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Research Article

On Stochastic Models Describing the Motions of Randomly Forced Linear Viscoelastic Fluids

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This paper is devoted to the analysis of stochastic equations describing the motions of a large class of incompressible linear viscoelastic fluids in two-dimensional subject to periodic boundary condition and driven by random external forces. To do so we distinguish two cases, and for each case a global existence result of probabilistic weak solution is expounded in this paper. We also prove that under suitable hypotheses on the external random forces the solution turns out to be unique. As concrete examples, we consider the stochastic equations for the Maxwell and Oldroyd fluids that are of great importance in the investigation towards the understanding of the elastic turbulence.

1. Introduction

The study of turbulent flows has attracted many prominent researchers from different fields of contemporary sciences for ages. For in-depth coverage of the deep and fascinating investigations undertaken in this field, the abundant wealth of results obtained, and remarkable advances achieved we refer to the monographs [1–3] and references therein. Recent study, see, for instance [4], has showed that the non-Newtonian elastic turbulence can be well understood on basis of known viscoelastic models such as the Oldroyd fluids or the Maxwell fluids. Indeed, by computational investigations of the two-dimensional periodic Oldroyd-B model the authors in [4] found that there is a considerable agreement between their numerical results and the experimental observations of elastic turbulence.

The hypothesis relating the turbulence to the "randomness of the background field" is one of the motivations of the study of stochastic version of equations governing the motion of fluids flows. The introduction of random external forces of noise type reflects (small) irregularities that give birth to a new random phenomenon, making the problem more realistic. Such approach in the mathematical investigation for the understanding of the

Newtonian turbulence phenomenon was pioneered by Bensoussan and Temam in [5] where they studied the Stochastic Navier-Stokes Equation (SNSE). Since then stochastic partial differential equations and stochastic models of fluid dynamics have been the object of intense investigations which have generated several important results. We refer to [6–13], just to cite a few. Similar investigations for Non-Newtonian elastic fluids have not almost been undertaken except in very few works; we refer, for instance, to [14–19] for some example of computational studies of stochastic models of polymeric fluids and to [20–23] for their mathematical analysis. It should be noted that the models investigated in these papers occur very naturally from the kinetic theory of polymer dynamics. Indeed they arise from the reformulation of Fokker-Planck or diffusion equations as stochastic differential equations (3.45). We also notice that they model some nonlinear viscoelastic models such as the FENE models which are very different from the models we shall treat in this paper. We refer to volume 2 of [24] for the conventional approach to kinetic theory which consists of deriving the deterministic partial differential equations for the polymer configurational distribution function (diffusion equation) and to volume 1 of [24] for the existing linear and nonlinear viscoelastic models.

In this paper we provide a detailed investigation of the system stochastic partial differential equations:

$$\begin{gathered} du + (u\cdot\nabla)u\,dt + \nabla\mathfrak{P}\,dt = \operatorname{div}\sigma\,dt + F(u,t)dt + G(u,t)dW, \\ \operatorname{div} u = 0, \qquad \int_D u(x)dx = 0, \qquad u_{|t=0} = u_0, \end{gathered} \tag{1.1}$$

$t \in [0,T]$, $T \in (0,\infty]$. This system describes the motion of a large class of incompressible linear viscoelastic fluids driven by random external forces and filling a periodic square $D = [0,L]^2 \subset \mathbb{R}^2$, $L > 0$. Here $u, \mathfrak{P}$,and W represent, respectively, a random periodic in space random velocity with period L in each direction, a random scalar pressure and an $\mathbb{R}^m$ -valued standard Wiener process, $m \in \{1,2,3,\ldots\}$. The tensor $\sigma = (\sigma_{ij})$ is the deviator of the stress tensor of the fluid; we assume throughout that it is a traceless tensor ($\operatorname{tr}\sigma = 0$). In this work we should distinguish the case

$$\sigma = \mathbf{K}\mathbf{D}, \tag{1.2}$$

$$\sigma = 2\nu\mathbf{D} + \mathbf{K}\mathbf{D}, \tag{1.3}$$

where

$$\mathbf{D} = \left(\frac{1}{2}\right)(\nabla u + \nabla^t u), \tag{1.4}$$

and the operator $\mathbf{K}$ is a continuous mapping satisfying some hypotheses (see (2.22)–(2.24)). The problem (1.1) also can be taken as a turbulent version of linear viscoelastic models for polymeric fluids. For some examples of classical models of turbulence, we refer to [2, 4, 19, 25] and references therein.

The mathematical works on some linear viscoelastic fluids undertaken by the Soviet mathematician Oskolkov in [26–28] and by Ladyzhenskaya in [29] have influenced the emergence of the paper [30] where a global solvability result of the deterministic counterpart of the system (1.1), (1.2) (resp., (1.1), (1.3)) subject to the periodic boundary condition (resp., nonslip boundary condition) was given. To the best of our knowledge similar investigations for the two general stochastic models (1.1), (1.2) and (1.1), (1.3) have not been undertaken yet. The purpose of this paper is to prove that under suitable conditions on $\mathbf{K}$, F, and G each of our stochastic model is well posed (see Theorems 3.3, 3.4, 4.2, and 4.3). In view of the technical difficulties involved, we provide full details of the proof of our results. Due to nontrivial difficulties that arise from the nature of the nonlinearities involved in (1.1) other mathematical issues such as existence, uniqueness of the invariant measure, and its ergodicity are beyond of the scope of this work; we leave these questions for future investigation.

The layout of this paper is as follows. In addition to the current introduction this article consists of three other sections. In Section 2 we give some notations, necessary backgrounds of probabilistic or analytic nature. Section 3 is devoted to the detailed analysis of the problem (1.1), (1.2). We prove the existence and pathwise uniqueness of its probabilistic weak solution which yields the existence of a unique probabilistic strong solution. In the very same section we consider the stochastic equations for randomly forced generalized Maxwell fluids as a concrete example. In Section 4 we only state the main theorems related to (1.1), (1.3) and apply the obtained results to the stochastic model for the generalized viscoelastic Oldroyd fluids; we refer to the previous section for the details of the proofs.

2. Preliminaries and Notations

This section is devoted to the presentation of notations and auxiliary results needed in the work. Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^2$, let $1 \leq p \leq \infty$, and let k be a nonnegative integer. We consider the well-known Lebesgue and Sobolev spaces $L^p(\mathcal{O})$ and $H^k(\mathcal{O})$, respectively. We refer to [31] for detailed information on Sobolev spaces. Let L be a nonnegative number and let $D = [0, L]^2$ be a periodic box of side length L. We denote by $H^k(D)$ the spaces consisting of those functions u that are in $H^k_{\text{Loc}}(\mathbb{R}^2)$ and that are periodic with period L:

$$u(x + Lr_i) = u(x), \quad i = 1, 2, \tag{2.1}$$

where $\{r_1, r_2\}$ represents the canonical basis of $\mathbb{R}^2$. Here the space $H^k_{\text{Loc}}(\mathbb{R}^2)$ is the space of functions u such that $u_{|\mathcal{O}}$ is an element of the Sobolev space $H^k(\mathcal{O})$ for every bounded set $\mathcal{O} \subset \mathbb{R}^2$. For functions v of zero space average, that is,

$$\int_D v \, dx = 0, \tag{2.2}$$

the following Poincaré's inequality holds:

$$|v|_{\text{sc}} \leq \wp \|v\|_{\text{sc}} \quad \forall v \in H^1(D), \tag{2.3}$$

where $|\cdot|_{\text{sc}}$ denotes the L^2-norm, $\wp > 0$ is Poincaré's constant, and $\|\cdot\|_{\text{sc}}$ denotes the seminorm generated by the scalar product:

$$((u,v))_{\text{sc}} = \int_D \nabla u \cdot \nabla v \, dx = \sum_{i=1}^{2} \int_D \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \tag{2.4}$$

in which ∇ is the gradient operator. From now we denote by $H_0^1(D)$ the space

$$H_0^1(D) := \left\{ u : u \in H^1(D) \text{ and } \int_D u \, dx = 0 \right\}. \tag{2.5}$$

Thanks to (2.3), we can endow $H_0^1(D)$ with the norm $\|\cdot\|_{\text{sc}}$. Besides Poincaré's inequality, we also have

$$c|u|_{H^1(D)} \leq |\operatorname{curl} u|_{\text{sc}} \leq c\|u\|_{\text{sc}}, \tag{2.6}$$

which holds for any divergence free fields. For $\beta \in \mathbb{R}$ we can define the space $H^\beta(D)$ via their expansion in Fourier series so that we also have the space

$$H_0^\beta(D) := \left\{ u : u \in H^\beta(D) \text{ and } \int_D u \, dx = 0 \right\}. \tag{2.7}$$

We refer to [32] (see also [1, 33]) for more details about these spaces. We proceed with the definitions of additional spaces frequently used in this paper.

