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*Research Article*

# $L^2$-Error Estimates of the Extrapolated Crank-Nicolson Discontinuous Galerkin Approximations for Nonlinear Sobolev Equations

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We analyze discontinuous Galerkin methods with penalty terms, namely, symmetric interior penalty Galerkin methods, to solve nonlinear Sobolev equations. We construct finite element spaces on which we develop fully discrete approximations using extrapolated Crank-Nicolson method. We adopt an appropriate elliptic-type projection, which leads to optimal $\ell^\infty(L^2)$ error estimates of discontinuous Galerkin approximations in both spatial direction and temporal direction.

## 1. Introduction

Let $\Omega$ be an open bounded domain in $R^d$, $d \geq 2$ with smooth boundary $\partial\Omega$, and let $0 < T < \infty$ be given. In this paper, we consider the problem of approximating $u(x,t)$ satisfying the following nonlinear Sobolev equations:

$$\begin{aligned} u_t - \nabla \cdot \{a(u)\nabla u + b(u)\nabla u_t\} &= f(u) \quad \text{in } \Omega \times (0,T], \\ (a(u)\nabla u + b(u)\nabla u_t)\cdot n &= 0 \quad \text{on } \partial\Omega \times (0,T], \\ u(x,0) &= u_0(x) \quad \text{on } \Omega, \end{aligned} \tag{1.1}$$

where $n$ denotes the unit outward normal vector to $\partial\Omega$ and $u_0(x)$ is a given function defined on $\Omega$. The initial data $u_0(x)$, $f$, $a$, and $b$ are assumed to be such that (1.1) admits a solution sufficiently smooth to guarantee the convergence results to be presented below. For details about the physical significance and various properties of existence and uniqueness of the Sobolev equations, see [1–6].

Early, in [7–9] the authors constructed the Galerkin approximations to the solution of (1.1) with periodic boundary conditions in one-dimensional space and obtained the optimal convergence in $L^2$ normed space and superconvergence results. Recently, Lin [10] constructed the Galerkin approximation of (1.1) with $d = 2$ using Crank-Nicolson method and proved the optimal convergence of error in $L^2$ normed space. In [11] the authors constructed the semidiscrete finite element approximations of (1.1) with nonlinear boundary condition and obtained the optimal $L^2$-error estimates.

In this work we will approximate the solution of (1.1) using a discontinuous symmetric Galerkin method with interior penalties for the spatial discretization and extrapolated Crank-Nicolson method for the time stepping. By implementing the extrapolated technique, we induce the linear systems which can be solved explicitly, and thus obviate the order reduction phenomenon which occurs when the system involved is nonlinear.

Compared to the classical Galerkin method, the discontinuous Galerkin method is very well suited for adaptive control of error and can deliver high orders of accuracy when the exact solution is sufficiently smooth. In [12] Rivière and Wheeler formulated and analyzed a family of discontinuous methods to approximate the solution of the transport problem with nonlinear reaction. They construct semidiscrete approximations which converge optimally in $h$ and suboptimally in $r$ for the energy norm and suboptimally for the $L^2$ norm. They also constructed fully discrete approximations and proved the optimal convergence in the temporal direction. Furthermore to solve reactive transport problems Sun and Wheeler in [13] analyzed three discontinuous Galerkin methods, namely, symmetric interior penalty Galerkin method, nonsymmetric interior penalty Galerkin method, and incomplete interior penalty Galerkin method. They obtained error estimates in $L^2(H^1)$ which are optimal in $h$ and nearly optimal in $p$ and they developed a parabolic lift technique for SIPG which leads to $h$-optimal and nearly $p$-optimal error estimates in $L^2(L^2)$ and negative norms. Recently in [14, 15] Sun and Yang adapted discontinuous Galerkin methods to nonlinear Sobolev equations and obtained the optimal $H^1$ error estimates. The main object of this paper is to obtain the optimal $\ell^\infty(L^2)$ error estimates in both the spatial direction and the temporal direction by adopting an appropriate elliptic-type projection.

This paper is organized as follows. In Section 2, we introduce some notations and preliminaries. In Section 3, we construct appropriate finite element spaces and define an auxiliary projection and prove its convergence. In Section 4, we construct the extrapolated discontinuous Galerkin fully discrete method which yields the second-order convergence in the temporal direction. The corresponding error estimates of the approximate solutions are also discussed.

## 2. Notations and Preliminaries

Let $\mathcal{E}_h = \{E_1, E_2, \ldots, E_{N_h}\}$ be a regular quasi-uniform subdivision of $\Omega$ where $E_i$ is a triangle or a quadrilateral if $d = 2$ and $E_i$ is a 3-simplex or 3-rectangle if $d = 3$. Let $h_j = \operatorname{diam}(E_j)$ and $h = \max_{1\le j\le N_h} h_j$. Here, the regular requirement is that there exists a constant $\rho > 0$ such that each $E_j$ contains a ball of radius $\rho h_j$. The quasi-uniformity requirement is that there is a constant $\gamma > 0$ such that

$$\frac{h}{h_j} \le \gamma \quad \text{for } j = 1, 2, \ldots, N_h. \tag{2.1}$$

We denote the edges (resp., faces for $d = 3$) of $\mathcal{E}_h$ by $\{e_1, e_2, \ldots, e_{P_h}, e_{P_h+1}, \ldots, e_{N_h}\}$ where $e_k$ has positive $d-1$ dimensional Lebesgue measure, $e_k \subset \Omega$, $1 \le k \le P_h$, and $e_k \subset \partial\Omega$, $P_h + 1 \le k \le N_h$. With each edge (or face) $e_k$, we associate a unit normal vector $n_k$ to $E_i$ if $e_k = \partial E_i \cap \partial E_j$ and $i < j$. For $k \ge P_h + 1$, $n_k$ is taken to be the unit outward vector normal to $\partial\Omega$.

For an $s \ge 0$ and a domain $E \subset \mathbb{R}^d$, we denote by $H^s(E)$ the Sobolev space of order $s$ equipped with the usual Sobolev norm $\|\cdot\|_{s,E}$. We simply write $\|\cdot\|_s$ instead of $\|\cdot\|_{s,\Omega}$ if $E = \Omega$ and $\|\cdot\|_E$ instead of $\|\cdot\|_{s,E}$ if $s = 0$. And also the usual seminorm defined on $H^s(E)$ is denoted by $|\cdot|_{s,E}$.

Now for an $s \ge 0$ and a given subdivision $\mathcal{E}_h$, we define the following space:

$$H^s(\mathcal{E}_h) = \left\{ v \in L^2(\Omega) | v|_{E_i} \in H^s(E_i),\ i = 1, 2, \ldots, N_h \right\}. \tag{2.2}$$

For $\phi \in H^s(\mathcal{E}_h)$ with $s > 1/2$, we define the average function $\{\phi\}$ and the jump function $[\phi]$ such that

$$\begin{aligned} \{\phi\} &= \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}\left(\phi|_{E_j}\right)|_{e_k}, \quad \forall x \in e_k,\ 1 \le k \le P_h, \\ [\phi] &= (\phi|_{E_i})|_{e_k} - \left(\phi|_{E_j}\right)|_{e_k}, \quad \forall x \in e_k,\ 1 \le k \le P_h, \end{aligned} \tag{2.3}$$

where $e_k = \partial E_i \cap \partial E_j$ with $i < j$.

We associated the following broken norms with the space $H^s(\mathcal{E}_h)$:

$$\begin{aligned} |||\phi|||^2 &= \sum_{i=1}^{N_h} \|\phi\|_{E_i}^2, \\ |||\phi|||_1^2 &= \sum_{i=1}^{N_h} \left( \|\phi\|_{1,E_i}^2 + h_i^2 \left\| \nabla^2 \phi \right\|_{E_i}^2 \right) + J_\beta^\sigma(\phi, \phi), \end{aligned} \tag{2.4}$$

where

$$J_\beta^\sigma(\phi, \psi) = \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [\phi]\,[\psi]\,ds, \quad \beta > 0 \tag{2.5}$$

is an interior penalty term and $\sigma$ is a discrete positive function that takes the constant value $\sigma_k$ on the edge $e_k$ and is bounded below by $\sigma_0 > 0$ and above by $\sigma^* > 0$.

