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Research Article

Local Regularity and Local Boundedness Results for Very Weak Solutions of Obstacle Problems

Gao Hongya,[1,2] Qiao Jinjing,[3] and Chu Yuming[4]

[1] *College of Mathematics and Computer Science, Hebei University, Baoding 071002, China*
[2] *Hebei Provincial Center of Mathematics, Hebei Normal University, Shijiazhuang 050016, China*
[3] *College of Mathematics and Computer Science, Hunan Normal University, Changsha 410081, China*
[4] *Faculty of Science, Huzhou Teachers College, Huzhou, Zhejiang 313000, China*

Correspondence should be addressed to Gao Hongya, hongya-gao@sohu.com

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Local regularity and local boundedness results for very weak solutions of obstacle problems of the $\mathcal{A}$-harmonic equation $\operatorname{div}\mathcal{A}(x,\nabla u(x)) = 0$ are obtained by using the theory of Hodge decomposition, where $|\mathcal{A}(x,\xi)| \approx |\xi|^{p-1}$.

1. Introduction and Statement of Results

Let Ω be a bounded regular domain in R^n, $n \geq 2$. By a regular domain we understand any domain of finite measure for which the estimates for the Hodge decomposition in (1.5) and (1.6) are satisfied; see [1]. A Lipschitz domain, for example, is a regular domain. We consider the second-order divergence type elliptic equation (also called $\mathcal{A}$-harmonic equation or Leray-Lions equation):

$$\operatorname{div}\mathcal{A}(x,\nabla u(x)) = 0, \tag{1.1}$$

where $\mathcal{A}(x,\xi) : \Omega \times \mathrm{R}^n \to \mathrm{R}^n$ is a Carathéodory function satisfying the following conditions:

(a) $\langle \mathcal{A}(x,\xi), \xi \rangle \geq \alpha |\xi|^p$,

(b) $|\mathcal{A}(x,\xi)| \leq \beta |\xi|^{p-1}$,

(c) $\mathcal{A}(x,0) = 0$,

where $p > 1$ and $0 < \alpha \leq \beta < \infty$. The prototype of (1.1) is the p-harmonic equation:

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0. \tag{1.2}$$

Suppose that ψ is an arbitrary function in Ω with values in $\mathrm{R} \cup \{\pm\infty\}$, and $\theta \in W^{1,r}(\Omega)$ with $\max\{1, p-1\} < r \leq p$. Let

$$\mathcal{K}^r_{\psi,\theta}(\Omega) = \left\{v \in W^{1,r}(\Omega) : v \geq \psi \text{ a.e., and } v - \theta \in W^{1,r}_0(\Omega)\right\}. \tag{1.3}$$

The function ψ is an obstacle and θ determines the boundary values.

For any $u, v \in \mathcal{K}^r_{\psi,\theta}(\Omega)$, we introduce the Hodge decomposition for $|\nabla(v-u)|^{r-p}\nabla(v-u) \in L^{r/(r-p+1)}(\Omega)$, see [1]:

$$|\nabla(v-u)|^{r-p}\nabla(v-u) = \nabla\phi_{v,u} + h_{v,u}, \tag{1.4}$$

where $\phi_{v,u} \in W_0^{1,r/(r-p+1)}(\Omega)$ and $h_{v,u} \in L^{r/(r-p+1)}(\Omega, \mathrm{R}^n)$ are a divergence-free vector field, and the following estimates hold:

$$\left\|\nabla\phi_{v,u}\right\|_{r/(r-p+1)} \leq c_1\|\nabla(v-u)\|_r^{r-p+1}, \tag{1.5}$$

$$\|h_{v,u}\|_{r/(r-p+1)} \leq c_1(p-r)\|\nabla(v-u)\|_r^{r-p+1}, \tag{1.6}$$

where $c_1 = c_1(n,p)$ is some constant depending only on n and p.

Definition 1.1 (see [2]). A very weak solution to the $\mathcal{K}^r_{\psi,\theta}$-obstacle problem is a function $u \in \mathcal{K}^r_{\psi,\theta}(\Omega)$ such that

$$\int_\Omega \langle \mathcal{A}(x,\nabla u), |\nabla(v-u)|^{r-p}\nabla(v-u)\rangle dx \geq \int_\Omega \langle \mathcal{A}(x,\nabla u), h_{v,u}\rangle dx, \tag{1.7}$$

whenever $v \in \mathcal{K}^r_{\psi,\theta}(\Omega)$.

