacian Δ_h is G-invariant, that is,

$$g \circ \Delta_h = \Delta_h \circ g, \quad \forall g \in G. \tag{2.5}$$

Example 2.1. Let N be an integer and $N \geq 2$. Since we need to consider the sum $i \neq j$ and i runs from 1 to N, this forces $N \geq 2$. Take the Coxeter group $G = S_N$, which is the symmetric group in N elements, acting on $\mathbb{R}^N$ by permuting the standard basis $e_1, \ldots, e_N$ (see [17, page 289]). We regard the transposition (ij) in S_N as a reflection σ_{ij} such that

$$\sigma_{ij}(e_i - e_j) = -(e_i - e_j). \tag{2.6}$$

Therefore, S_N is a finite reflection generated by σ_{ij} with a root system

$$R = \{\pm(e_i - e_j) : 1 \leq i < j \leq N\}. \tag{2.7}$$

As all transpositions are conjugate in S_N, the vector space of multiplicity function is one dimensional. The complex Dunkl operators associated with the multiplicity parameters $\kappa \in \mathbb{C}$ are given by

$$\mathfrak{D}_j f(z) = \frac{\partial f}{\partial z_j}(z) + \kappa \sum_{i\neq j} \frac{f(z) - f(\sigma_{ij}z)}{z_i - z_j}, \tag{2.8}$$

where σ_{ij} acts on $\mathbb{C}^N$ by interchanging z_i and z_j; more precisely, $\sigma_{ij}z = (\sigma_{ij}z_1, \ldots, \sigma_{ij}z_n)$ with $\sigma_{ij}z_i = z_j$, $\sigma_{ij}z_j = z_i$, and $\sigma_{ij}z_k = z_k$ for any $k \neq i, j$.

In this case, the condition (1.5) of the main theorem reduces to

$$\kappa > -\frac{1}{N-1}. \tag{2.9}$$

Example 2.2. In the one-dimensional case $N = 1$, the root system R is of type A_1, the reflection group $G = \mathbb{Z}_2$, and the multiplicity function is given by a single parameter $\kappa \in \mathbb{C}$. The Dunkl operator $\mathfrak{D} := \mathfrak{D}_1$ and the Dunkl Laplacian Δ_h are given, respectively, by

$$\begin{gathered}\mathfrak{D}f(z) = f'(z) + \kappa\frac{f(z) - f(-z)}{z}, \\ \Delta_h f(z) = f''(z) + 2\kappa\frac{f'(z)}{z} - 2\kappa\frac{f(z) - f(-z)}{z^2}.\end{gathered} \tag{2.10}$$

If f is an even function, then the third term in the formula of $\Delta_h f$ vanishes, while the sum of the first two items provides a singular Sturm-Liouville operator.

3. Proof of the Main Theorem

Before proving Theorem 1.2, we need some lemmas.

Denote

$$R_s = sI + \sum_{j=1}^{N} z_j \frac{\partial}{\partial z_j}. \tag{3.1}$$

We write R instead of R_0 when $s = 0$.

Lemma 3.1. *If $s \in \mathbb{C}$, $\operatorname{Re} s > 0$, and $f \in H(\Omega)$, then*

$$f(z) = I_s R_s f(z) = R_s I_s f(z). \tag{3.2}$$

Proof. For any $s \in \mathbb{C}$, $\operatorname{Re} s > 0$ and $f \in H(\Omega)$,

$$f(z) = \int_0^1 \frac{d}{dt}\left(t^s f(tz)\right) dt. \tag{3.3}$$

By direct calculation

$$\frac{d}{dt}\left(t^s f(tz)\right) = s t^{s-1} f(tz) + t^{s-1}\left(\sum_{i=1}^{N} w_i \frac{\partial f}{\partial w_i}\right)(tz), \tag{3.4}$$

where $w_i = tz_i$. Therefore

$$f(z) = \int_0^1 \left(sf(tz) + \left(\sum_{i=1}^{N} w_i \frac{\partial f}{\partial w_i} \right)(tz) \right) t^{s-1} dt;$$
$$f(z) = s\int_0^1 f(tz)t^{s-1}dt + \left(\sum_{i=1}^{N} w_i \frac{\partial}{\partial w_i} \right) \int_0^1 f(tz)t^{s-1}dt. \tag{3.5}$$