## 3. Finite Element Spaces and Convergence of Auxiliary Projection

For a positive integer $r$, we construct the following finite element spaces:

$$D_r(\mathcal{E}_h) = \left\{ v \in L^2(\Omega) | v|_{E_i} \in P_r(E_i),\ i = 1, 2, \ldots, N_h \right\}, \tag{3.1}$$

where $P_r(E_i)$ denotes the set of polynomials of degree less than or equal to $r$ on $E_i$.

Now we state the following *hp*-approximation properties and trace inequalities whose proofs can be found in [16, 17].

**Lemma 3.1.** *Let $E_j \in \mathcal{E}_h$ and $\phi \in H^s(E_j)$. Then there exist a positive constant $C$ depending on $s$, $\gamma$, and $\rho$ but independent of $\phi$, $r$, and $h$ and a sequence $z_r^h \in P_r(E_j)$, $r = 1, 2, \ldots$ such that, for any $0 \le q \le s$,*

$$\begin{aligned}
\left\| \phi - z_r^h \right\|_{q,E_j} &\le C \frac{h_j^{\mu-q}}{r^{s-q}} \|\phi\|_{s,E_j} \quad s \ge 0, \\
\left\| \phi - z_r^h \right\|_{0,e_j} &\le C \frac{h_j^{\mu-1/2}}{r^{s-1/2}} \|\phi\|_{s,E_j} \quad s > \frac{1}{2}, \\
\left\| \phi - z_r^h \right\|_{1,e_j} &\le C \frac{h_j^{\mu-3/2}}{r^{s-3/2}} \|\phi\|_{s,E_j} \quad s > \frac{3}{2},
\end{aligned} \tag{3.2}$$

*where $\mu = \min(r+1, s)$ and $e_j$ is an edge or a face of $E_j$.*

**Lemma 3.2.** *For each $E_j \in \mathcal{E}_h$, there exists a positive constant $C$ depending only on $\gamma$ and $\rho$ such that the following trace inequalities hold:*

$$\begin{aligned}
\|\phi\|_{0,e_j}^2 &\le C \left( \frac{1}{h_j} |\phi|_{0,E_j}^2 + h_j |\phi|_{1,E_j}^2 \right), \quad \forall \phi \in H^1(E_j), \\
\left\| \frac{\partial \phi}{\partial n_j} \right\|_{0,e_j}^2 &\le C \left( \frac{1}{h_j} |\phi|_{1,E_j}^2 + h_j |\phi|_{2,E_j}^2 \right), \quad \forall \phi \in H^2(E_j),
\end{aligned} \tag{3.3}$$

*where $e_j$ is an edge or a face of $E_j$ and $n_j$ is the unit outward normal vector to $E_j$.*

Now we introduce the following bilinear mappings $A(\rho;\cdot,\cdot)$ and $B(\rho;\cdot,\cdot)$ defined on $H^s(\mathcal{E}_h) \times H^s(\mathcal{E}_h)$ as

$$\begin{aligned}
A(\rho;\phi,\psi) &= (a(\rho)\nabla\phi, \nabla\psi) - \sum_{k=1}^{P_h} \int_{e_k} \{a(\rho)\nabla\phi \cdot n_k\}[\psi] - \sum_{k=1}^{P_h} \int_{e_k} \{a(\rho)\nabla\psi \cdot n_k\}[\phi] + J_\beta^\sigma(\phi,\psi), \\
B(\rho;\phi,\psi) &= (b(\rho)\nabla\phi, \nabla\psi) - \sum_{k=1}^{P_h} \int_{e_k} \{b(\rho)\nabla\phi \cdot n_k\}[\psi] - \sum_{k=1}^{P_h} \int_{e_k} \{b(\rho)\nabla\psi \cdot n_k\}[\phi] + J_\beta^\sigma(\phi,\psi).
\end{aligned} \tag{3.4}$$

Using the bilinear mappings $A$ and $B$, we construct the weak formulation of problem (1.1) as follows:

$$(u_t(t), v) + A(u(t); u(t), v) + B(u(t); u_t(t), v) = \left( f(u(t)), v \right), \quad \forall v \in H^s(\mathcal{E}_h). \tag{3.5}$$

Now for a $\lambda > 0$ we define the following bilinear forms $A_\lambda(\rho;\cdot,\cdot)$ and $B_\lambda(\rho;\cdot,\cdot)$ on $H^s(\mathcal{E}_h) \times H^s(\mathcal{E}_h)$ such that

$$\begin{aligned} A_\lambda(\rho;\phi,\psi) &= A(\rho;\phi,\psi) + \lambda(\phi,\psi), \\ B_\lambda(\rho;\phi,\psi) &= B(\rho;\phi,\psi) + \lambda(\phi,\psi). \end{aligned} \tag{3.6}$$

$A_\lambda$ and $B_\lambda$ satisfy the following boundedness and coercivity properties, respectively. The proofs can be found in [18, 19].

**Lemma 3.3.** *For a $\lambda > 0$, there exists a constant $C > 0$ satisfying*

$$\begin{aligned} |A_\lambda(\rho;\phi,\psi)| &\le C|||\phi|||_1|||\psi|||_1, \\ |B_\lambda(\rho;\phi,\psi)| &\le C|||\phi|||_1|||\psi|||_1 \quad \forall\phi,\psi \in H^s(\mathcal{E}_h). \end{aligned} \tag{3.7}$$

**Lemma 3.4.** *For a $\lambda > 0$, there exists a constant $\underset{\sim}{c} > 0$ satisfying*

$$\begin{aligned} A_\lambda(\rho;\phi,\phi) &\ge \underset{\sim}{c}|||\phi|||_1^2, \\ B_\lambda(\rho;\phi,\phi) &\ge \underset{\sim}{c}|||\phi|||_1^2 \quad \forall\phi \in H^s(\mathcal{E}_h). \end{aligned} \tag{3.8}$$

Wheeler [20] introduced an elliptic projection to prove the optimal $L^2$-error estimates for Galerkin approximation to parabolic differential equations. Adopting this idea we construct a projection $\tilde{u}(t) : [0,T] \to D_r(\mathcal{E}_h)$ such that

$$\begin{aligned} A_\lambda(u;u-\tilde{u},v) + B_\lambda(u;u_t-\tilde{u}_t,v) &= 0 \quad \forall v \in D_r(\mathcal{E}_h), \\ (\tilde{u}(0),v) &= (u(0),v). \end{aligned} \tag{3.9}$$

By Lemmas 3.3 and 3.4, $\tilde{u}(t)$ is well defined.

## 4. The Optimal $\ell^\infty(L^2)$ Error Estimates of Fully Discrete Approximations

In this section we construct fully discrete discontinuous Galerkin approximations using extrapolated Crank-Nicolson method and prove the optimal convergence in $L^2$ normed space.

For a positive integer $N > 0$ we let $\Delta t = T/N$ and for $0 \le j \le N$ and we define $t_j = j(\Delta t)$ and $g_j = g(x,t_j)$. For $0 \le j \le N-1$, we define $\partial_t g_j = (g_{j+1}-g_j)/\Delta t$, $t_{j+1/2} = (1/2)(t_j+t_{j+1})$, and $g_{j+1/2} = (1/2)(g(t_j) + g(t_{j+1}))$.