For any Banach space X and any integer $M > 0$ we set

$$X^{\otimes M} = \underbrace{X \times \cdots \times X}_{M \text{ times}}, \quad \mathbb{X} = X \times X. \tag{2.8}$$

If $|\cdot|_X$ is the norm on X, then $|u|^2_{X^{\otimes M}} = \sum_{i=1}^M |u_i|_X^2$.

We introduce the spaces

$$\begin{gathered} \mathcal{V} = \left\{ u \in \left[\mathcal{C}^\infty_{\text{per}}(D)\right]^{\otimes 2} : \operatorname{div} u = 0 \text{ and } \int_D u \, dx = 0 \right\}, \\ \mathbb{V} = \text{closure of } \mathcal{V} \text{ in } \mathbb{H}_0^1(D), \\ \mathbb{H} = \text{closure of } \mathcal{V} \text{ in } \mathbb{L}^2(D), \end{gathered} \tag{2.9}$$

where $\mathcal{C}^\infty_{\text{per}}(D)$ denotes the space of infinitely differentiable periodic function with period L.

We denote by $(\cdot,\cdot)$ (resp., $|\cdot|$) the inner product (resp., the norm) induced by the inner product (resp., the norm) in $\mathbb{L}^2(D)$ on $\mathbb{H}$. Thanks to Poincaré's inequality (2.3), we can endow $\mathbb{V}$ with the norm $\|\cdot\|$, which is defined by $\|u\|^2 = \sum_{i=1}^2 \|u_i\|^2_{\text{sc}}$. From now on, we identify the space $\mathbb{H}$ with its dual space $\mathbb{H}^\star$ via the Riesz representation and we have the Gelfand triple:

$$\mathbb{V} \subset \mathbb{H} \subset \mathbb{V}^\star, \tag{2.10}$$

where each space is dense in the next one and the inclusions are continuous. It follows that we can make the identification

$$(v, w) = \langle v, w \rangle, \tag{2.11}$$

for any $v \in \mathbb{H}$ and $w \in \mathbb{V}$. Here $\langle \cdot, \cdot \rangle$ denotes the duality product $\mathbb{V}^*, \mathbb{V}$.

Next we define some probabilistic evolution spaces necessary throughout the paper. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$ be a given stochastic basis; that is, $(\Omega, \mathcal{F}, P)$ is a complete probability space and $(\mathcal{F}_t)_{0 \le t \le T}$ is an increasing sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains every P-null subset of Ω. For any real Banach space $(X, |\cdot|_X)$, for any $r, p \ge 1$ we denote by $L^p(\Omega, \mathcal{F}, P; L^r(0, T; X))$ the space of processes $u = u(\omega, t)$ with values in X defined on $\Omega \times [0, T]$ such that

(1) u is measurable with respect to (ω, t) and for each t, $\omega \mapsto u(\omega, t)$ is $\mathcal{F}^t$-measurable;

(2) $u(\omega, t) \in X$ for almost all (ω, t) and

$$\|u\|_{L^p(\Omega, \mathcal{F}, P; L^r(0,T;X))} = \left(E\left(\int_0^T \|u\|_X^r dt \right)^{p/r} \right)^{1/p} < \infty, \tag{2.12}$$

where E denotes the mathematical expectation with respect to the probability measure P.

When $r = \infty$, we write

$$\|u\|_{L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;X))} = \left(E \sup_{0 \le t \le T} \|u\|_X^p \right)^{1/p} < \infty. \tag{2.13}$$

For $p \ge 1$, we also consider the space $L^p(0, T; X)$ of X-valued measurable functions u defined on $[0, T]$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u\|_X^p dt \right)^{1/p} < \infty. \tag{2.14}$$

Let W be a standard Wiener process defined on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$ and taking its values in $\mathbb{R}^m$. Given a measurable and $\mathcal{F}_t$-adapted $X^{\otimes m}$-valued process f such that

$$E \int_0^T |f(t)|_{X^{\otimes m}}^2 dt < \infty, \tag{2.15}$$

the process

$$I(f)(t) = \int_0^t f(s) dW(s), \quad 0 \le t \le T, \tag{2.16}$$

is well defined and is a continuous martingale. Moreover it satisfies

$$EI(f)(t) = 0, \quad 0 \le t \le T,$$
$$E\left|I(f)(t)\right|_X^2 = E\int_0^t \left|f(s)\right|_X^2 ds, \quad 0 \le t \le T. \tag{2.17}$$

We refer to [34, 35] (see also [36]) for further reading on probability theory and stochastic calculus.

Let X be a separable complete metric space and $\mathcal{B}(X)$ its Borel σ-field. A family Π_k of probability measures on $(X, \mathcal{B}(X))$ is relatively compact if every sequence of elements of Π_k contains a subsequence Π_{k_j} which converges weakly to a probability measure Π, that is, for any ϕ bounded and continuous function on Ω,

$$\int \phi(s)d\Pi_{k_j} \longrightarrow \int \phi(s)d\Pi. \tag{2.18}$$

The family Π_k is said to be tight if, for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \Omega$ such that $P(K_\varepsilon) \ge 1 - \varepsilon$, for every $P \in \Pi_k$.

We have the well-known result.

Theorem 2.1 (Prokhorov). *Assume that X is a Polish space; then the family Π_k is relatively compact if and only if it is tight.*

We will use the following useful theorem due to Skorokhod.

Theorem 2.2 (Skorokhod). *For any sequence of probability measures Π_k on Ω which converges to a probability measure Π, there exists a probability space $(\Omega', \mathcal{F}', P')$ and random variables X_k, X with values in Ω such that the probability law of X_k (resp., X) is Π_k (resp., Π) and $\lim_{k\to\infty} X_k = XP'$-a.s.*

We refer to [36] for the proofs of these two theorems.

The following result is very important in Section 3.2.2 where we prove a probabilistic compactness result; its proof can be found in [37].

Theorem 2.3. *Let X, B, Y be three Banach spaces such that the following embedding are continuous:*

$$X \subset B \subset Y. \tag{2.19}$$

Moreover, assume that the embedding $X \subset B$ is compact; then the set $\mathfrak{F}$ consisting of functions $v \in L^q(0,T;B) \cap L^1_{\text{loc}}(0,T;X)$, $1 \le q \le \infty$ such that

$$\sup_{0 \le h \le 1} \int_{t_1}^{t_2} |v(t+h) - v(t)|_Y dt \longrightarrow 0, \quad \text{as } h \longrightarrow 0, \tag{2.20}$$

for any $0 < t_1 < t_2 < T$ is compact in $L^p(0,T;B)$ for any $p < q$.

Throughout the symbol $\sigma : \sigma'$ denotes the summation

$$\sigma : \sigma' = \operatorname{tr}(\sigma\sigma') = \sum_{i,k=1}^{2} \sigma_{ik}\sigma'_{ki}. \tag{2.21}$$

We assume that $\mathbf{K}$ is a symmetric tensor-valued continuous mapping which satisfies the following.

(i) $\mathbf{K}$ is bounded, that is,

$$E\int_0^T |\mathbf{K}\mathbf{D}|^2 dt \leq CE\int_0^T |\mathbf{D}|^2 dt. \tag{2.22}$$

(ii) For any $\mathbf{D}_1$ and $\mathbf{D}_2$ we have

$$0 \leq E\int_{[0,T]\times D} (\mathbf{K}\mathbf{D}_1 : \mathbf{D}_1)dx\,dt, \tag{2.23}$$

$$0 \leq E\int_{D\times[0,T]} (\mathbf{K}\mathbf{D}_1 - \mathbf{K}\mathbf{D}_2 : \mathbf{D}_1 - \mathbf{D}_2)dx\,dt, \tag{2.24}$$

Remark 2.4. The hypothesis (2.23) has a physical meaning since it implies that the dissipation of energy is positive (see [30, Section 1] and [38, Chapters 2-3]). The assumption (2.24) is a mathematical assumption which allows us to prove the well posedness of the models. It is fulfilled at least for general viscoelastic flows generated by the linear rheological equations of the type (see [24, Section 5.2])

$$\sigma = \int_0^t \mathbf{K}(t-\tau)\mathbf{D}(x,\tau)d\tau. \tag{2.25}$$

We also notice that (2.24) and (2.23) are equivalent if $\mathbf{K}$ is linear.

3. Analysis of the Stochastic Equation of the Type (1.1), (1.2)

In this section we investigate the stochastic equations (1.1), (1.2). The first subsection is devoted to the statement of the main results which is going to be proved in the second subsection.

3.1. Hypotheses and Statement of the Main Results

Throughout this section we suppose the following.