From the above two identities and the definitions of I_s and R_s, we have $f(z) = I_s R_s f(z)$ and $f(z) = R_s I_s f(z)$. □

Lemma 3.2. *If $f \in H(\Omega)$, then for any $s \in \mathbb{C}$, $\operatorname{Re} s > 0$, and $z \in \Omega$*

$$\Delta_h I_s f(z) = I_{s+2} \Delta_h f(z). \tag{3.6}$$

Proof. By definition, we have for a.e. $z \in \Omega$

$$\begin{aligned} G_h I_s f(z) &= 2 \sum_{v \in R_+} \kappa_v \frac{\langle \nabla (I_s f(z)), v \rangle}{\langle z, v \rangle} \\ &= 2 \sum_{v \in R_+} \kappa_v \frac{1}{\langle z, v \rangle} \int_0^1 \sum_{i=1}^{N} \frac{\partial f}{\partial z_i}(tz) \overline{v}_i t^s dt \\ &= \int_0^1 2 \sum_{v \in R_+} \kappa_v \frac{\langle \nabla f(tz), v \rangle}{\langle tz, v \rangle} t^{s+1} dt \\ &= \int_0^1 G_h f(tz) t^{s+1} dt \\ &= I_{s+2} G_h f(z). \end{aligned} \tag{3.7}$$

Similarly

$$\begin{aligned} D_h I_s f(z) &= -2 \sum_{v \in R_+} \kappa_v \frac{(I_s f)(z) - (I_s f)(\sigma_v z)}{\langle z, v \rangle^2} |v|^2 \\ &= -2 \sum_{v \in R_+} \kappa_v \frac{|v|^2}{\langle z, v \rangle^2} \int_0^1 (f(tz) - f(t\sigma_v z)) t^{s-1} dt \\ &= \int_0^1 -2 \sum_{v \in R_+} \kappa_v \frac{|v|^2}{\langle tz, v \rangle^2} (f(tz) - f(t\sigma_v z)) t^{s+1} dt \\ &= I_{s+2} D_h f(z). \end{aligned} \tag{3.8}$$

It is also easy to see

$$\Delta I_s f(z) = I_{s+2}\Delta f(z). \tag{3.9}$$

Since $\Delta_h = \Delta + G_h + D_h$, it follows that $\Delta_h I_s f(z) = I_{s+2}\Delta_h f(z)$ for a.e. $z \in \Omega$. From the regularity property of Dunkl operators, Δ_h maps $C^2(\Omega)$ into $C(\Omega)$. By the continuity, Lemma 3.2 follows. □

Lemma 3.3. *Let $H_1 = \{f \in H(\Omega) : \Delta_h f = 0\}$. If $s > 0$ and Q_j as in* (1.12), *then*

$$R_s H_1 = H_1, \qquad I_s H_1 = H_1, \qquad Q_j H_1 = H_1. \tag{3.10}$$

Proof. Note that (3.6) implies

$$R_{s+2}\Delta_h f(z) = \Delta_h R_s f(z), \quad z \in \Omega. \tag{3.11}$$

Indeed, from Lemma 3.1, $R_{s+2}\Delta_h = R_{s+2}\Delta_h I_s R_s = R_{s+2} I_{s+2}\Delta_h R_s = \Delta_h R_s$. As direct consequence of (3.6) and (3.11), we find that $I_s f$ and $R_s f$ are Dunkl harmonic, whenever f is Dunkl harmonic. From the definition of Q_j, we thus obtain $Q_j H_1 = H_1$. □