The extrapolated Crank-Nicolson discontinuous Galerkin approximation $\{U_j\}_{j=0}^{N} \subset D_r(\mathcal{E}_h)$ is defined by

$$(\partial_t U_j, v) + A\big(EU_j : U_{j+1/2}, v\big) + B\big(EU_j; \partial_t U_j, v\big) = \big(f\big(EU_j\big), v\big), \quad \forall v \in D_r(\mathcal{E}_h), \tag{4.1}$$

where $EU_j = (3/2)U_j - (1/2)U_{j-1}$, $U_{j+1/2} = (1/2)(U_j + U_{j+1})$.

To apply (4.1), we need two initial stages $U_0$ and $U_1$ to be defined in the following:

$$\begin{gathered}(\partial_t U_0, v) + A(U_{1/2}; U_{1/2}, v) + B(U_{1/2}; \partial_t U_0, v) = \big(f(U_{1/2}), v\big), \\ U_0 = \widetilde{u}(0),\end{gathered} \tag{4.2}$$

where $U_{1/2} = (1/2)(U_0 + U_1)$.

To prove the optimal convergence of $u(t_j) - U_j$ in $L^2$ normed space we denote $\eta(x,t) = u(x,t) - \widetilde{u}(x,t)$ and $\xi(x,t_j) = \widetilde{u}(x,t_j) - U_j(x)$, $j = 0, 1, \ldots, N$.

Now we state the following approximations for $\eta$ whose proofs can be found in [18, 19].

**Theorem 4.1.** *If $u_t \in L^2(H^s)$ and $u_0 \in H^s$, then there exists a constant $C$ independent of $h$ and $\Delta t$ satisfying*

(i) $|||\eta_t||| + h|||\eta_t|||_1 \leq Ch^s(\|u_t\|_{H^s} + \|u_0\|_s)$

(ii) $|||\eta||| + h|||\eta|||_1 \leq Ch^s(\|u_t\|_{L^2(H^s)} + \|u_0\|_s)$.

**Theorem 4.2.** *If $u_t \in L^2(H^s)$, $u_{tt} \in H^s$, $u_{ttt} \in H^s$ and $u_0 \in H^s$ then there exists a constant $C$ independent of $h$ and $\Delta t$ satisfying*

(i) $|||\eta_{tt}|||_1 \leq Ch^{s-1}\{\|u_t\|_{L^2(H^s)} + \|u_{tt}\|_s + \|u_0\|_s\}$,

(ii) $|||\eta_{ttt}|||_1 \leq Ch^{s-1}\{\|u_t\|_{L^2(H^s)} + \|u_{tt}\|_s + \|u_{ttt}\|_s + \|u_0\|_s\}$

*provided that $\beta \geq 1/(d-1)$.*

By simple computations and the applications of Theorem 4.2 we obtain the following lemmas.

**Lemma 4.3.** *If $\rho$ satisfies*

$$\partial_t \widetilde{u}_j - \widetilde{u}_t\big(t_{j+1/2}\big) = (\Delta t)\rho_{j+1/2}, \tag{4.3}$$

*then there exists a constant $C$ independent of $h$ and $\Delta t$ such that*

$$\begin{gathered}\big|\big|\big|\rho_{j+1/2}\big|\big|\big| \leq C\Delta t\Big(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)} + \|u_{ttt}\|_{L^\infty(H^s)}\Big), \\ \big|\big|\big|\rho_{j+1/2}\big|\big|\big|_1 \leq C\Delta t\Big(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)} + \|u_{ttt}\|_{L^\infty(H^s)}\Big).\end{gathered} \tag{4.4}$$

Consequently from Lemma 4.3 there exists a constant $C$ independent of $\Delta t$ and $h$ such that

$$|||\rho_{j+1/2}||| \leq C\Delta t, \qquad |||\rho_{j+1/2}|||_1 \leq C\Delta t \tag{4.5}$$

if $u$ is sufficiently smooth.

**Lemma 4.4.** *If $r_{j+1/2} = \tilde{u}(t_{j+1/2}) - \tilde{u}_{j+1/2}$, then there exists a constant $C$ independent of $h$ and $\Delta t$ such that*

$$\begin{aligned} |||r_{j+1/2}||| &\leq C(\Delta t)^2\Big(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)}\Big), \\ |||r_{j+1/2}|||_1 &\leq C(\Delta t)^2\Big(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)}\Big). \end{aligned} \tag{4.6}$$

Consequently from Lemma 4.4 there exists a constant $C$ independent of $\Delta t$ and $h$ such that

$$|||r_{j+1/2}||| \leq C(\Delta t)^2, \qquad |||r_{j+1/2}|||_1 \leq C(\Delta t)^2 \tag{4.7}$$

if $u$ is sufficiently smooth.

**Lemma 4.5.** *If we let $\varphi_{j+1/2} = \tilde{u}(t_{j+1/2}) - ((3/2)\tilde{u}(t_j) - (1/2)\tilde{u}(t_{j-1}))$, then there exists a constant $C$ independent of $h$ and $\Delta t$ such that*

$$\begin{aligned} |||\varphi_{j+1/2}||| &\leq C(\Delta t)^2\Big(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)}\Big), \\ |||\varphi_{j+1/2}|||_1 &\leq C(\Delta t)^2\Big(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(H^s)}\Big). \end{aligned} \tag{4.8}$$

Consequently from Lemma 4.5, we induce that there exists a constant $C$ independent of $\Delta t$ and $h$ such that

$$|||\varphi_{j+1/2}||| \leq C(\Delta t)^2, \qquad |||\varphi_{j+1/2}|||_1 \leq C(\Delta t)^2 \tag{4.9}$$

if $u$ is sufficiently smooth.

**Theorem 4.6.** *For $0 < \lambda < 1$ and $\delta > 0$, if $u_t \in L^\infty(H^s)$, $u_{tt} \in L^\infty(H^s)$, then there exists a constant $C > 0$ independent of $h$ and $\Delta t$ such that for $j = 1, 2, \ldots, N$*

$$|||u(t_j) - U_j||| \leq C\Big(h^\mu + (\Delta t)^2\Big)\Big(\|u_0\|_s + \|u_t\|_{L^\infty(H^s)} + \|\nabla u_t\|_{L^\infty} + \|u_{tt}\|_{L^\infty(H^s)} + \|u_{ttt}\|_{L^\infty(H^s)}\Big) \tag{4.10}$$

*hold where $s = d/2 + 1 + \delta$ and $\mu = \min(r+1, s)$.*

*Proof.* From (4.1) and (1.1), we have

$$\begin{aligned}
&(u_t(t_{j+1/2}) - \partial_t U_j, v) + A_\lambda(u(t_{j+1/2}); u(t_{j+1/2}), v) - A_\lambda(EU_j; U_{j+1/2}, v) \\
&\quad + B_\lambda(u(t_{j+1/2}); u_t(t_{j+1/2}), v) - B_\lambda(EU_j; \partial_t U_j, v) \\
&\quad = (f(u(t_{j+1/2})) - f(EU_j), v) + \lambda(u(t_{j+1/2}) - U_{j+1/2}, v) + \lambda(u_t(t_{j+1/2}) - \partial_t U_j, v).
\end{aligned} \tag{4.11}$$

By the notations of $\eta$ and $\xi$, we get

$$\begin{aligned}
u_t(t_{j+1/2}) - \partial_t U_j &= u_t(t_{j+1/2}) - \partial_t \tilde{u}_j + \partial_t \tilde{u}_j - \partial_t U_j \\
&= \eta_t(t_{j+1/2}) + \Delta t \rho_{j+1/2} + \partial_t \xi_j.
\end{aligned} \tag{4.12}$$