(HYP 1) The mapping F induces a nonlinear operator from $\mathbb{H} \times [0,T]$ into $\mathbb{V}$ which is assumed to be measurable (resp., continuous) with respect to its second (resp., first) variable. We require that there exists constant $C_F > 0$ such that for almost all $t \in [0, T]$ and for each $u \in \mathbb{H}$

$$\|F(u,t)\| \leq C_F(1+|u|). \tag{3.1}$$

(HYP 2) The $\mathbb{V}^{\otimes m}$-valued function G defined on $\mathbb{H} \times [0, T]$ is measurable (resp., continuous) with respect to its second (resp., first) argument, and it verifies

$$|G(u,t)|_{\mathbb{V}^{\otimes m}} \leq C_G(1+|u|), \tag{3.2}$$

for almost everywhere $t \in [0, T]$ and for any $u \in \mathbb{H}$.

(HYP 3) We assume as well that there exist two positive constants C'_F and C'_G such that

$$\begin{aligned} \|F(u,t) - F(v,t)\| &\leq C'_F|u-v|, \\ |G(u,t) - G(v,t)|_{\mathbb{V}^{\otimes m}} &\leq C'_G|u-v|, \end{aligned} \tag{3.3}$$

for any $u, v \in \mathbb{H}$.

(HYP 4) In addition to (2.22)–(2.24) we assume furthermore that

$$0 \leq -(\operatorname{curl}\ \operatorname{div}(\mathbf{KD}), \operatorname{curl}\ u). \tag{3.4}$$

Remark 3.1. For a vector $u \in \mathbb{R}^2$, the operator curl is defined by

$$\operatorname{curl} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}. \tag{3.5}$$

The divergence of a tensor field $\mathbf{D}$ is defined using the recursive relation

$$\operatorname{div}(\mathbf{D}) \cdot \mathbf{c} = \operatorname{div}(\mathbf{c} \cdot \mathbf{D}), \qquad \operatorname{div}\ v = \operatorname{tr}(\nabla v), \tag{3.6}$$

where $\mathbf{c}$ is an arbitrary constant vector, and v is a vector field.

Karazeeva remarked in [30, Section 5.2] that when $\mathbf{K}$ and $\partial/\partial x_k$, $k = 1, 2$, commute, then (3.4) is a consequence of (2.23). The condition (3.4) is met when $\mathbf{K}$ is given by the equation in Remark 2.4.

We introduce the concept of the solution of the problem (1.1), (1.2).

Definition 3.2. By a probabilistic weak solution of the problem (1.1), (1.2), one means a system

$$(\Omega, \mathcal{F}, P, \mathcal{F}^t, W, u), \tag{3.7}$$

where

(1) $(\Omega, \mathcal{F}, P)$ is a complete probability space, and $\mathcal{F}^t$ is a filtration on $(\Omega, \mathcal{F}, P)$;

(2) $W(t)$ is an m-dimensional $\mathcal{F}^t$-standard Wiener process;

(3) $u \in L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;\mathbb{V})) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;\mathbb{H}))$;

(4) for almost all t, $u(t)$ is $\mathcal{F}^t$-measurable;

(5) P-a.s. the following integral equation of Itô type holds:

$$\begin{aligned}(u(t)-u(0),\phi)+\int_0^t\int_D(\mathbf{KD}:\nabla\phi)\,dx\,ds-\int_0^t((u.\nabla\phi),u)\,ds\\=\int_0^t(F(u(s),s),\phi)\,ds+\int_0^t(G(u(s),s),\phi)\,dW(s)\end{aligned}\tag{3.8}$$

for any $t \in [0,T]$ and $\phi \in \mathcal{V}$.

We have the following.

Theorem 3.3. *If $u_0 \in \mathbb{V}$ and if the hypotheses (HYP 1), (HYP 2), and (HYP 4) hold, then the problem* (1.1), (1.2) *has a solution in the sense of the above definition. Moreover, almost surely the paths of the solution are $\mathbb{V}$- (resp., $\mathbb{H}$-), valued weakly (resp., strongly) continuous.*

Theorem 3.4. *Assume that (HYP 1)–(HYP 4) hold and let u_1 and u_2 be two probabilistic weak solutions of* (1.1), (1.2) *starting with the same initial condition and defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}^t, P)$ with the same Winer process W. If one sets $v = u_1 - u_2$, then one has $v = 0$ almost surely.*

3.2. Proof of Theorems 3.3 and 3.4

This subsection is devoted to the proof of the existence and uniqueness results stated in the previous subsection. We split the proof into four subsections. The proof of the existence theorem is inspired by the works [6, 30] (see also [10]). Throughout this subsection C will designate a positive constant which depends only on the data (u_0, T, C_F, C_G).

3.2.1. The Approximate Solution and Some A Priori Estimates

In this subsection we derive crucial a priori estimates from the Galerkin approximation. They will serve as a toolkit for the proof of the Theorem 3.3.

The operator $-\Delta$ is a self-adjoint and positive definite on $\mathbb{H}$, and its inverse is completely continuous. Therefore $\mathbb{H}$ has a complete orthonormal basis consisting of the eigenfunctions $(e_i)_{i\geq 1} \in [C^\infty(D)]^{\otimes 2}$ of $-\Delta$. The family $(e_i)_{i\geq 1} \in [C^\infty(D)]^{\otimes 2}$ forms an orthogonal basis in $\mathbb{V}$. We now introduce the Galerkin approximation for the problem (1.1)-(1.2). We consider the subset $\mathbb{H}_N = \text{Span}(e_1, \dots, e_N) \subset \mathbb{H}$ and we look for a finite-dimensional approximation of a solution of our problem as a vector $u^N \in \mathbb{H}_N$ that can be written as

$$u^N(t) = \sum_{i=1}^{N} c_{iN}(t) e_i(x). \tag{3.9}$$

We set

$$\mathbf{D}^N = (1/2)\left(\nabla u^N + \nabla^t u^N\right). \tag{3.10}$$

Let us consider a complete probabilistic system $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{P}, \overline{\mathcal{F}}^t, \overline{W})$ such that the filtration $\{\overline{\mathcal{F}}_t\}$ satisfies the usual condition and $\overline{W}$ is an m-dimensional standard Wiener process taking values in $\mathbb{R}^m$. We denote by $\overline{E}$ the mathematical expectation with respect to $\overline{P}$. We require u^N to satisfy the following:

$$\begin{aligned} d\left(u^N, e_i\right) + \left(\int_D \left(\mathbf{K}\mathbf{D}^N : \nabla e_i\right) dx\right) dt - \left(\left(u^N . \nabla\right) e_i, u^N\right) dt \\ = \left(F\left(u^N, t\right), e_i\right) dt + \left(G\left(u^N, t\right), e_i\right) d\overline{W}, \end{aligned} \tag{3.11}$$

$i \in \{1, \dots, N\}$. Here u_0^N is the orthogonal projection of u_0 onto the space $\mathbb{H}_N$:

$$u_0^N \longrightarrow u_0 \quad \text{strongly in as } N \longrightarrow \infty. \ \mathbb{H} \tag{3.12}$$

The sequence of continuous functions u^N exists at least on a short (possibly random) interval $(0, T_N)$. Indeed the coefficients C_{iN} satisfy

$$\begin{aligned} dC_{iN} + \sum_{k,j=1}^{N} \left(\sum_{l=1}^{2} \int_D e_j e_k \frac{\partial e_i}{\partial x_l} dx\right) C_{kN}(t) C_{jN}(t) dt + \sum_{k=1}^{N} (\mathbf{K}(C_{kN} \nabla e_k), e_i) dt \\ = \left(F\left(u^N, t\right), e_i\right) dt + \left(G\left(u^N, t\right), e_i\right) d\overline{W}, \end{aligned} \tag{3.13}$$

which is a system of stochastic ordinary differential equations with continuous coefficients. From the existence theorem stated in [39, page 59] (see also [35, Theorem 4.22, page 323]) we infer the existence of continuous functions C_{iN} on $(0, T_N)$. Global existence will follow from a priori estimates for u^N.

Lemma 3.5. *One has*

$$\overline{E} \sup_{0 \le t \le T} \left| u^N(t) \right|^p < C, \tag{3.14}$$

for any $2 \le p < \infty$ *and* $1 \le N < \infty$.