Lemma 3.4. *Let $g \in H(\Omega)$, $j \in \mathbb{N}$, Then for any $z \in \Omega$*

$$\Delta_h\left(z^{2j} g(x)\right) = z^{2j}\Delta_h g(x) + 4jz^{2(j-1)} R_{(N+2j-2)/2+\gamma} + 2jg(z). \tag{3.12}$$

Proof. For any $f, g \in H(\Omega)$

$$\Delta(fg) = (\Delta f)g + 2\langle \nabla f, \nabla g\rangle + f(\Delta g). \tag{3.13}$$

Take $f(z) = z^{2j}$ and apply identities $(\partial/\partial z_i)(z^{2j}) = 2jz_i z^{2(j-1)}$ and $\Delta(z^{2j}) = 2j(N+2j-2)z^{2(j-1)}$ to yield

$$\Delta\left(z^{2j} g\right) = z^{2j}\Delta g + 4jz^{2(j-1)} R_{(N+2j-2)/2}\, g. \tag{3.14}$$

By our assumption $R_+ \subset \mathbb{R}^N$. Therefore

$$v = \overline{v}, \quad v \in R_+. \tag{3.15}$$

As a result,

$$z^2 = (\sigma_v z)^2 \tag{3.16}$$

for any $z \in \Omega$ and $v \in R_+$. Indeed

$$\begin{aligned}(\sigma_v z)^2 &= \sum_{j=1}^{N}\left(z_j - 2\frac{\langle z, v\rangle}{|v|^2}v_j\right)^2\\ &= z^2 - 4\frac{\langle z, v\rangle\langle z, \overline{v}\rangle}{|v|^2} + 4\frac{\langle z, v\rangle^2\langle v, \overline{v}\rangle}{|v|^4}\\ &= z^2.\end{aligned} \tag{3.17}$$

Then

$$\begin{aligned}D_h\left(z^{2j}g(z)\right) &= -2\sum_{v\in R_+}\kappa_v\frac{z^{2j}g(z) - (\sigma_v z)^{2j}g(\sigma_v z)}{\langle z, v\rangle^2}|v|^2\\ &= z^{2j}D_h g(z).\end{aligned} \tag{3.18}$$

By definition, we have

$$\begin{aligned}G_h\left(z^{2j}g\right) &= 2\sum_{v\in R_+}\kappa_v\frac{\langle\nabla\left(z^{2j}g\right), v\rangle}{\langle z, v\rangle}\\ &= z^{2j}G_h(g) + 2\sum_{v\in R_+}\kappa_v\frac{\langle\nabla\left(z^{2j}\right), v\rangle}{\langle z, v\rangle}g\\ &= z^{2j}G_h(g) + 4j\gamma z^{2(j-1)}g(z).\end{aligned} \tag{3.19}$$

Since $\Delta_h = \Delta + G_h + D_h$, summing up the above identity leads to identity (3.12) for $z \in \Omega$. □

Lemma 3.5. *For any complex Dunkl harmonic function f in Ω,*

$$\Delta_h^n z^{2n} Q_n f(z) = f(z), \quad z \in \Omega. \tag{3.20}$$

Proof. From (1.12) and Lemma 3.1, we know that

$$Q_n^{-1} = 4^n n! R_{(N+2(n-1))/2+\gamma} R_{(N+2(n-2))/2+\gamma} \cdots R_{N/2+\gamma}. \tag{3.21}$$

Denote $g = Q_n f$. Then g is Dunkl harmonic in Ω due to (3.10), and

$$f(z) = 4^n n! R_{(N+2(n-1))/2+\gamma} R_{(N+2(n-2))/2+\gamma} \cdots R_{N/2+\gamma} g(z). \tag{3.22}$$

We need to show

$$\Delta_h^n z^{2n} g(z) = 4^n n! R_{(N+2(n-1))/2+\gamma} R_{(N+2(n-2))/2+\gamma} \cdots R_{N/2+\gamma} g(z) \tag{3.23}$$

for any Dunkl harmonic function g in Ω and $n \in \mathbb{N}$.