By the definition of $\eta$, we obtain

$$\begin{aligned}
&A_\lambda(u(t_{j+1/2}); u(t_{j+1/2}), v) - A_\lambda(EU_j; U_{j+1/2}, v) \\
&\quad = A_\lambda(EU_j; \xi_{j+1/2}, v) - A_\lambda(EU_j; \tilde{u}_{j+1/2}, v) + A_\lambda(u(t_{j+1/2}); u(t_{j+1/2}), v) \\
&\quad = A_\lambda(EU_j; \xi_{j+1/2}, v) + A_\lambda(u(t_{j+1/2}); \eta(t_{j+1/2}), v) + A_\lambda(u(t_{j+1/2}); \tilde{u}(t_{j+1/2}) - \tilde{u}_{j+1/2}, v) \\
&\qquad + A_\lambda(u(t_{j+1/2}); \tilde{u}_{j+1/2}, v) - A_\lambda(EU_j; \tilde{u}_{j+1/2}, v).
\end{aligned} \tag{4.13}$$

From the definition of $\eta$, we have

$$\begin{aligned}
&B_\lambda(u(t_{j+1/2}); u_t(t_{j+1/2}), v) - B_\lambda(EU_j; \partial_t U_j, v) \\
&\quad = B_\lambda(EU_j; \partial_t \xi_j, v) + B_\lambda(u(t_{j+1/2}); u_t(t_{j+1/2}), v) - B_\lambda(EU_j; \partial_t \tilde{u}_j, v) \\
&\quad = B_\lambda(EU_j; \partial_t \xi_j, v) + B_\lambda(u(t_{j+1/2}); u_t(t_{j+1/2}) - \partial_t \tilde{u}_j, v) + B_\lambda(u(t_{j+1/2}); \partial_t \tilde{u}_j, v) \\
&\qquad - B_\lambda(EU_j; \partial_t \tilde{u}_j, v) \\
&\quad = B_\lambda(EU_j; \partial_t \xi_j, v) + B_\lambda(u(t_{j+1/2}); \eta_t(t_{j+1/2}) - \Delta t \rho_{j+1/2}, v) + B_\lambda(u(t_{j+1/2}); \partial_t \tilde{u}_j, v) \\
&\qquad - B_\lambda(EU_j; \partial_t \tilde{u}_j, v).
\end{aligned} \tag{4.14}$$

Substituting (4.12)–(4.14) in (4.11) and choosing $v = \xi_{j+1/2} + \partial_t \xi_j$ imply that

$$\begin{aligned}
&(\partial_t \xi_j, \xi_{j+1/2} + \partial_t \xi_j) + A_\lambda(EU_j; \xi_{j+1/2}, \xi_{j+1/2} + \partial_t \xi_j) + B_\lambda\left(EU_j; \partial_t \tilde{\xi}_j, \xi_{j+1/2} + \partial_t \xi_j\right) \\
&\quad = -(\eta_t(t_{j+1/2}) + \Delta t \rho_{j+1/2}, \xi_{j+1/2} + \partial_t \xi_j) - A_\lambda(u(t_{j+1/2}); \eta(t_{j+1/2}), \xi_{j+1/2} + \partial_t \xi_j) \\
&\qquad - A_\lambda(u(t_{j+1/2}); r_{j+1/2}, \xi_{j+1/2} + \partial_t \xi_j) - A_\lambda(u(t_{j+1/2}); \tilde{u}_{j+1/2}, \xi_{j+1/2} + \partial_t \xi_j) \\
&\qquad + A_\lambda(EU_j; \tilde{u}_{j+1/2}, \xi_{j+1/2} + \partial_t \xi_j) - B_\lambda(u(t_{j+1/2}); \eta_t(t_{j+1/2}), \xi_{j+1/2} + \partial_t \xi_j)
\end{aligned}$$

$$+ B_\lambda(u(t_{j+1/2}); \Delta t \rho_{j+1/2}, \xi_{j+1/2} + \partial_t \xi_j) - B_\lambda(u(t_{j+1/2}); \partial_t \widetilde{u}_j, v)$$
$$+ B_\lambda(EU_j; \partial_t \widetilde{u}_j, \xi_{j+1/2} + \partial_t \xi_j) + (f(u(t_{j+1/2})) - f(EU_j), \xi_{j+1/2} + \partial_t \xi_j)$$
$$+ \lambda(u(t_{j+1/2}) - U_{j+1/2}, \xi_{j+1/2} + \partial_t \xi_j) + \lambda(u_t(t_{j+1/2}) - \partial_t U_j, \xi_{j+1/2} + \partial_t \xi_j). \tag{4.15}$$

By Cauchy-Schwarz's inequality clearly we have

$$(\partial_t \xi_j, \xi_{j+1/2}) \geq \frac{1}{2\Delta t}\left( |||\xi_{j+1}|||^2 - |||\xi_j|||^2 \right). \tag{4.16}$$

By the definition of $A_\lambda$ we have

$$\begin{aligned}
A_\lambda(EU_j; \xi_{j+1/2}, \partial_t \xi_j) &= (a(EU_j)\nabla \xi_{j+1/2}, \nabla(\partial_t \xi_j)) - \sum_{k=1}^{P_h} \int_{e_k} \{a(EU_j)\nabla \xi_{j+1/2} \cdot n_k\}[\partial_t \xi_j] \\
&\quad - \sum_{k=1}^{P_h} \int_{e_k} \{a(EU_j)\nabla(\partial_t \xi_j) \cdot n_k\}[\xi_{j+1/2}] + J_\beta^\sigma(\xi_{j+1/2}, \partial_t \xi_j) + \lambda(\xi_{j+1/2}, \partial_t \xi_j) \\
&\geq a_0 \frac{1}{2\Delta t}\left( |||\nabla \xi_{j+1}|||^2 - |||\nabla \xi_j|||^2 \right) + \frac{1}{2\Delta t}\left[ J_\beta^\sigma(\xi_{j+1}, \xi_{j+1}) - J_\beta^\sigma(\xi_j, \xi_j) \right] \\
&\quad + \lambda \frac{1}{2\Delta t}\left( |||\xi_{j+1}|||^2 - |||\xi_j|||^2 \right) - \sum_{k=1}^{P_h} \int_{e_k} \{a(EU_j)\nabla \xi_{j+1/2} \cdot n_k\}[\partial_t \xi_j] \\
&\quad - \sum_{k=1}^{P_h} \int_{e_k} \{a(EU_j)\nabla(\partial_t \xi_j) \cdot n_k\}[\xi_{j+1/2}].
\end{aligned} \tag{4.17}$$

For the definition of $B_\lambda$ we get

$$\begin{aligned}
&B_\lambda(EU_j; \partial_t \xi_j, \xi_{j+1/2}) \\
&\quad \geq \frac{1}{2\Delta t}\left[ a_0\left( |||\nabla \xi_{j+1}|||^2 - |||\nabla \xi_j|||^2 \right) + \left( J_\beta^\sigma(\xi_{j+1}, \xi_{j+1}) - J_\beta^\sigma(\xi_j, \xi_j) \right) + \lambda\left( |||\xi_{j+1}|||^2 - |||\xi_j|||^2 \right) \right] \\
&\qquad - \sum_{k=1}^{P_h} \int_{e_k} \{b(EU_j)\nabla \xi_{j+1/2} \cdot n_k\}[\partial_t \xi_j] - \sum_{k=1}^{P_h} \int_{e_k} \{b(EU_j)\nabla(\partial_t \xi_j) \cdot n_k\}[\xi_{j+1/2}].
\end{aligned} \tag{4.18}$$