Remark 1.2. If $r = p$ in Definition 1.1, then $h_{v,u} = 0$ by the uniqueness of the Hodge decomposition (1.4), and (1.7) becomes

$$\int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla(v-u)\rangle dx \geq 0. \tag{1.8}$$

This is the classical definition for $\mathcal{K}^p_{\psi,\theta}$-obstacle problem; see [3] for some details of solutions of $\mathcal{K}^p_{\psi,\theta}$-obstacle problem.

This paper deals with local regularity and local boundedness for very weak solutions of obstacle problems. Local regularity and local boundedness properties are important among the regularity theories of nonlinear elliptic systems; see the recent monograph [4] by Bensoussan and Frehse. Meyers and Elcrat [5] first considered the higher integrability for weak solutions of (1.1) in 1975; see also [6]. Iwaniec and Sbordone [1] obtained the regularity result for very weak solutions of the $\mathcal{A}$-harmonic (1.1) by using the celebrated Gehring's Lemma. The local and global higher integrability of the derivatives in obstacle problem was first considered by Li and Martio [7] in 1994 by using the so-called reverse Hölder inequality. Gao et al. [2] gave the definition for very weak solutions of obstacle problem of $\mathcal{A}$-harmonic (1.1) and obtained the local and global higher integrability results. The local regularity results for minima of functionals and solutions of elliptic equations have been obtained in [8]. For some new results related to $\mathcal{A}$-harmonic equation, we refer the reader to [9–11]. Gao and Tian [12] gave the local regularity result for weak solutions of obstacle problem with the obstacle function $\psi \geq 0$. Li and Gao [13] generalized the result of [12] by obtaining the local integrability result for very weak solutions of obstacle problem. The main result of [13] is the following proposition.

Proposition 1.3. *There exists r_1 with $\max\{1, p-1\} < r_1 < p$, such that any very weak solution u to the $\mathcal{K}^r_{\psi,\theta}$-obstacle problem belongs to $L^{s^*}_{\mathrm{loc}}(\Omega)$, $s^* = 1/(1/s - 1/n)$, provided that $0 \leq \psi \in W^{1,s}_{\mathrm{loc}}(\Omega)$, $r < s < n$, and $r_1 < r < \min\{p, n\}$.*

Notice that in the above proposition we have restricted ourselves to the case $r < n$, because when $r \geq n$, every function in $W^{1,r}_{\mathrm{loc}}(\Omega)$ is trivially in $L^t_{\mathrm{loc}}(\Omega)$ for every $t > 1$ by the classical Sobolev imbedding theorem.

In the first part of this paper, we continue to consider the local regularity theory for very weak solutions of obstacle problem by showing that the condition $\psi \geq 0$ in Proposition 1.3 is not necessary.

Theorem 1.4. *There exists r_1 with $\max\{1, p-1\} < r_1 < p$, such that any very weak solution u to the $\mathcal{K}^r_{\psi,\theta}$-obstacle problem belongs to $L^{s^*}_{\mathrm{loc}}(\Omega)$, provided that $\psi \in W^{1,s}_{\mathrm{loc}}(\Omega)$, $r < s < n$, and $r_1 < r < \min\{p, n\}$.*

As a corollary of the above theorem, if $r = p$, that is, if we consider weak solutions of $\mathcal{K}^p_{\psi,\theta}$-obstacle problem, then we have the following local regularity result.

Corollary 1.5. *Suppose that $\psi \in W^{1,s}_{\mathrm{loc}}(\Omega)$, $1 < p < s < n$. Then a solution u to the $\mathcal{K}^p_{\psi,\theta}$-obstacle problem belongs to $L^{s^*}_{\mathrm{loc}}(\Omega)$.*

We omit the proof of this corollary. This corollary shows that the condition $\psi \geq 0$ in the main result of [12] is not necessary.