Let g be Dunkl harmonic in Ω and $n \in \mathbb{N}$. Then Lemma 3.4 shows

$$\Delta_h\left(z^{2n}g(z)\right) = 4nz^{2(n-1)}R_{(N+2(n-1))/2+\gamma}g(z). \tag{3.24}$$

We use induction on n to prove (3.23). It is easy to prove when $n = 1$. For the general case, from (3.24) we have

$$\begin{aligned}\Delta_h^n\left(z^{2n}g(z)\right) &= \Delta_h^{n-1}\left(\Delta_h\left(z^{2n}g(z)\right)\right)\\ &= 4n\Delta_h^{n-1}\left(z^{2(n-1)}R_{(N+2(n-1))/2+\gamma}g(z)\right).\end{aligned} \tag{3.25}$$

Equation (3.23) follows directly from the assumption of induction. □

Now we come to the proof of our main theorem.

Proof of Theorem 1.2. Denote $H_n = \{f \in H(\Omega) : (\Delta_h)^n f = 0\}$. It is sufficient to show that

$$H_n = H_{n-1} + T_{n-1}H_1, \quad n \in \mathbb{N}, \tag{3.26}$$

where $T_n = z^{2n}I$. Notice that Lemma 3.5 states that

$$\Delta_h^n T_n Q_n = I. \tag{3.27}$$

We split the proof into two parts.

(i) $H_n \supset H_{n-1} + T_{n-1}H_1$. Since $H_{n-1} \subset H_n$, we need only to show $T_{n-1}H_1 \subset H_n$. For any $g \in H_1$, by (3.27) and (3.10) we have

$$\Delta_h^n(T_{n-1}g) = \Delta_h\left(\Delta_h^{n-1}T_{n-1}Q_{n-1}\right)Q_{n-1}^{-1}g = \Delta_h Q_{n-1}^{-1}g = 0. \tag{3.28}$$

(ii) $H_n \subset H_{n-1} + T_{n-1}H_1$. For any $f \in H_n$, we have the decomposition

$$f = \left(I - T_{n-1}Q_{n-1}\Delta_h^{n-1}\right)f + T_{n-1}\left(Q_{n-1}\Delta_h^{n-1}f\right). \tag{3.29}$$

We will show that the first summand above is in H_{n-1} and the item in the braces of the second summand is in H_1. This can be verified directly. First,

$$\begin{aligned}\Delta_h^{n-1}\left(I - T_{n-1}Q_{n-1}\Delta^{n-1}\right)f &= \left(\Delta_h^{n-1} - \left(\Delta_h^{n-1}T_{n-1}Q_{n-1}\right)\Delta_h^{n-1}\right)f\\ &= \left(\Delta_h^{n-1} - \Delta_h^{n-1}\right)f = 0.\end{aligned} \tag{3.30}$$

Next, since $\Delta_h^{n-1}f \in H_1$ and $Q_{n-1}H_1 \subset H_1$, we have $Q_{n-1}\Delta_h^{n-1}f \in H_1$, as desired.This proves that $H_n = H_{n-1}+T_{n-1}H_1$. By induction, we can easily deduce that $H_n = H_1+T_1H_1+\cdots+T_{n-1}H_1$.

Next we prove that for any $f \in H_n$ the decomposition

$$f = g + T_{n-1} f_n, \quad g \in H_{n-1},\ f_n \in H_1 \tag{3.31}$$

is unique. In fact, for such a decomposition, applying Δ_h^{n-1} on both sides we obtain

$$\begin{aligned}
\Delta_h^{n-1} f &= \Delta_h^{n-1} g + \Delta_h^{n-1} T_{n-1} f_n \\
&= \Delta_h^{n-1} T_{n-1} Q_{n-1} Q_{n-1}^{-1} f_1 \\
&= Q_{n-1}^{-1} f_n.
\end{aligned} \tag{3.32}$$

Therefore

$$f_n = Q_{n-1} \Delta_h^{n-1} f, \tag{3.33}$$

so that

$$g = f - T_{n-1} f_n = \left(I - T_{n-1} Q_{n-1} \Delta^{n-1}\right) f. \tag{3.34}$$

Thus the uniqueness follows by induction.