Applying (4.17) and (4.18) in (4.15) we conclude that

$$\frac{1}{2\Delta t}\left[ (1+2\lambda)\left( |||\xi_{j+1}|||^2 - |||\xi_j|||^2 \right) + 2a_0\left( |||\nabla(\xi_{j+1})|||^2 - |||\nabla \xi_j|||^2 \right) + 2(J(\xi_{j+1}, \xi_{j+1}) - J(\xi_j, \xi_j)) \right]$$
$$+ |||\partial_t \xi_j|||^2 + \underset{\sim}{c}|||\xi_{j+1/2}|||_1^2 + \underset{\sim}{c}|||\partial_t \xi_j|||_1^2$$

$$\begin{aligned}
&\leq -(\eta_t(t_{j+1/2}) + \Delta t\rho_{j+1/2}, \xi_{j+1/2} + \partial_t\xi_j) + (f(u(t_{j+1/2})) - f(EU_j), \xi_{j+1/2} + \partial_t\xi_j) \\
&\quad + \lambda(u(t_{j+1/2}) - U_{j+1/2}, \xi_{j+1/2} + \partial_t\xi_j) + \lambda(\eta_t(t_{j+1/2}) + \Delta t\rho_{j+1/2}, \xi_{j+1/2} + \partial_t\xi_j) \\
&\quad - A_\lambda(u(t_{j+1/2}); r_{j+1/2}, \xi_{j+1/2} + \partial_t\xi_j) - A_\lambda(u(t_{j+1/2}); \widetilde{u}_{j+1/2}, \xi_{j+1/2} + \partial_t\xi_j) \\
&\quad + A_\lambda(EU_j; \widetilde{u}_{j+1/2}, \xi_{j+1/2} + \partial_t\xi_j) + B_\lambda(u(t_{j+1/2}); \Delta t\rho_{j+1/2}, \xi_{j+1/2} + \partial_t\xi_j) \\
&\quad - B_\lambda(u(t_{j+1/2}); \partial_t\widetilde{u}_j, \xi_{j+1/2} + \partial_t\xi_j) + B_\lambda(EU_j; \partial_t\widetilde{u}_j, \xi_{j+1/2} + \partial_t\xi_j) \\
&\quad + \sum_{k=1}^{P_h}\int_{e_k}\{a(EU_j)\nabla\xi_{j+1/2}\cdot n_k\}[\partial_t\xi_j] + \sum_{k=1}^{P_h}\int_{e_k}\{b(EU_j)\nabla\xi_{j+1/2}\cdot n_k\}[\partial_t\xi_j] \\
&\quad + \sum_{k=1}^{P_h}\int_{e_k}\{a(EU_j)\nabla(\partial_t\xi_j)\cdot n_k\}[\xi_{j+1/2}] + \sum_{k=1}^{P_h}\int_{e_k}\{b(EU_j)\nabla(\partial_t\xi_j)\cdot n_k\}[\xi_{j+1/2}] \\
&= \sum_{i=1}^{10} I_i.
\end{aligned} \tag{4.19}$$

For sufficiently small $\varepsilon > 0$ by applying Lemma 4.3 there exists a constant $C > 0$ such that

$$\begin{aligned}
|I_1| &\leq (|||\eta_t(t_{j+1/2})||| + |||\Delta t\rho_{j+1/2}|||)(|||\xi_{j+1/2}||| + |||\partial_t\xi_j|||) \\
&\leq C\Big(|||\eta_t(t_{j+1/2})|||^2 + (\Delta t)^2|||\rho_{j+1/2}|||^2 + |||\xi_{j+1}|||^2 + |||\xi_j|||^2\Big) + \varepsilon|||\partial_t\xi_j|||^2 \\
&\leq C\Big(|||\eta_t(t_{j+1/2})|||^2 + (\Delta t)^4 + |||\xi_{j+1}|||^2 + |||\xi_j|||^2\Big) + \varepsilon|||\partial_t\xi_j|||^2.
\end{aligned} \tag{4.20}$$

Applying Lemmas 4.3 and 4.4, $I_2$ can be estimated as follows:

$$\begin{aligned}
|I_2| &\leq C|||\mathrm{u}(t_{j+1/2}) - EU_j|||(|||\xi_{j+1/2}||| + |||\partial_t\xi_j|||) \\
&\leq C\Big(|||\eta(t_{j+1/2})|||^2 + (\Delta t)^4 + |||\xi_j|||^2 + |||\xi_{j+1}|||^2 + |||\xi_{j-1}|||^2\Big) + \varepsilon|||\partial_t\xi_j|||^2.
\end{aligned} \tag{4.21}$$

We obtain the following estimates of $I_i$ for each $3 \leq i \leq 5$ :

$$\begin{aligned}
|I_3| &\leq \lambda(|||\eta(t_{j+1/2})||| + |||r_{j+1/2}||| + |||\xi_{j+1/2}|||)(|||\xi_{j+1/2}||| + |||\partial_t\xi_j|||) \\
&\leq C\Big(|||\eta(t_{j+1/2})|||^2 + (\Delta t)^4 + |||\xi_j|||^2 + |||\xi_{j+1}|||^2\Big) + \varepsilon|||\partial_t\xi_j|||^2. \\
|I_4| &\leq \lambda(|||\eta_t(t_{j+1/2})||| + (\Delta t)|||\rho_{j+1/2}|||)(|||\xi_{j+1/2}||| + |||\partial_t\xi_j|||) \\
&\leq C\Big(|||\eta_t(t_{j+1/2})|||^2 + (\Delta t)^4 + |||\xi_{j+1}|||^2 + |||\xi_j|||^2\Big) + \varepsilon|||\partial_t\xi_j|||^2. \\
|I_5| &\leq C|||r_{j+1/2}|||_1\Big(|||\xi_{j+1/2}|||_1 + |||\partial_t\xi_j|||_1\Big) \leq C(\Delta t)^4 + \varepsilon|||\xi_{j+1/2}|||_1^2 + \varepsilon|||\partial_t\xi_j|||_1^2.
\end{aligned} \tag{4.22}$$

From the definition of $I_6$, we can separate $I_6$ as follows:

$$\begin{aligned} I_6 &= \left(\left(a(EU_j) - a(u(t_{j+1/2}))\right)\nabla\widetilde{u}_{j+1/2}, \nabla\left(\xi_{j+1/2} + \partial_t\xi_j\right)\right) \\ &\quad - \sum_{k=1}^{P_h}\int_{e_k}\left\{a(EU_j) - a(u(t_{j+1/2}))\nabla\widetilde{u}_{j+1/2}\cdot n_k\right\}\left[\xi_{j+1/2} + \partial_t\xi_j\right] \\ &\quad - \sum_{k=1}^{P_h}\int_{e_k}\left\{a(EU_j) - a(u(t_{j+1/2}))\nabla\left(\xi_{j+1/2} + \partial_t\xi_j\right)\cdot n_k\right\}\left[\widetilde{u}_{j+1/2}\right] \\ &= \sum_{i=1}^{3} I_{6i}. \end{aligned} \tag{4.23}$$

By applying Lemma 4.5, $I_{61}$ can be estimated in the following way:

$$\begin{aligned} I_{61} &\leq C\left\|\nabla\widetilde{u}_{j+1/2}\right\|_{\infty}\left(|||\eta(t_{j+1/2})||| + (\Delta t)^2 + |||\xi_{j-1}||| + |||\xi_j|||\right)\cdot\left(|||\nabla\xi_{j+1/2}||| + |||\nabla\partial_t\xi_j|||\right) \\ &\leq C\left(|||\eta(t_{j+1/2})|||^2 + (\Delta t)^4 + |||\xi_{j-1}|||^2 + |||\xi_j|||^2\right) + \varepsilon|||\xi_{j+1/2}|||_1^2 + \varepsilon|||\partial_t\xi_j|||_1^2. \end{aligned} \tag{4.24}$$