Proof. Thanks to Itô's formula we derive from (3.11) that

$$d\left|u^N\right|^2 + 2\left(\int_D \left(\mathbf{K}\mathbf{D}^N : \mathbf{D}^N\right)dx\right)dt$$
$$= 2\left(\left(F\left(u^N,t\right),u^N\right)\right)dt + \sum_{i=1}^{N}\left(G\left(u^N,t\right),e_i\right)^2 dt + 2\left(G\left(u^N,t\right),u^N\right)d\overline{W}, \tag{3.15}$$

where we have used the fact that $((u\cdot\nabla)v,w) = -((u\cdot\nabla w),v)$ for any $u,v,w\in\mathbb{V}$. Thanks to (2.23) we get

$$d\left|u^N\right|^2 \le 2\left|\left(F\left(u^N,t\right)\right)\right|\left|u^N\right|dt + \sum_{i=1}^{N}\left(G\left(u^N,t\right),e_i\right)^2 dt + 2\left(G\left(u^N,t\right),u^N\right)d\overline{W}. \tag{3.16}$$

More generally we have

$$d\left|u^N\right|^p \le p\left|\left(F\left(u^N,t\right)\right)\right|\left|u^N\right|^{p-1}dt + \left(\frac{1}{2}\right)(p(p-1))\sum_{i=1}^{N}\left|u^N\right|^{p-2}\left(G\left(u^N,t\right),e_i\right)^2 dt$$
$$+ p|u^N|^{p-2}\left(G\left(u^N,t\right),u^N\right)d\overline{W}, \tag{3.17}$$

for all $2\le p<\infty$. For any integer $M\ge 1$, we introduce the sequence of increasing stopping-times:

$$\tau_M = \begin{cases} \inf\{t\ge 0; \left|u^N(t)\right| \ge M\}, \\ T \quad \text{if } \{t\ge 0; \left|u^N(t)\right| \ge M\} = \emptyset. \end{cases} \tag{3.18}$$

Owing to Schwarz's inequality and the assumptions (3.1)-(3.2) we have that

$$\sup_{0\le s\le t\wedge\tau_M}\left|u^N(s)\right|^p \le \left|u_0^N\right|^p + pC_F\int_0^{t\wedge\tau_M}\left(1+\left|u^N\right|\right)\left|u^N(s)\right|^{p-1}ds$$
$$+ \left(\frac{1}{2}\right)p(p-1)\sum_{i=1}^{N}\int_0^{t\wedge\tau_M}\left|u^N\right|^{p-2}\left(G\left(u^N,t\right),e_i\right)^2 ds \tag{3.19}$$
$$+ p\sup_{0\le s\le t\wedge\tau_M}\left|\int_0^s\left|u^N\right|^{p-2}\left(G\left(u^N,s\right),u^N\right)d\overline{W}\right|.$$

Since

$$\sum_{i=1}^{N}\left(G\left(u^N,t\right),e_i\right)^2 \le \left|G\left(u^N,t\right)\right|^2_{\mathbb{H}^{\otimes m}}, \tag{3.20}$$

we derive from (3.19) and (3.2) that

$$\sup_{0\le s\le t\wedge\tau_M}\left|u^N(s)\right|^p \le pC\int_0^{t\wedge\tau_M}\left|u^N(s)\right|^p ds + C_G^2 p(p-1)\int_0^{t\wedge\tau_M}\left|u^N\right|^{p-2}\left(1+\left|u^N\right|^2\right)ds$$
$$+\left|u_0^N\right|^p + p\sup_{0\le s\le t\wedge\tau_M}\left|\int_0^s \left|u^N\right|^{p-2}\left(G\left(u^N,s\right),u^N\right)d\overline{W}\right| + CT. \tag{3.21}$$

Using Hölder's inequality and taking the mathematical expectation in both sides of this estimate yield

$$\overline{E}\sup_{0\le s\le t\wedge\tau_M}\left|u^N(s)\right|^p \le \overline{E}\left|u_0^N\right|^p + C\overline{E}\int_0^{t\wedge\tau_M}\left|u^N(s)\right|^p ds$$
$$+p\overline{E}\sup_{0\le s\le t\wedge\tau_M}\left|\int_0^s \left|u^N\right|^{p-2}\left(G\left(u^N,s\right),u^N\right)d\overline{W}\right| + pC_F. \tag{3.22}$$

Let us set

$$\gamma^N = \overline{E}\sup_{0\le s\le t\wedge\tau_M}\left|\int_0^s \left|u^N\right|^{p-2}\left(G\left(u^N,s\right),u^N\right)d\overline{W}\right|. \tag{3.23}$$

By Burkhölder-Davis-Gundy's inequality we obtain

$$p\gamma^N \le pC\overline{E}\left(\int_0^{t\wedge\tau_m}\left|u^N\right|^{2p-4}\left(G\left(u^N,s\right),u^N\right)^2 ds\right)^{1/2},$$
$$p\gamma^N \le pC\overline{E}\sup_{0\le s\le t\wedge\tau_M}\left|u^N\right|^{p/2}\left(\int_0^{t\wedge\tau_M}\left|u^N\right|^{p-4}\left(G\left(u^N,s\right),u^N\right)^2 ds\right)^{1/2}, \tag{3.24}$$

which with the assumption (3.2) implies that

$$p\gamma^N \le \left(\frac{1}{2}\right)\overline{E}\sup_{0\le s\le t\wedge\tau_M}\left|u^N\right|^p + \left(\frac{1}{2}\right)C\overline{E}\int_0^{t\wedge\tau_M}\left|u^N\right|^{p-2}\left(1+\left|u^N\right|\right)^2 ds. \tag{3.25}$$

Out of this and (3.22) we infer that

$$\overline{E}\sup_{0\le s\le t\wedge\tau_M}\left|u^N(s)\right|^p \le \overline{E}\left|u_0^N\right|^p + C(p,C_F,C_G)\int_0^t \overline{E}\sup_{0\le r\le s\wedge\tau_M}\left|u^N(r)\right|^p ds. \tag{3.26}$$

Now by Gronwall's lemma applied to $h(t) := \overline{E}\sup_{0\le s\le t\wedge\tau_M}|u^N(s)|^p$, we obtain that

$$\overline{E}\sup_{0\le s\le t\wedge\tau_M}\left|u^N(s)\right|^p \le C(p,u_0,C_F,C_G,T), \quad t\in(0,T). \tag{3.27}$$

It remains to prove that $T_N = T$; to do so we must prove that $\tau_M \nearrow T$ a.s.; This is classic but we prefer to give the details. From the continuity of u^N we infer that $u^N(\tau_M) \geq M$. For any $t \in (0,T)$, $\overline{E}(1_{\tau_M<t}) = \overline{P}(\tau_M < t)$. We also have that

$$\overline{E} \sup_{0\leq s\leq t\wedge\tau_M} \left|u^N(s)\right|^2 \geq \overline{E}\left(\sup_{0\leq s\leq \tau_M}\left(\left|u^N(s)\right|^2 1_{\tau_M<t}\right)\right) \geq M^2\overline{P}(\tau_M < t), \quad t \in (0,T). \tag{3.28}$$

We infer from this, (3.27), and the monotonicity of τ_M that $\tau_M \nearrow T$ a.s. as was required. Since the constant C in (3.27) is independent of N and M, Fatou's theorem completes the proof of the lemma. □

The estimate of Lemma 3.5 is not sufficient to pass to the limit in the nonlinear term. We still need to derive some additional crucial but nontrivial inequalities.

Lemma 3.6. *One has*

$$\overline{E}\sup_{0\leq t\leq T}\left\|u^N(t)\right\|^p \leq C, \tag{3.29}$$

for any $2 \leq p < \infty$ *and* $1 \leq N < \infty$.

Proof. Let P^N be the orthogonal projection of $\mathbb{V}^*$ onto the span $\{e_1,\ldots,e_N\}$ that is

$$P^N h = \sum_{j=1}^{N}\langle h, e_j\rangle e_j. \tag{3.30}$$

Because $P^N u^N = u^N$, we can rewrite (3.11) in the following form which should be understood as the equality between random variables with values in $\mathbb{V}^*$:

$$du^N - P^N\left(\operatorname{div}\left(\mathbf{KD}^N\right)\right)dt + P^N\left(u^N\cdot\nabla u^N\right)dt = P^N F\left(u^N,t\right)dt + P^N G\left(u^N,t\right)d\overline{W}. \tag{3.31}$$