To prove the converse, we see from (3.23) that, for any $n \in \mathbb{N}$, $\Delta_h^{n+1} z^{2n} H_1 = 0$. Replacing n by j, we have

$$\Delta_h^n z^{2j} H_1 = 0 \tag{3.35}$$

for any $n > j$. □

4. Gauss Decomposition

As a direct consequence of Theorem 1.2, we can get the extended Fischer decomposition theorem. Let $\mathcal{P}_m$ denote the space of homogeneous polynomials of degree m in $\mathbb{C}^N$. Notice that $\mathfrak{D}_j$ maps $\mathcal{P}_m$ into $\mathcal{P}_{m-1}$, so that $\Delta_h \mathcal{P}_m \subset \mathcal{P}_{m-2}$. If $f \in \mathcal{P}_m$, then

$$\Delta_h^{[m/2]+1} f(z) = 0, \tag{4.1}$$

and $I_s f(z) = (1/(m+s)) f(z)$ so that

$$Q_j f(z) = d_{j,m} f(z), \tag{4.2}$$

where $d_{j,n}^{-1} = 4^j j! (N/2 + \gamma + n)_j$ and $(a)_n = a(a+1)\cdots(a+n-1)$. Denote

$$c_j = d_{j,m-2j} = \frac{1}{4^j j! \left(N/2 + \gamma + m - 2j\right)_j}. \tag{4.3}$$

Corollary 4.1. *Let f be a homogeneous polynomial of degree m in $\mathbb{C}^N$. Then there exist uniquely Dunkl harmonic homogeneous polynomials f_j of degree $m-2j$ such that*

$$f(z) = f_0(z) + z^2 f_1(z) + \cdots + z^{2[m/2]} f_{[m/2]}(z), \quad \forall z \in \Omega. \tag{4.4}$$

Moreover the Dunkl harmonic functions $f_0, \ldots, f_{[m/2]}$ are given by the following formulae:

$$\begin{aligned}
f_0 &= \left(I - c_1 z^2 \Delta_h\right)\left(I - c_2 z^4 \Delta_h^2\right) \cdots \left(I - c_{[m/2]} z^{2[m/2]} \Delta_h^{[m/2]}\right) f(z)\\
f_1 &= c_1 \Delta_h \left(I - c_2 z^4 \Delta_h^2\right) \cdots \left(I - c_{[m/2]} z^{2[m/2]} \Delta_h^{[m/2]}\right) f(z)\\
&\vdots\\
f_{[m/2]-1} &= c_{[m/2]-1} \Delta_h^{[m/2]-1} \left(I - c_{[m/2]} z^{2[m/2]} \Delta_h^{[m/2]}\right) f(z)\\
f_{[m/2]} &= c_{[m/2]} \Delta_h^{[m/2]} f(z).
\end{aligned} \tag{4.5}$$

Proof. Let $f \in \mathcal{P}_m$, then f is Dunkl harmonic of degree $[m/2]+1$, so that Theorem 1.2 gives the decomposition of f as in (4.4). It remains to check the formulae of $f_0, \ldots, f_{[m/2]}$. We only consider the formula of f_0, since the others are similar. That is, we need to show

$$\begin{aligned}
f_0 &= \left(I - z^2 Q_1 \Delta_h\right)\left(I - z^4 Q_2 \Delta_h^2\right) \cdots \left(I - z^{2[m/2]} Q_{[m/2]} \Delta_h^{[m/2]}\right) f(z)\\
&= \left(I - c_1 z^2 \Delta_h\right)\left(I - c_2 z^4 \Delta_h^2\right) \cdots \left(I - c_{[m/2]} z^{2[m/2]} \Delta_h^{[m/2]}\right) f(z).
\end{aligned} \tag{4.6}$$