Similarly there exists a constant $C > 0$ such that

$$\begin{aligned} I_{62} &\leq C\sum_{k=1}^{P_h}\left\|\nabla\widetilde{u}_{j+1/2}\right\|_{\infty,e_k}\left(\left\|\eta(t_{j+1/2})\right\|_{0,e_k} + \left\|\varphi_{j+1/2}\right\|_{0,e_k} + \left\|\xi_j\right\|_{0,e_k} + \left\|\xi_{j-1}\right\|_{0,e_k}\right) \\ &\quad \times\left(\left\|\left[\xi_{j+1/2}\right]\right\|_{0,e_k} + \left\|\left[\partial_t\xi_j\right]\right\|_{0,e_k}\right) \\ &\leq C\sum_{i=1}^{N_h}\left\|\nabla\widetilde{u}_{j+1/2}\right\|_{\infty,E_i} \\ &\quad \times\left(h^{-1/2}\left\|\eta(t_{j+1/2})\right\|_{0,E_i} + h^{1/2}\left\|\nabla\eta(t_{j+1/2})\right\|_{0,E_i} + h^{-1/2}\left\|\varphi_{j+1/2}\right\|_{0,E_i} + h^{-1/2}\left\|\xi_j\right\|_{0,E_i}\right. \\ &\qquad \left. + h^{-1/2}\left\|\xi_{j-1}\right\|_{0,E_i}\right)\cdot\left(|||\xi_{j+1/2}|||_1 + |||\partial_t\xi_j|||_1\right)\cdot h^{\beta(d-1)/2} \\ &\leq C\left(|||\eta(t_{j+1/2})|||^2 + h^2|||\nabla\eta(t_{j+1/2})|||^2 + (\Delta t)^4 + |||\xi_j|||^2 + |||\xi_{j-1}|||^2\right) \\ &\quad + \varepsilon|||\xi_{j+1/2}|||_1^2 + \varepsilon|||\partial_t\xi_j|||_1^2. \end{aligned} \tag{4.25}$$

By applying the trace inequality we have

$$\begin{aligned} I_{63} &\leq C\sum_{k=1}^{P_h}\left(\left\|\nabla(\xi_{j+1/2})\right\|_{\infty,e_k} + \left\|\nabla(\partial_t\xi_j)\right\|_{\infty,e_k}\right) \\ &\quad \times\left(\left\|\eta(t_{j+1/2})\right\|_{0,e_k} + \left\|\varphi_{j+1/2}\right\|_{0,e_k} + \left\|\xi_j\right\|_{0,e_k} + \left\|\xi_{j-1}\right\|_{0,e_k}\right)\cdot\left\|\left[\eta_{j+1/2}\right]\right\|_{0,e_k} \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{N_h}\Big(\|\nabla(\xi_{j+1/2})\|_{\infty,E_i}+\|\nabla\partial_t\xi_j\|_{\infty,E_i}\Big)h^{-1/2}\\
&\quad\times\Big[\|\eta(t_{j+1/2})\|_{0,E_i}+h\|\nabla\eta(t_{j+1/2})\|_{0,E_i}+\|\varphi_{j+1/2}\|_{0,E_i}+\|\xi_j\|_{0,E_i}+\|\xi_{j-1}\|_{0,E_i}\Big]\\
&\quad\cdot h^{-1/2}\Big(\|\eta_{j+1/2}\|_{0,E_i}+h\|\nabla\eta_{j+1/2}\|_{0,E_i}\Big)\\
&\le C\sum_{i=1}^{N_h}\Big(\|\nabla(\xi_{j+1/2})\|_{0,E_i}+\|\nabla\partial_t\xi_j\|_{0,E_i}\Big)\\
&\quad\times\Big(\|\eta(t_{j+1/2})\|_{0,E_i}+h\|\nabla\eta(t_{j+1/2})\|_{0,E_i}+\|\varphi_{j+1/2}\|_{0,E_i}+\|\xi_j\|_{0,E_i}+\|\xi_{j-1}\|_{0,E_i}\Big)h^{-1-d/2+s}\\
&\le C\Big(|||\eta(t_{j+1/2})|||^2+h^2|||\nabla\eta(t_{j+1/2})|||^2+(\Delta t)^4+|||\xi_j|||^2+|||\xi_{j-1}|||^2\Big)\\
&\quad+\varepsilon|||\xi_{j+1/2}|||_1^2+\varepsilon|||\partial_t\xi_j|||_1^2.
\end{aligned}
\tag{4.26}
$$

From the estimation of $I_{6i}$, $1\le i\le 3$, we have

$$
\begin{aligned}
|I_6|&\le C\Big(|||\eta(t_{j+1/2})|||^2+h^2|||\nabla\eta(t_{j+1/2})|||^2+|||\xi_j|||^2+|||\xi_{j-1}|||^2+(\Delta t)^4\Big)\\
&\quad+3\varepsilon|||\xi_{j+1/2}|||_1^2+3\varepsilon|||\partial_t\xi_j|||_1^2.
\end{aligned}
\tag{4.27}
$$

By applying Lemma 4.3 we obviously obtain

$$
|I_7|\le C(\Delta t)|||\rho_{j+1/2}|||_1\Big(|||\xi_{j+1/2}|||_1+|||\partial_t\xi_j|||_1\Big)\le C(\Delta t)^4+\varepsilon|||\xi_{j+1/2}|||_1^2+\varepsilon|||\partial_t\xi_j|||_1^2.
\tag{4.28}
$$

Now we can separate $I_8$ as follows:

$$
\begin{aligned}
I_8&\le\big((b(EU_j)-b(u(t_{j+1/2})))\nabla(\partial_t\tilde{u}_j),\nabla(\xi_{j+1/2}+\partial_t\xi_j)\big)\\
&\quad-\sum_{k=1}^{P_h}\int_{e_k}\{b(EU_j)-b(u(t_{j+1/2}))\nabla(\partial_t\tilde{u}_j)\cdot n_k\}[\xi_{j+1/2}+\partial_t\xi_j]\\
&\quad-\sum_{k=1}^{P_h}\int_{e_k}\{b(EU_j)-b(u(t_{j+1/2}))\nabla(\xi_{j+1/2}+\partial_t\xi_j)\cdot n_k\}[\partial_t\tilde{u}_j]\\
&=\sum_{j=1}^{3}I_{8j}.
\end{aligned}
\tag{4.29}
$$

Since

$$\begin{aligned} \|\nabla(\partial_t \widetilde{u}_j)\|_\infty &= \left\| \frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} \nabla \widetilde{u}_t(\tau)d\tau \right\|_\infty \le \|\nabla \widetilde{u}_t\|_{L^\infty(L^\infty)} \\ &\le C\|\widetilde{u}_t\|_{L^\infty(H^s)} \le C\Big(\|u_t - \widetilde{u}_t\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)}\Big), \end{aligned} \tag{4.30}$$

$I_{81}$ can be estimated as follows

$$\begin{aligned} I_{81} &\le C\|\nabla(\partial_t\widetilde{u}_j)\|_\infty (|||\eta(t_{j+1/2})||| + |||\varphi_{j+1/2}||| + |||\xi_j||| + |||\xi_{j-1}|||)(|||\nabla\xi_{j+1/2}||| + |||\nabla\partial_t\xi_j|||) \\ &\le C\Big(|||\eta(t_{j+1/2})|||^2 + |||\xi_j|||^2 + |||\xi_{j-1}|||^2 + (\Delta t)^4\Big) + \varepsilon|||\xi_{j+1/2}|||_1^2 + \varepsilon|||\partial_t\xi_j|||_1^2. \end{aligned} \tag{4.31}$$