The second part of this paper considers local boundedness for very weak solutions of $\mathcal{K}^r_{\psi,\theta}$-obstacle problem. The local boundedness for solutions of obstacle problems plays a central role in many aspects. Based on the local boundedness, we can further study the regularity of the solutions. For the local boundedness results of weak solutions of nonlinear elliptic equations, we refer the reader to [4]. In this paper we consider very weak solutions and show that if the obstacle function is $\psi \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, then a very weak solution u to the $K^r_{\psi,\theta}$-obstacle problem is locally bounded.

Theorem 1.6. *There exists* r_1 *with* $\max\{1, p-1\} < r_1 < p$, *such that for any* r *with* $r_1 < r < \min\{p, n\}$ *and any* $\psi \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, *a very weak solution* u *to the* $\mathcal{K}^r_{\psi,\theta}$*-obstacle problem is locally bounded.*

Remark 1.7. As far as we are aware, Theorem 1.6 is the first result concerning local boundedness for *very weak* solutions of obstacle problems.

In the remaining part of this section, we give some symbols and preliminary lemmas used in the proof of the main results. If $x_0 \in \Omega$ and $t > 0$, then B_t denotes the ball of radius t centered at x_0. For a function $u(x)$ and $k > 0$, let $A_k = \{x \in \Omega : |u(x)| > k\}$, $A_k^+ = \{x \in \Omega : u(x) > k\}$, $A_{k,t} = A_k \cap B_t$, $A_{k,t}^+ = A_k^+ \cap B_t$. Moreover if $s < n$, s^* is always the real number satisfying $1/s^* = 1/s - 1/n$. Let $T_k(u)$ be the usual truncation of u at level $k > 0$, that is,

$$T_k(u) = \max\{-k, \min\{k, u\}\}. \tag{1.9}$$

Let $t_k(u) = \min\{u, k\}$.

We recall two lammas which will be used in the proof of Theorem 1.4.

Lemma 1.8 (see [8]). *Let* $u \in W^{1,r}_{\mathrm{loc}}(\Omega)$, $\varphi_0 \in L^q_{\mathrm{loc}}(\Omega)$, *where* $1 < r < n$ *and* q *satisfies*

$$1 < q < \frac{n}{r}. \tag{1.10}$$

Assume that the following integral estimate holds:

$$\int_{A_{k,t}} |\nabla u|^r dx \le c_0 \left[\int_{A_{k,t}} \varphi_0 dx + (t-\tau)^{-\alpha} \int_{A_{k,t}} |u|^r dx \right], \tag{1.11}$$

for every $k \in N$ *and* $R_0 \le \tau < t \le R_1$, *where* c_0 *is a real positive constant that depends only on* $N, q, r, R_0, R_1, |\Omega|$ *and* α *is a real positive constant. Then* $u \in L^{(qr)^*}_{\mathrm{loc}}(\Omega)$.

Lemma 1.9 (see [14]). *Let* $f(\tau)$ *be a nonnegative bounded function defined for* $0 \le R_0 \le t \le R_1$. *Suppose that for* $R_0 \le \tau < t \le R_1$ *one has*

$$f(\tau) \le A(t-\tau)^{-\alpha} + B + \theta f(t), \tag{1.12}$$

where A, B, α, θ *are nonnegative constants and* $\theta < 1$. *Then there exists a constant* $c_2 = c_2(\alpha, \theta)$, *depending only on* α *and* θ, *such that for every* $\rho, R, R_0 \le \rho < R \le R_1$ *one has*

$$f(\rho) \le c_2 \left[A(R-\rho)^{-\alpha} + B \right]. \tag{1.13}$$

We need the following definition.

Definition 1.10 (see [15]). A function $u(x) \in W^{1,m}_{\text{loc}}(\Omega)$ belongs to the class $\mathcal{B}(\Omega, \gamma, m, k_0)$, if for all $k > k_0$, $k_0 > 0$ and all $B_\rho = B_\rho(x_0)$, $B_{\rho-\rho\sigma} = B_{\rho-\rho\sigma}(x_0)$, $B_R = B_R(x_0)$, one has

$$\int_{A^+_{k,\rho-\rho\sigma}} |\nabla u|^m dx \le \gamma \left\{ \sigma^{-m} \rho^{-m} \int_{A^+_{k,\rho}} (u-k)^m dx + \left| A^+_{k,\rho} \right| \right\}, \tag{1.14}$$

for $R/2 \le \rho - \rho\sigma < \rho < R$, $m < n$, where $|A^+_{k,\rho}|$ is the n-dimensional Lebesgue measure of the set $A^+_{k,\rho}$.