Notice that for any $f \in \mathcal{P}_m$, $\Delta_h^{[m/2]} f \in \mathcal{P}_{m-2[m/2]} \cap H_1$, so that (4.2) implies

$$Q_{[m/2]} \Delta_h^{[m/2]} f(z) = c_{[m/2]} \Delta_h^{[m/2]} f(z). \tag{4.7}$$

Therefore

$$z^{2[m/2]} Q_{[m/2]} \Delta_h^{[m/2]} f(z) = c_{[m/2]} z^{2[m/2]} \Delta_h^{[m/2]} f(z) \in \mathcal{P}_m, \tag{4.8}$$

and also

$$\left(I - z^{2[m/2]} Q_{[m/2]} \Delta_h^{[m/2]}\right) f(z) = \left(I - c_{[m/2]} z^{2[m/2]} \Delta_h^{[m/2]}\right) f(z) \in \mathcal{P}_m. \tag{4.9}$$

The remaining proof follows by induction. □

Acknowledgment

The research is partially supported by the *Unidade de Investigação "Matemática e Aplicações"* of University of Aveiro and by the NNSF of China (no. 10771201).

References

[1] N. Aronszajn, T. M. Creese, and L. J. Lipkin, *Polyharmonic Functions*, The Clarendon Press, Oxford University Press, New York, NY, USA, 1983.

[2] W. K. Hayman and B. Korenblum, "Representation and uniqueness theorems for polyharmonic functions," *Journal d'Analyse Mathématique*, vol. 60, pp. 113–133, 1993.

[3] H. R. Malonek and G. Ren, "Almansi-type theorems in Clifford analysis," *Mathematical Methods in the Applied Sciences*, vol. 25, no. 16–18, pp. 1541–1552, 2002.

[4] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Mathematical Series, no. 32, Princeton University Press, Princeton, NJ, USA, 1971.

[5] N. Ja. Vilenkin and A. U. Klimyk, *Representation of Lie Groups and Special Functions. Recent Advances*, vol. 316 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1995.

[6] S. A. Lurie and V. V. Vasiliev, *The Biharmonic Problem in the Theory of Elasticity*, Gordon and Breach, Luxembourg, UK, 1995.

[7] L.-E. Andersson, T. Elfving, and G. H. Golub, "Solution of biharmonic equations with application to radar imaging," *Journal of Computational and Applied Mathematics*, vol. 94, no. 2, pp. 153–180, 1998.

[8] D. H. Armitage, S. J. Gardiner, W. Haussmann, and L. Rogge, "Characterization of best harmonic and superharmonic L_1-approximants," *Journal für die Reine und Angewandte Mathematik*, vol. 478, pp. 1–15, 1996.

[9] O. Kounchev, *Multivariate Polysplines: Applications to Numerical and Wavelet Analysis*, Academic Press, San Diego, Calif, USA, 2001.

[10] C. F. Dunkl, "Differential-difference operators associated to reflection groups," *Transactions of the American Mathematical Society*, vol. 311, no. 1, pp. 167–183, 1989.

[11] C. F. Dunkl, "Reflection groups and orthogonal polynomials on the sphere," *Mathematische Zeitschrift*, vol. 197, no. 1, pp. 33–60, 1988.

[12] C. F. Dunkl, "Integral kernels with reflection group invariance," *Canadian Journal of Mathematics*, vol. 43, no. 6, pp. 1213–1227, 1991.

[13] C. F. Dunkl, "Intertwining operators associated to the group S_3," *Transactions of the American Mathematical Society*, vol. 347, no. 9, pp. 3347–3374, 1995.

[14] J. F. van Diejen and L. Vinet, *Calogero-Sutherland-Morser Models*, Springer, New York, NY, USA, 2000.

[15] G. Ren, "Almansi decomposition for Dunkl operators," *Science in China Series A*, vol. 48, supplement 1, pp. 333–342, 2005.

[16] G. B. Ren and H. R. Malonek, "Complex Dunkl operators," http://arxiv.org/abs/0912.5196.

[17] C. F. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, vol. 81, Cambridge University Press, Cambridge, UK, 2001.

[18] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, vol. 29 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.