We apply Lemma 3.2 to estimate $I_{82}$ as follows:

$$\begin{aligned} I_{82} &\le C\sum_{k=1}^{P_h}\|\nabla(\partial_t\widetilde{u}_j)\|_{\infty,e_k}\Big(\|\eta(t_{j+1/2})\|_{0,e_k} + \|\varphi_{j+1/2}\|_{0,e_k} + \|\xi_j\|_{0,e_k} + \|\xi_{j-1}\|_{0,e_k}\Big) \\ &\quad\times\Big(\|[\xi_{j+1/2}]\|_{0,e_k} + \|[\partial_t\xi_j]\|_{0,e_k}\Big) \\ &\le C\sum_{i=1}^{N_h}\|\nabla(\partial_t\widetilde{u}_j)\|_{\infty,E_i}\cdot h^{-1/2}\Big(\|\eta(t_{j+1/2})\|_{0,E_i} + h\|\nabla\eta(t_{j+1/2})\|_{0,E_i} + \|\varphi_{j+1/2}\|_{0,e_k} \\ &\qquad + \|\xi_j\|_{0,E_i} + \|\xi_{j-1}\|_{0,E_i}\Big)\cdot h^{1/2}\Big(|||\xi_{j+1/2}|||_1 + |||\partial_t\xi_j|||_1\Big) \\ &\le C\Big(|||\eta(t_{j+1/2})|||^2 + h^2|||\nabla\eta(t_{j+1/2})|||^2 + |||\xi_j|||^2 + |||\xi_{j-1}|||^2 + (\Delta t)^4\Big) \\ &\quad + \varepsilon|||\xi_{j+1/2}|||_1^2 + \varepsilon|||\partial_t\xi_j|||_1^2. \end{aligned} \tag{4.32}$$

From the result of approximation of $\eta$ of Theorem 4.1

$$\begin{aligned} I_{83} &\le C\sum_{k=1}^{P_h}\Big(\|\nabla\xi_{j+1/2}\|_{\infty,e_k} + \|\nabla\partial_t\xi_j\|_{\infty,e_k}\Big)\Big(\|\eta(t_{j+1/2})\|_{0,e_k} + \|\varphi_{j+1/2}\|_{0,e_k} + \|\xi_j\|_{0,e_k} + \|\xi_{j-1}\|_{0,e_k}\Big) \\ &\quad\cdot\|[\eta_t(t_{j+1/2}) + \Delta t\varphi_{j+1/2}]\|_{0,e_k} \\ &\le C\sum_{i=1}^{N_h}h^{-d/2}\Big(\|\nabla\xi_{j+1/2}\|_{0,E_i} + \|\nabla\partial_t\xi_j\|_{0,E_i}\Big)h^{-1/2} \\ &\quad\times\Big(\|\eta(t_{j+1/2})\|_{0,E_i} + h\|\nabla\eta(t_{j+1/2})\|_{0,E_i} + \|\varphi_{j+1/2}\|_{0,E_i} + \|\xi_j\|_{0,E_i} + \|\xi_{j-1}\|_{0,E_i}\Big) \\ &\quad\times h^{-1/2}\cdot\Big(\|\eta_t(t_{j+1/2})\|_{0,E_i} + h\|\nabla\eta_t(t_{j+1/2})\|_{0,E_i} + (\Delta t)h|||\nabla\varphi_{j+1/2}|||_{0,E_i}\Big) \end{aligned}$$

$$
\begin{aligned}
&\leq C\Big(|||\xi_{j+1/2}|||_1 + |||\partial_t\xi_j|||_1\Big)\big(|||\eta(t_{j+1/2})||| + h|||\nabla\eta(t_{j+1/2})||| + |||\varphi_{j+1/2}||| + |||\xi_j||| + |||\xi_{j-1}|||\big)\\
&\leq C\Big(|||\eta(t_{j+1/2})|||^2 + h^2|||\nabla\eta(t_{j+1/2})|||^2 + |||\xi_j|||^2 + |||\xi_{j-1}|||^2 + (\Delta t)^4\Big)\\
&\quad + \varepsilon|||\xi_{j+1/2}|||_1^2 + \varepsilon|||\partial_t\xi_j|||_1^2.
\end{aligned}
\tag{4.33}
$$

Therefore we get

$$
\begin{aligned}
I_8 \leq{}& C\Big(|||\eta(t_{j+1/2})|||^2 + h^2|||\nabla\eta(t_{j+1/2})|||^2 + |||\xi_j|||^2 + |||\xi_{j-1}|||^2 + (\Delta t)^4\Big)\\
&+ 3\varepsilon|||\xi_{j+1/2}|||_1^2 + 3\varepsilon|||\partial_t\xi_j|||_1^2.
\end{aligned}
\tag{4.34}
$$

Similarly, $I_9$ and $I_{10}$ are estimated as follows:

$$
\begin{aligned}
I_9 &\leq \sum_{k=1}^{P_h}\int_{e_k}\{(a(EU_j) + b(EU_j))\nabla\xi_{j+1/2}\cdot n_k\}[\partial_t\xi_j]\\
&\leq C\sum_{k=1}^{P_h}\|\nabla\xi_{j+1/2}\|_{0,e_k}\|[\partial_t\xi_j]\|_{0,e_k}\\
&\leq C\sum_{i=1}^{N_h}\|\nabla\xi_{j+1/2}\|_{0,E_i}|||\partial_t\xi_j|||_1\\
&\leq C\Big(|||\nabla\xi_{j+1}|||^2 + |||\nabla\xi_j|||^2\Big) + \varepsilon|||\partial_t\xi_j|||_1^2.\\
I_{10} &\leq \sum_{k=1}^{P_h}\int_{e_k}\{(a(EU_j) + b(EU_j))\nabla(\partial_t\xi_j)\cdot n_k\}[\xi_{j+1/2}]\\
&\leq C\sum_{k=1}^{P_h}\|\nabla(\partial_t\xi_j)\|_{0,e_k}\|[\xi_{j+1/2}]\|_{0,e_k}\\
&\leq C\Big(J_\beta^\sigma(\xi_{j+1},\xi_{j+1}) + J_\beta^\sigma(\xi_j,\xi_j)\Big) + \varepsilon|||\partial_t\xi_j|||_1^2.
\end{aligned}
\tag{4.35}
$$

Substituting the estimates of $I_i$, $1 \leq i \leq 10$ into (4.19), we get

$$
\begin{aligned}
&\frac{1}{2\Delta t}\Big[\Big(|||\xi_{j+1}|||^2 - |||\xi_j|||^2\Big) + \Big(|||\nabla(\xi_{j+1})|||^2 - |||\nabla\xi_j|||^2\Big) + (J(\xi_{j+1},\xi_{j+1}) - J(\xi_j,\xi_j))\Big]\\
&\qquad + |||\partial_t\xi_j|||^2 + |||\xi_{j+1/2}|||_1^2 + |||\partial_t\xi_j|||_1^2\\
&\quad\leq C\Big[|||\eta_t(t_{j+1/2})|||^2 + (\Delta t)^4 + |||\xi_{j+1}|||^2 + |||\xi_j|||^2 + |||\eta(t_{j+1/2})|||^2 + |||\xi_{j-1}|||^2\\
&\qquad\quad + h^2|||\nabla\eta(t_{j+1/2})|||^2 + |||\nabla\xi_{j+1}|||^2 + |||\nabla\xi|||^2 + J(\xi_{j+1},\xi_{j+1}) + J(\xi_j,\xi_j)\Big].
\end{aligned}
\tag{4.36}
$$