Applying the operator curl $(=\nabla\wedge)$ to both sides of this equation implies

$$\begin{aligned} d\zeta^N - \nabla\wedge\left(P^N\left(\operatorname{div}\left(\mathbf{KD}^N\right)\right)\right)dt + \nabla\wedge\left(P^N\left(u^N\cdot\nabla u^N\right)\right)dt \\ = \nabla\wedge\left(P^N F\left(u^N,t\right)\right)dt + \nabla\wedge\left(P^N G\left(u^N,t\right)\right)d\overline{W}, \end{aligned} \tag{3.32}$$

where $\zeta^N = \nabla\wedge u^N$. Thanks to the regularity of the e_i-s, the function ζ^N is periodic at the boundary of the square D. Itô's formula for the function $|\zeta^N|^2$ implies that

$$\begin{aligned} d\left|\zeta^N\right|^2 - 2\left(\nabla\wedge\left(P^N\left(\operatorname{div}\left(\mathbf{KD}^N\right)\right)\right),\zeta^N\right)dt - 2\left(\nabla\wedge\left(P^N F\left(u^N,t\right)\right),\zeta^N\right)dt \\ = \left|\nabla\wedge\left(P^N G\left(u^N,t\right)\right)\right|^2 dt + 2\left(\nabla\wedge\left(P^N G\left(u^N,t\right)\right),\zeta^N\right)d\overline{W}, \end{aligned} \tag{3.33}$$

where we have used the fact that

$$2\left\langle \nabla \wedge \left(P^N\left(u^N.\nabla u^N\right)\right), \zeta^N \right\rangle = 0 \tag{3.34}$$

in the periodic boundary condition setting. More generally, the following holds:

$$\begin{aligned} d\left|\zeta^N\right|^p &- p\left|\zeta^N\right|^{p-2}\left(\nabla \wedge \left(P^N\left(\operatorname{div}\left(\mathbf{K}\mathbf{D}^N\right)\right)\right), \zeta^N\right)dt - p\left|\zeta^N\right|^{p-2}\left(\nabla \wedge \left(P^N F\left(u^N,t\right)\right), \zeta^N\right)dt \\ &= \left(\frac{1}{2}\right)p(p-1)\left|\zeta^N\right|^{p-2}\left|\nabla \wedge \left(P^N G\left(u^N,t\right)\right)\right|^2 dt + p\left|\zeta^N\right|^{p-2}\left(\nabla \wedge \left(P^N G\left(u^N,t\right)\right), \zeta^N\right)d\overline{W}, \end{aligned} \tag{3.35}$$

for $2 \leq p < \infty$. We use the divergence freeness of u^N, the periodicity of ζ^N, and the identities

$$\begin{aligned} (\operatorname{curl} v, \phi) &= (v, \operatorname{curl} \phi) + \int_{\partial D} (v \times \mathbf{n})\phi \, dx, \\ \operatorname{curl}(\operatorname{curl} v) &= -\Delta v + \nabla(\operatorname{div} v), \qquad P^N \Delta u^N = \Delta\left(P^N u^N\right) = \Delta u^N, \end{aligned} \tag{3.36}$$

to reach

$$\left(\nabla \wedge \left(P^N\left(\operatorname{div}\left(\mathbf{K}\mathbf{D}^N\right)\right)\right), \zeta^N\right) = \left(\operatorname{div}\left(\mathbf{K}\mathbf{D}^N\right), \nabla \wedge \zeta^N\right). \tag{3.37}$$

By utilizing this, (3.4), and Schwarz's inequality in (3.35), we obtain that

$$\begin{aligned} \left|\zeta^N(t)\right|^p \leq \left|\zeta_0^N\right|^p &+ \left(\frac{1}{2}\right)p(p-1)\int_0^t \left|\zeta^N\right|^{p-2}\left|\nabla \wedge \left(P^N G\left(u^N,t\right)\right)\right|^2 ds \\ &+ p\left|\int_0^t \left|\zeta^N\right|^{p-2}\left(\nabla \wedge \left(P^N G\left(u^N,t\right)\right), \zeta^N\right)d\overline{W}\right| \\ &+ p\int_0^t \left|\zeta^N\right|^{p-1}\left|\nabla \wedge \left(P^N F\left(u^N,t\right)\right)\right| ds. \end{aligned} \tag{3.38}$$

Thanks to the estimates (2.6), (3.1), and (3.2) we deduce from the above estimate that

$$\begin{aligned} \overline{E}\sup_{0\leq s\leq t}\left|\zeta^N(s)\right|^p &\leq p\overline{E}\sup_{0\leq s\leq t}\left|\int_0^s \left|\zeta^N\right|^{p-2}\left(\nabla \wedge \left(P^N G\left(u^N,t\right)\right), \zeta^N\right)d\overline{W}\right| \\ &+ \overline{E}\left|\zeta_0^N\right|^p + p\overline{E}\int_0^t \left|\zeta^N\right|^p ds + CT. \end{aligned} \tag{3.39}$$

Let us set

$$\Gamma^N = p\overline{E}\sup_{0\leq s\leq t}\left|\int_0^s \left|\zeta^N\right|^{p-2}\left(\nabla \wedge \left(P^N G\left(u^N,t\right)\right), \zeta^N\right)d\overline{W}\right|. \tag{3.40}$$

By using Burkhölder-Davis-Gundy's inequality and Schwarz's inequality we obtain

$$p\Gamma^N \leq \overline{E}\left(\int_0^s \left|\zeta^N\right|^{2p-4} \left|\nabla \wedge \left(P^N G\left(u^N, t\right)\right)\right|^2 \left|\zeta^N\right|^2 ds\right)^{1/2}. \tag{3.41}$$

We derive from this and the estimates (2.6) (this is allowed since $P^N G(u^N, t)) \in \mathbb{H}_N$) and (3.2) that

$$p\Gamma^N \leq \left(\frac{1}{2}\right)\overline{E}\sup_{0\leq s\leq t}\left|\zeta^N\right|^p + C\overline{E}\int_0^t \left|\zeta^N\right|^p ds. \tag{3.42}$$

From this, (3.39), and Gronwall's lemma we deduce that

$$\overline{E}\sup_{0\leq s\leq t}\left|\zeta^N(s)\right|^p \leq C. \tag{3.43}$$

Owing to (2.6) the proof of the lemma is finished. □

The following result is central in the proof of the forthcoming tightness property of the Galerkin solution.

Lemma 3.7. *For any* $0 \leq \delta < 1$ *one has*

$$\overline{E}\sup_{|\theta|\leq\delta}\int_0^{T-\delta} \left|u^N(s+\theta) - u^N(s)\right|_{\mathbb{V}^*}^2 \leq C\delta. \tag{3.44}$$

Proof. We can rewrite (3.11) in an integral form as the equality between random variables with values in $\mathbb{V}^*$

$$\begin{aligned} u^N &- \int_0^t P^N\left(\operatorname{div}\left(\mathbf{K}\mathbf{D}^N\right)\right)ds + \int_0^t P^N\left(u^N \cdot \nabla u^N\right)ds \\ &= u_0^N + \int_0^t P^N F\left(u^N, t\right)ds + \int_0^N P^N G\left(u^N, t\right)d\overline{W}. \end{aligned} \tag{3.45}$$

By using the triangle inequality for the norm $|\cdot|_{\mathbb{V}^*}$, we deduce from (3.45) that

$$\begin{aligned} \left|u^N(t+\theta) - u^N(t)\right|_{\mathbb{V}^*}^2 \leq C\theta\int_t^{t+\theta}\left|\operatorname{div}\left(\mathbf{K}\mathbf{D}^N\right)\right|_{\mathbb{V}^*}^2 ds + C\theta\int_t^{t+\theta}\left|\left(u^N.\nabla u^N\right)\right|_{\mathbb{V}^*}^2 ds \\ + C\theta\int_t^{t+\theta}\left|F\left(u^N, s\right)\right|^2 ds + C\left|\int_t^{t+\theta} P^N G\left(u^N, s\right)d\overline{W}\right|^2, \end{aligned} \tag{3.46}$$

for any $0 \leq \theta \leq \delta$. The continuity of div as linear operator along with (2.22), (3.1), and

Lemmas 3.5 and 3.6 implies that

$$\overline{E}\sup_{0\leq\theta\leq\delta}\int_0^{T-\delta}\left|u^N(t+\theta)-u^N(t)\right|_{\mathbb{V}^*}^2dt\leq C\delta+C\delta\overline{E}\int_0^{T-\delta}\int_t^{t+\delta}\left|\left(u^N.\nabla u^N\right)\right|_{\mathbb{V}^*}^2ds\,dt$$
$$+C\int_0^{T-\delta}\overline{E}\sup_{0\leq\theta\leq\delta}\left|\int_t^{t+\theta}P^NG\left(u^N,s\right)d\overline{W}\right|^2dt. \tag{3.47}$$

By making use of Martingale inequality, (3.2), and Lemma 3.5 we have that

$$\overline{E}\sup_{0\leq\theta\leq\delta}\int_0^{T-\delta}\left|u^N(t+\theta)-u^N(t)\right|_{\mathbb{V}^*}^2dt\leq C\delta\overline{E}\int_0^{T-\delta}\int_t^{t+\delta}\left|\left(u^N.\nabla u^N\right)\right|_{\mathbb{V}^*}^2ds\,dt+C\delta+C\delta^2. \tag{3.48}$$

By the well-known inequality

$$\left|\left(u^N.\nabla u^N\right)\right|_{\mathbb{V}^*}^2\leq C_B\left|u^N\right|^2\left\|u^N\right\|^2, \tag{3.49}$$

which holds in the 2-dimensional case, we obtain that

$$\overline{E}\sup_{0\leq\theta\leq\delta}\int_0^{T-\delta}\left|u^N(t+\theta)-u^N(t)\right|_{\mathbb{V}^*}^2dt\leq C\delta. \tag{3.50}$$

To complete the proof we use the same argument for negative values of θ. □

3.2.2. Tightness Property and Application of Prokhorov's and Skorohod's Theorems

We denote by $\mathfrak{Z}$ the following subset of $L^2(0,T;\mathbb{H})$:

$$\mathfrak{Z}=\left\{z\in L^\infty(0,T;\mathbb{V});\ \sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M}|z(t+\theta)-z(t)|_{\mathbb{V}^*}^2\leq C\nu_M\right\}, \tag{3.51}$$

for any sequences ν_M,μ_M such that $\nu_M,\mu_M\to 0$ as $M\to\infty$ and $\sum_{M\geq 0}\mu_M/\nu_M<\infty$. The following result is a particular case of Theorem 2.3 (see also [40, Proposition 3.1, page 45] for a similar result).

Lemma 3.8. *The set $\mathfrak{Z}$ is compact in $L^2(0,T;\mathbb{H})$.*

Next we consider the space $\mathfrak{S}=C(0,T;\mathbb{R}^m)\times L^2(0,T;\mathbb{H})$ endowed with its Borel σ-algebra $\mathcal{B}(\mathfrak{S})$ and the family of probability measures Π^N on $\mathfrak{S}$, which is the probability measure induced by the following mapping:

$$\phi:\omega\longmapsto\left(\overline{W}(\omega,\cdot),u^N(\omega,\cdot)\right). \tag{3.52}$$

That is, for any $A\in\mathcal{B}(\mathfrak{S})$, $\Pi^N(A)=\overline{P}(\phi^{-1}(A))$.

Lemma 3.9. *The family* $(\Pi^N)_{N\geq 1}$ *is tight in* $\mathfrak{S}$.

Proof. For any $\varepsilon > 0$ and $M \geq 1$, we claim that there exists a compact subset $\mathfrak{K}_\varepsilon$ of $\mathfrak{S}$ such that $\Pi^N(\mathfrak{K}_\varepsilon) \geq 1-\varepsilon$. To back our claim we define the sets

$$\mathfrak{W}_\varepsilon = \left\{ W : \sup_{\substack{t,s\in[0,T]\\ |t-s|<T/2^M}} 2^{M/8}|W(t)-W(s)| \leq J_\varepsilon,\ \forall M \right\},$$
$$\mathfrak{Z}_\varepsilon = \left\{ z;\ \sup_{0\leq t\leq T}|z(t)|^2 \leq K_\varepsilon,\ \sup_{0\leq t\leq T}\|z(t)\|^2 \leq L_\varepsilon,\ \sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M} |z(t+\theta)-z(t)|^2_{\mathbb{V}^*}dt \leq R_\varepsilon \nu_M \right\}, \tag{3.53}$$

where $J_\varepsilon, K_\varepsilon, L_\varepsilon$, and R_ε are positive constants to be fixed in the course of the proof. The sequences ν_M and μ_M are chosen so that they are independent of ε, $\nu_M, \mu_M \to 0$ as $M \to \infty$ and $\sum_M \mu_M/\nu_M < \infty$. It is clear by Ascoli-Arzela's theorem that $\mathfrak{W}_\varepsilon$ is a compact subset of $C(0,T;\mathbb{R}^m)$, and by Lemma 3.8 $\mathfrak{Z}_\varepsilon$ is a compact subset of $L^2(0,T;\mathbb{H})$. We have to show that $\mathfrak{P}_\varepsilon = \Pi^N((\overline{W},u^N) \notin \mathfrak{W}_\varepsilon \times \mathfrak{Z}_\varepsilon) < \varepsilon$. Indeed, we have

$$\begin{aligned}
\mathfrak{P}_\varepsilon \leq{}& \overline{P}\left[\bigcup_{M=1}^{\infty}\bigcup_{j=1}^{2^M}\left(\sup_{t,s\in I_j}\left|\overline{W}(t)-\overline{W}(s)\right| \geq J_\varepsilon \frac{1}{2^{M/8}}\right)\right] + \overline{P}\left(\sup_{0\leq t\leq T}\left|u^N(t)\right|^2 \geq K_\varepsilon\right) \\
&+ \overline{P}\left(\bigcup_M\left\{\sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M}\left|u^N(t+\theta)-u^N(t)\right|^2_{\mathbb{V}^*}dt \geq R_\varepsilon\nu_M\right\}\right) \\
&+ \overline{P}\left(\sup_{0\leq t\leq T}\left\|u^N(t)\right\|^2 \geq L_\varepsilon\right),
\end{aligned} \tag{3.54}$$

where $\{I_j : 1 \leq j \leq 2^M\}$ is a family of intervals of length $T/2^M$ which forms a partition of the interval $[0,T]$. It is well known that for any Wiener process B

$$\overline{E}|B(t)-B(s)|^{2m} = C_m|t-s|^m \quad \text{for any } m \geq 1, \tag{3.55}$$

where C_m is a constant depending only on m. From this and Markov's Inequality

$$\overline{P}(\omega : \zeta(w) \geq \alpha) \leq \frac{1}{\alpha^k}\overline{E}\left(|\zeta(\omega)|^k\right), \tag{3.56}$$

where $\zeta(\omega)$ is a random variable on $(\overline{\Omega}, \overline{\mathfrak{F}}, \overline{P})$ and positive numbers k and α, we obtain

$$\begin{aligned}\mathfrak{P}_\varepsilon \leq& \sum_{M=1}^{\infty}\sum_{j=1}^{2^M} C_m\left(2^{M/8}\right)^{2m}\frac{1}{J_\varepsilon^{2m}}\left(\frac{T}{2^M}\right)^m + \frac{1}{K_\varepsilon}\overline{E}\sup_{t\leq T}\left|u^N(t)\right|^2 + \frac{1}{L_\varepsilon}\overline{E}\sup_{0\leq t\leq T}\left\|u^N(t)\right\|^2 \\ &+ \sum_M \frac{1}{R_\varepsilon \nu_M}\overline{E}\sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M}\left|u^N(t+\theta)-u^N(t)\right|_{\mathbb{V}^*}^2 dt.\end{aligned} \tag{3.57}$$

Owing to the Lemmas 3.5 and 3.7 and by choosing $m = 2$, we have

$$\begin{aligned}\mathfrak{P}_\varepsilon &\leq \frac{C_2T^2}{J_\varepsilon^4}\sum_{M=1}^{\infty}2^{-(1/2)M} + C\left(\frac{1}{K_\varepsilon}+\frac{1}{L_\varepsilon}+\frac{1}{R_\varepsilon}\sum_M\frac{\mu_M}{\nu_M}\right) \\ &\leq \frac{C_2T^2}{J_\varepsilon^4}\left(1+\sqrt{2}\right) + C\left(\frac{1}{K_\varepsilon}+\frac{1}{L_\varepsilon}+\frac{1}{R_\varepsilon}\sum_M\frac{\mu_M}{\nu_M}\right).\end{aligned} \tag{3.58}$$

A convenient choice of $J_\varepsilon, K_\varepsilon, L_\varepsilon$, and R_ε completes the proof of the claim, and hence the proof of the lemma. □

Now it follows by Prokhorov's theorem that the family $(\Pi^N)_{N\geq 1}$ is relatively compact in the set of probability measures (equipped with the weak convergence topology) on $\mathfrak{S}$. Then, we can extract a subsequence Π^{N_μ} that weakly converges to a probability measure Π. By Skorohod's theorem, there exists a probability space $(\Omega, \mathfrak{F}, P)$ and random variables (W^{N_μ}, u^{N_μ}) and (W, u) on $(\Omega, \mathfrak{F}, P)$ with values in $\mathfrak{S}$ such that

$$\begin{aligned}W^{N_\mu} &\longrightarrow W \quad \text{in } C(0,T;\mathbb{R}^m)\ P\text{-a.s.}, \\ u^{N_\mu} &\longrightarrow u \quad \text{in } L^2(0,T;\mathbb{H})\ P\text{-a.s.}\end{aligned} \tag{3.59}$$

Moreover, the probability law of (W^{N_μ}, u^{N_μ}) is Π^{N_μ} and that of (W, u) is Π.

For the filtration $\mathfrak{F}^t$, it is enough to choose $\sigma(W(s), u(s) : 0 \leq s \leq t)$.

By the same argument as in [40, Section 3.3] (see also [41, Section 4.3]) we can prove that the limit process W is a standard m-dimensional Wiener process defined on $(\Omega, \mathfrak{F}, \{\mathfrak{F}^t\}_{0\leq t\leq T}, P)$.

Theorem 3.10. *The pair u^{N_μ}, W^{N_μ} satisfies the following equation:*

$$\begin{aligned}&\left(u^{N_\mu}(t), e_i\right) + \int_0^t\int_D (\mathbf{KD} : \nabla e_i)dx\,ds + \int_0^t\left(\left(u^{N_\mu}.\nabla e_i\right), u^{N_\mu}\right)ds \\ &\quad = \left(u_0^{N_\mu}, e_i\right) + \int_0^t\left(F\left(u^{N_\mu}(s), s\right), e_i\right)ds + \int_0^t\left(G\left(u^{N_\mu}(s), s\right), e_i\right)dW^{N_\mu}\end{aligned} \tag{3.60}$$

for almost all $\omega \in \Omega$, for any $t \in [0,T]$ and $1 \leq i \leq N_\mu$.