If we sum both sides of (4.36) from $j = 0$ to $N - 1$, then we obtain

$$
\begin{aligned}
&|||\xi_N|||^2 - |||\xi_0|||^2 + |||\nabla\xi_N|||^2 - |||\nabla\xi_0|||^2 + J(\xi_N,\xi_N) + J(\xi_0,\xi_0) \\
&\quad + 2\Delta t \sum_{j=0}^{N-1}\left( |||\partial_t\xi_j|||^2 + |||\xi_{j+1/2}|||_1^2 + |||\partial_t\xi_j|||_1^2 \right) \\
&\quad \leq C\left\{ (\Delta t)\sum_{j=0}^{N-1}\left[ |||\eta(t_{j+1/2})|||^2 + |||\eta_t(t_{j+1/2})|||^2 + h^2|||\nabla\eta(t_{j+1/2})|||^2 + (\Delta t)^4 \right] \right. \\
&\qquad \left. +(\Delta t)\sum_{j=0}^{N}\left( |||\xi_j|||^2 + J(\xi_j,\xi_j) + |||\nabla\xi_j|||^2 \right) \right\},
\end{aligned}
\tag{4.37}
$$

which implies

$$
\begin{aligned}
&|||\xi_N|||^2 + |||\nabla\xi_N|||^2 + J(\xi_N,\xi_N) + 2\Delta t\sum_{j=0}^{N-1}\left( |||\partial_t\xi_j|||^2 + |||\xi_{j+1/2}|||_1^2 + |||\partial_t\xi_j|||_1^2 \right) \\
&\quad \leq |||\xi_0|||^2 + |||\nabla\xi_0|||^2 + J(\xi_0,\xi_0) \\
&\qquad + C(\Delta t)\sum_{j=0}^{N-1}\left[ |||\eta(t_{j+1/2})|||^2 + |||\eta_t(t_{j+1/2})|||^2 + h^2|||\nabla\eta(t_{j+1/2})|||^2 + (\Delta t)^4 \right] \\
&\qquad + C(\Delta t)\sum_{j=0}^{N}\left( |||\xi_j|||^2 + |||\nabla\xi_j|||^2 + J(\xi_j,\xi_j) \right),
\end{aligned}
\tag{4.38}
$$

where $\Delta t$ is sufficiently small. By applying the discrete version of Gronwall's inequality, we have

$$
\begin{aligned}
&|||\xi_N|||^2 + |||\nabla\xi_N|||^2 + J(\xi_N,\xi_N) + \Delta t\sum_{j=0}^{N-1}\left( |||\partial_t\xi_j|||^2 + |||\xi_{j+1/2}|||_1^2 + |||\partial_t\xi_j|||_1^2 \right) \\
&\quad \leq C\left\{ |||\xi_0|||^2 + |||\nabla\xi_0|||^2 + J(\xi_0,\xi_0) \right. \\
&\qquad \left. +\Delta t\sum_{j=0}^{N-1}\left( |||\eta(t_{j+1/2})|||^2 + |||\eta_t(t_{j+1/2})|||^2 + h^2|||\nabla\eta(t_{j+1/2})|||^2 + (\Delta t)^4 \right) \right\}.
\end{aligned}
\tag{4.39}
$$

Therefore by applying the result of Lemma 4.7 we have

$$\begin{aligned} |||\xi|||_{\ell^\infty(L^2)} + |||\nabla\xi|||_{\ell^\infty(L^2)} &\le C\Big(h^s + (\Delta t)^2\Big), \\ |||e|||_{\ell^\infty(L^2)} &\le C\Big(h^s + (\Delta t)^2\Big), \\ |||e|||_{\ell^\infty(H^1)} &\le C\Big(h^{s-1} + (\Delta t)^2\Big), \end{aligned} \tag{4.40}$$

which proves the optimal $\ell^\infty(L^2)$ error estimation of the fully discrete solutions. □

Lemma 4.7 can be proved by the similar process of Theorem 4.6. as follows

**Lemma 4.7.** *For* $0 < \lambda < 1$ *and* $\delta > 0$*, if* $u_t \in L^\infty(H^{d/2+1+\delta})$*,* $u_{tt} \in L^\infty(H^{d/2+1})$*, and* $h^{-d/2}\Delta t \le C_0$*, for some constant* $C_0$ *then there exists a constant* $C > 0$ *independent of* $h$ *and* $\Delta t$

$$\begin{aligned} |||\xi_1|||_{L^2} + |||\nabla\xi_1|||_{L^2} &\le C\Big(h^s + (\Delta t)^2\Big), \\ |||e_1|||_{L^2} &\le C\Big(h^s + (\Delta t)^2\Big). \end{aligned} \tag{4.41}$$

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## References

[1] G. I. Barenblatt, I. P. Zheltov, and I. N. Kochina, "Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks," *Journal of Applied Mathematics and Mechanics*, vol. 24, no. 5, pp. 1286–1303, 1960.

[2] R. W. Carroll and R. E. Showalter, *Singular and Degenerate Cauchy Problems*, vol. 127 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1976.

[3] P. J. Chen and M. E. Gurtin, "On a theory of heat conduction involving two temperatures," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 19, no. 4, pp. 614–627, 1968.

[4] P. L. Davis, "A quasilinear parabolic and a related third order problem," *Journal of Mathematical Analysis and Applications*, vol. 40, no. 2, pp. 327–335, 1972.

[5] R. E. Ewing, "Time-stepping Galerkin methods for nonlinear Sobolev partial differential equations," *SIAM Journal on Numerical Analysis*, vol. 15, no. 6, pp. 1125–1150, 1978.

[6] T. W. Ting, "A cooling process according to two-temperature theory of heat conduction," *Journal of Mathematical Analysis and Applications*, vol. 45, pp. 23–31, 1974.

[7] D. N. Arnold, J. Douglas, Jr., and V. Thomée, "Superconvergence of a finite element approximation to the solution of a Sobolev equation in a single space variable," *Mathematics of Computation*, vol. 36, no. 153, pp. 53–63, 1981.

[8] D. N. Arnold, "An interior penalty finite element method with discontinuous elements," *SIAM Journal on Numerical Analysis*, vol. 19, no. 4, pp. 742–760, 1982.

[9] M. T. Nakao, "Error estimates of a Galerkin method for some nonlinear Sobolev equations in one space dimension," *Numerische Mathematik*, vol. 47, no. 1, pp. 139–157, 1985.

[10] Y. Lin, "Galerkin methods for nonlinear Sobolev equations," *Aequationes Mathematicae*, vol. 40, no. 1, pp. 54–66, 1990.

[11] Y. Lin and T. Zhang, "Finite element methods for nonlinear Sobolev equations with nonlinear boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 165, no. 1, pp. 180–191, 1992.

[12] B. Rivière and M. F. Wheeler, "Non conforming methods for transport with nonlinear reaction," in *Fluid Flow and Transport in Porous Media: Mathematical and Numerical Treatment*, vol. 295 of *Contemporary Mathematics*, pp. 421–432, American Mathematical Society, Providence, RI, USA, 2002.

[13] S. Sun and M. F. Wheeler, "Symmetric and nonsymmetric discontinuous Galerkin methods for reactive transport in porous media," *SIAM Journal on Numerical Analysis*, vol. 43, no. 1, pp. 195–219, 2005.

[14] T. Sun and D. Yang, "A priori error estimates for interior penalty discontinuous Galerkin method applied to nonlinear Sobolev equations," *Applied Mathematics and Computation*, vol. 200, no. 1, pp. 147–159, 2008.

[15] T. Sun and D. Yang, "Error estimates for a discontinuous Galerkin method with interior penalties applied to nonlinear Sobolev equations," *Numerical Methods for Partial Differential Equations*, vol. 24, no. 3, pp. 879–896, 2008.

[16] I. Babuška and M. Suri, "The $h$-$p$ version of the finite element method with quasi-uniform meshes," *RAIRO Modélisation Mathématique et Analyse Numérique*, vol. 21, no. 2, pp. 199–238, 1987.

[17] I. Babuška and M. Suri, "The optimal convergence rate of the $p$-version of the finite element method," *SIAM Journal on Numerical Analysis*, vol. 24, no. 4, pp. 750–776, 1987.

[18] M. R. Ohm, H. Y. Lee, and J. Y. Shin, "Error estimates for discontinuous Galerkin method for nonlinear parabolic equations," *Journal of Mathematical Analysis and Applications*, vol. 315, no. 1, pp. 132–143, 2006.

[19] M. R. Ohm, H. Y. Lee, and J. Y. Shin, "$L^2$-error analysis of discontinuous Galerkin approximations for nonlinear Sobolev equations," submitted.

[20] M. F. Wheeler, "A priori $L_2$ error estimates for Galerkin approximations to parabolic partial differential equations," *SIAM Journal on Numerical Analysis*, vol. 10, pp. 723–759, 1973.