Proof. The proof follows the same lines as in [6, Section 4.3.4] (see also [41, Section 4.3]), and so we omit it. □

3.2.3. Passage to the Limit

To back our goal we need to pass to the limit in the terms of the estimate (3.60). From the tightness property we have

$$u^{N_\mu} \longrightarrow u \quad \text{in } L^2(0,T;\mathbb{H}) \ P\text{-a.s.,} \tag{3.61}$$

as $N_\mu \to \infty$. Since u^{N_μ} agrees with (3.60), then it verifies the same estimates as u^N. In particular the estimate

$$E\sup_{0\leq t\leq T}\left|u^{N_\mu}\right|^p \leq C \tag{3.62}$$

for $p \geq 2$ implies that the norm $|u^{N_\mu}|_{L^2(0,T;\mathbb{H})}$ is uniformly integrable with respect to the probability measure. Therefore, we can deduce from Vitali's Convergence Theorem that

$$u^{N_\mu} \longrightarrow u \quad \text{in } L^2\Big(\Omega, \mathcal{F}, P, L^2(0,T;\mathbb{H})\Big), \tag{3.63}$$

as $N_\mu \to \infty$. It is readily seen that

$$\left(u^{N_\mu}, e_i\right)_{\mathbb{V}} \longrightarrow (u, e_i)_{\mathbb{V}} \quad \text{weakly in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T)\Big). \tag{3.64}$$

Thanks to the convergence (3.63) and the continuity of **K** we see that

$$\int_D \left(\mathbf{KD}^{N_\mu} : \nabla e_i\right)dx \longrightarrow \int_D (\mathbf{KD} : \nabla e_i)dx \quad \text{strongly in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T)\Big), \tag{3.65}$$

as $N_\mu \to \infty$. Let χ be an element of $L^\infty(\Omega \times [0,T], dP \otimes dt)$. Thanks to (3.63) we can prove by arguing as in [42] that

$$E\int_0^T \left(\left(u^{N_\mu} \cdot \nabla e_i\right), u^{N_\mu}\right)\chi dt \longrightarrow E\int_0^T ((u \cdot \nabla e_i), u)\chi dt, \tag{3.66}$$

as $N_\mu \to \infty$. The dense injection

$$L^\infty(\Omega \times [0,T], dP \otimes dt) \subset L^2(\Omega \times [0,T], dP \otimes dt) \tag{3.67}$$

together with the relation (3.66) shows that

$$\left(\left(u^{N_\mu} \cdot \nabla\right)e_i, u^{N_\mu}\right) \rightharpoonup ((u \cdot \nabla)e_i, u) \quad \text{weakly in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T)\Big), \tag{3.68}$$

as $N_\mu \to \infty$.

It follows from the continuity of F, (3.63), and Vitali's theorem that

$$P^{N_\mu} F\left(u^{N_\mu}(\cdot),\cdot\right) \longrightarrow F(u(\cdot),\cdot) \quad \text{strongly in } L^2\left(\Omega,\mathcal{F},P;L^2(0,T;\mathbb{V})\right), \tag{3.69}$$

as $N_\mu \to \infty$. This implies in particular that

$$\left(F\left(u^{N_\mu}(\cdot),\cdot\right),e_i\right) \longrightarrow (F(u(\cdot),\cdot),e_i) \quad \text{strongly in } L^2\left(\Omega,\mathcal{F},P;L^2(0,T)\right), \tag{3.70}$$

as $N_\mu \to \infty$. We can use the argument in [6, Section 4.3.5] (see also [41, Section 5.1]) to prove that

$$\int_0^t \left(G\left(u^{N_\mu},s\right),e_i\right)dW^{N_\mu} \rightharpoonup \int_0^t (G(u,s),e_i)dW \quad \text{weakly in } L^2\left(\Omega,\mathcal{F},P;L^2(0,T)\right), \tag{3.71}$$

for any $t \in (0,T)$ and as $N_\mu \to \infty$.

Combining all these results and passing to the limit in (3.60), we see that u satisfies (3.8) which holds for almost all $\omega \in \Omega$, for all $t \in [0,T]$. This proves the first part of Theorem 3.3. By arguing as in [43] (Chapter 2, Lemma 1.2) we get the continuity result stated in Theorem 3.3.

3.2.4. Proof of the Uniqueness of the Solution

Let u_1 and u_2 be two probabilistic weak solutions of (1.1), (1.2) starting with the same initial condition and defined on the same stochastic basis $(\Omega,\mathcal{F},\mathcal{F}^t,P)$ with the same Wiener process W. Set $v = u_1 - u_2$ and

$$\begin{gathered}\mathbf{D}_v = \left(\frac{1}{2}\right)(\nabla v + \nabla^t v), \\ \mathbf{D}_i = \left(\frac{1}{2}\right)(\nabla u_i + \nabla^t u_i), \quad i = 1,2.\end{gathered} \tag{3.72}$$

It can be shown that the process v satisfies the following equation:

$$\begin{aligned}&dv(t) - \mathbb{P}\operatorname{div}(\mathbf{KD}_1 - \mathbf{KD}_2)dt + \mathbb{P}((u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2)dt \\ &\quad = (F(u_1(t),t) - F(u_2(t),t))dt + G(u_1(t),t) - G(u_2(t),t)dW,\end{aligned} \tag{3.73}$$

where $\mathbb{P}$ is the projector from $\mathbb{L}^2(D)$ onto $\mathbb{H}$. Thanks to Itô's formula for $|v|^2$ we have

$$\begin{aligned}
&|v(t)|^2 + 2\int_0^t \int_D (\mathbf{KD}_1 - \mathbf{KD}_2 : \mathbf{D}_v)dx\,dt + 2\int_0^t ((v\cdot\nabla)u_1, v)ds \\
&\quad = \int_0^t \Big(2(F(u_1(s),s) - F(u_2(s),s), v(s)) + |G(u_1(s),s) - G(u_2(s),s)|^2\Big)ds \\
&\qquad + 2\int_0^t (G(u_1(s),s) - G(u_2(s),s), v(s))dW.
\end{aligned} \tag{3.74}$$

Setting $\sigma(t) = \exp(\int_0^t -\eta\|u_1(s)\|^2 ds)$, for all $\eta > 0$, we have that

$$\begin{aligned}
&\sigma(t)|v(t)|^2 + 2\int_0^t \sigma(s)\int_D (\mathbf{KD}_1 - \mathbf{KD}_2 : \mathbf{D}_v)dx\,dt + 2\int_0^t \sigma(s)((v\cdot\nabla)u_1, v)ds \\
&\quad = \int_0^t \sigma(s)\Big(2(F(u_1(s),s) - F(u_2(s),s), v(s)) + |G(u_1(s),s) - G(u_2(s),s)|^2\Big)ds \\
&\qquad + 2\int_0^t \sigma(s)(G(u_1(s),s) - G(u_2(s),s), v(s))dW - \eta\int_0^t \sigma(s)\|u_1(s)\|^2|v(s)|^2 ds.
\end{aligned} \tag{3.75}$$

By the assumptions on $\mathbf{K}$, F, and G and (3.49) we have

$$\begin{aligned}
E\sigma(t)|v(t)|^2 \le{}& 2C_B E\int_0^t \sigma(s)|v(s)|^2\|u_1(s)\|ds + 2C'_F E\int_0^t \sigma(s)|v(s)|^2 ds \\
&+ C'^2_G E\int_0^t \sigma(s)|v(s)|^2 ds - \eta E\int_0^t \sigma(s)\|u_1(s)\|^2|v(s)|^2 ds.
\end{aligned} \tag{3.76}$$

which implies

$$\begin{aligned}
E\sigma(t)|v(t)|^2 \le{}& C_B^2 E\int_0^t \sigma(s)|v(s)|^2\|u_1(s)\|^2 ds + CE\int_0^t \sigma(s)|v(s)|^2 ds \\
&- \eta E\int_0^t \sigma(s)\|u_1(s)\|^2|v(s)|^2 ds
\end{aligned} \tag{3.77}$$

By choosing $\eta = C_B^2$ and by making use of Gronwall's lemma we have

$$E\sigma(t)|v(t)|^2 = 0, \tag{3.78}$$

for any $t \ge 0$. Since $0 \le \sigma(t) < \infty$, then this completes the proof of Theorem 3.4.

3.3. Example: The Stochastic Equation for the Maxwell Fluids

The motion of a randomly forced Maxwell fluids is given by the system (1.1)-(1.2). The tensor σ for the Maxwell fluids is given by

$$\left(1+\sum_{l=1}^{L}\lambda_l\frac{\partial^l}{\partial t^l}\right)\sigma=2\mu\left(1+\sum_{l=1}^{L-1}k_l\mu^{-1}\frac{\partial^l}{\partial t^l}\right)\mathbf{D},\quad L=1,2,3,\ldots, \tag{3.79}$$

where $\lambda_l > 0$ and $k_l > 0$ represent the relaxation and retardation times, respectively. Considering the polynomials

$$\begin{aligned} P_m(p)&=\mu+\sum_{i=1}^{L}(k_l-\lambda_l)p^l,\\ Q(p)&=1+\sum_{l-1}^{L}\lambda_l p^l. \end{aligned} \tag{3.80}$$

It is shown in [30] that the operator $\mathbf{K}$ for the Maxwell fluids is given by

$$\mathbf{KD}=\sum_{l=1}^{L}\int_0^t\beta_l^{(m)}e^{-\alpha_l(t-\tau)}\mathbf{D}(x,\tau)d\tau, \tag{3.81}$$

where

$$\beta_l^{(m)}=P_m(-\alpha_l)\left[Q'(-\alpha_l)\right]^{-1} \tag{3.82}$$

is assumed to be positive. Here the numbers $-\alpha_l$ designate the roots of the polynomial Q. The result in [30] shows that $\mathbf{K}$ satisfies (2.22)–(2.24) and (3.4). Hence the results in Theorems 3.3 and 3.4 can be applied to the stochastic equations (1.1)-(1.2) and (3.81) for the Maxwell fluids provided that the assumptions (HYP 1)–(HYP 4) hold.

4. Stochastic Equation of Type (1.1), (1.3)

This section is devoted to the investigation of (1.1), (1.3). We omit the details of the proofs since they can be derived from similar ideas used in Section 3. We state our main results in the first subsection and we give a concrete example in the second subsection. For this section we suppose following.

(AF) the mapping F induces a nonlinear operator from $\mathbb{H}\times[0,T]$ into $\mathbb{H}$ which is assumed to be measurable (resp., continuous) with respect to its second (resp., first) variable. We require that for almost all $t\in[0,T]$ and for each u

$$|F(u,t)|\le C_F(1+|u|). \tag{4.1}$$

(AG) The $\mathbb{H}^{\otimes m}$-valued function G defined on $\mathbb{H} \times [0,T]$ is measurable (resp., continuous) with respect to its second (resp., first) argument, and it verifies

$$|G(u,t)|_{\mathbb{H}^{\otimes m}} \leq C_G(1+|u|), \tag{4.2}$$

for all $t \in [0,T]$ and for any $u \in \mathbb{H}$.

(ASFG) We assume as well that

$$\begin{aligned} |F(u,t) - F(v,t)| &\leq C'_F|u-v|, \\ |G(u,t) - G(v,t)|_{\mathbb{H}^{\otimes m}} &\leq C'_G|u-v|, \end{aligned} \tag{4.3}$$

for any $u, v \in \mathbb{H}$.

4.1. Statement of the Main Results

We introduce the concept of the solution of the problem (1.1), (1.3).

Definition 4.1. By a probabilistic weak solution of the problem (1.1), (1.3), we mean a system

$$(\Omega, \mathcal{F}, P, \mathcal{F}^t, W, u), \tag{4.4}$$

where

(1) $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\mathcal{F}^t$ is a filtration on $(\Omega, \mathcal{F}, P)$;

(2) $W(t)$ is an m-dimensional $\mathcal{F}^t$-standard Wiener process;

(3) $u \in L^p(\Omega, \mathcal{F}, P; L^2(0,T;\mathbb{V})) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;\mathbb{H}))$ for all $2 \leq p < \infty$;

(4) for almost all t, $u(t)$ is $\mathcal{F}^t$-measurable;

(5) P-a.s. the following integral equation of Itô type holds.

$$\begin{aligned} (u(t) - u(0), \phi) + \nu \int_0^t ((u,\phi))ds + \int_0^t \int_D (\mathbf{KD} : \nabla\phi) dx\, ds + \int_0^t ((u \cdot \nabla\phi), u) ds \\ = \int_0^t (F(u(s),s),\phi)ds + \int_0^t (G(u(s),s),\phi)dW(s) \end{aligned} \tag{4.5}$$

for any $t \in [0,T]$ and $\phi \in \mathcal{V}$.

We have the following.

Theorem 4.2. *If $u_0 \in \mathbb{H}$ and if the hypotheses (AF)-(AG) hold, then the problem* (1.1), (1.3) *has a solution in the sense of the above definition. Moreover u is strongly (resp., weakly) continuous in $\mathbb{H}$ (resp., $\mathbb{V}$) with probability one.*

Proof. The proof follows from the Galerkin method; and the compactness method, the procedure is very similar to the proof of Theorem 3.3, and it is even easier. We just formally derive the crucial estimates.

The application of Itô's formula for $|u|^2$ yields

$$\begin{aligned} &|u|^2 + 2\nu\int_0^t \|u\|^2 ds + 2\int_0^t\int_D (\mathbf{KD} : \mathbf{D})dx\,dt \\ &\quad \le 2\int_0^t (F(u,t),u)ds + \int_0^t |G(u,t)|^2 ds + |u_0|^2 + 2\int_0^t (G(u,t),u)dW. \end{aligned} \tag{4.6}$$

More generally

$$\begin{aligned} &|u|^p + p\nu\int_0^t |u|^{p-2}\|u\|^2 ds + p\int_0^t |u|^{p-2}\int_D (\mathbf{KD} : \mathbf{D})dx\,dt - p\int_0^t |u|^{p-2}(F(u,t),u)ds \\ &\quad \le \left(\frac{1}{2}\right) p(p-1)\int_0^t |u|^{p-2}|G(u,t)|^2 ds + |u_0|^p + p\int_0^t |u|^{p-2}(G(u,t),u)dW, \end{aligned} \tag{4.7}$$

for any $2 \le p < \infty$. Thanks to the assumptions on $\mathbf{K}$, F, and G we obtain that

$$|u|^p + p\nu\int_0^t |u|^{p-2}\|u\|^2 ds \le |u_0|^p + C\int_0^t |u|^p ds + p\int_0^t |u|^{p-2}(G(u,t),u)dW. \tag{4.8}$$

Standard arguments of Martingale inequality and Gronwall's inequality yield

$$E\sup_{0\le t\le T} |u(t)|^p \le C. \tag{4.9}$$

Coming back to (4.6) we can show that

$$E\left(\int_0^T \|u(s)\|^2 ds\right)^{p/2}. \tag{4.10}$$

□

We also have the uniqueness result whose proof follows from similar arguments used in Theorem 3.4.

Theorem 4.3. *Assume that (AF)–(ASFG) hold and let u_1 and u_2 be two probabilistic weak solutions of* (1.1), (1.3) *starting with the same initial condition and defined on the same stochastic basis* $(\Omega, \mathcal{F}, \mathcal{F}^t, P)$. *If one sets $v = u_1 - u_2$, then one has $v = 0$ almost surely.*

4.2. Application to the Oldroyd Fluids

The tensor σ for the Oldroyd fluids is given by

$$\left(1+\sum_{l=1}^{L}\lambda_l\frac{\partial^l}{\partial t^l}\right)\sigma=2\mu\left(1+\sum_{l=1}^{L}k_l\mu^{-1}\frac{\partial^l}{\partial t^l}\right)\mathbf{D},\quad L=1,2,3,\ldots, \tag{4.11}$$

where $\lambda_l > 0$ and $k_l > 0$ represent the relaxation and retardation times, respectively. Let

$$\begin{aligned} P_o(p) &= \mu-\nu+\sum_{i=1}^{L}(k_l-\nu\lambda_l)p^l, \\ \beta_l^{(o)} &= P_o(-\alpha_l)\left[Q'(-\alpha_l)\right]^{-1}. \end{aligned} \tag{4.12}$$

The latter quantity is assumed to be positive. It is shown in [30] that the operator $\mathbf{K}$ for the Oldroyd fluids is given by

$$\mathbf{KD}=\sum_{l=1}^{L}\int_0^t\beta_l^{(o)}e^{-\alpha_l(t-\tau)}\mathbf{D}(x,\tau)d\tau, \tag{4.13}$$

and that $\mathbf{K}$ satisfies the assumption (2.22)–(2.24). Therefore Theorems 4.2 and 4.3 hold for the Oldroyd fluid provided that the assumptions on F and G (see (AF)–(ASFG)) are valid.

Remark 4.4. Theorem 4.2 (resp., Theorem 3.3) holds true for those viscoelastic fluids which do not satisfy the assumption (ASFG) (resp., (HYP 3)). One example we can consider is the third-order fluids whose tensor is given by

$$\sigma=2\nu\mathbf{D}+\mu\mathbf{D}^3. \tag{4.14}$$

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