We recall a lemma from [15] which will be used in the proof of Theorem 1.6.

Lemma 1.11 (see [15]). *Suppose that $u(x)$ is an arbitrary function belonging to the class $\mathcal{B}(\Omega, \gamma, m, k_0)$ and $B_R \subset\subset \Omega$. Then one has*

$$\max_{B_{R/2}} u(x) \le c, \tag{1.15}$$

in which the constant c is determined only by the quantities $\gamma, m, k_0, R, \|\nabla u\|_{m_1}$.

2. Local Regularity

Proof of Theorem 1.4. Let u be a very weak solution to the $\mathcal{K}^r_{\psi,\theta}$-obstacle problem. By Lemma 1.8, it is sufficient to prove that u satisfies the inequality (1.11) with $\alpha = r$. Let $B_{R_1} \subset\subset \Omega$ and $0 < R_0 \le \tau < t \le R_1$ be arbitrarily fixed. Fix a cut-off function $\phi \in C^\infty_0(B_{R_1})$ such that

$$\operatorname{supp} \phi \subset B_t, \quad 0 \le \phi \le 1, \ \phi = 1 \text{ in } B_\tau, \left|\nabla\phi\right| \le 2(t-\tau)^{-1}. \tag{2.1}$$

Consider the function

$$v = u - T_k(u) - \phi^r(u - \psi_k), \tag{2.2}$$

where $T_k(u)$ is the usual truncation of u at level $k \ge 0$ defined in (1.9) and $\psi_k = \max\{\psi, T_k(u)\}$. Now $v \in \mathcal{K}^r_{\psi - T_k(u), \theta - T_k(u)}(\Omega)$; indeed, since $u \in \mathcal{K}^r_{\psi,\theta}(\Omega)$ and $\phi \in C^\infty_0(\Omega)$, then

$$\begin{aligned} v - (\theta - T_k(u)) &= u - \theta - \phi^r(u - \psi_k) \in W^{1,r}_0(\Omega), \\ v - (\psi - T_k(u)) &= u - \psi - \phi^r(u - \psi_k) \ge (1 - \phi^r)(u - \psi) \ge 0, \end{aligned} \tag{2.3}$$

a.e. in Ω. Let

$$E(v,u)=\left|\phi^r\nabla u\right|^{r-p}\phi^r\nabla u+|\nabla(v-u+T_k(u))|^{r-p}\nabla(v-u+T_k(u)), \tag{2.4}$$

By an elementary inequality [16, Page 271, (4.1)],

$$\begin{gathered}\left||X|^{-\varepsilon}X-|Y|^{-\varepsilon}Y\right|\le 2^{\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}|X-Y|^{1-\varepsilon},\quad X,Y\in\mathbb{R}^n,\ 0\le\varepsilon<1,\\ \nabla v=\nabla(u-T_k(u))-\phi^r\nabla(u-\psi_k)-r\phi^{r-1}\nabla\phi(u-\psi_k),\end{gathered} \tag{2.5}$$

one can derive that

$$|E(v,u)|\le 2^{p-r}\frac{p-r+1}{r-p+1}\left|\phi^r\nabla\psi_k-r\phi^{r-1}\nabla\phi(u-\psi_k)\right|^{r-p+1}. \tag{2.6}$$

We get from the definition of $E(v,u)$ that

$$\begin{aligned}&\int_{\mathcal{A}_{k,t}}\left\langle\mathcal{A}(x,\nabla u),\left|\phi^r\nabla u\right|^{r-p}\phi^r\nabla u\right\rangle dx\\ &\quad=\int_{A_{k,t}}\langle\mathcal{A}(x,\nabla u),E(v,u)\rangle dx\\ &\qquad-\int_{A_{k,t}}\langle\mathcal{A}(x,\nabla u),|\nabla(v-u+T_k(u))|^{r-p}\nabla(v-u+T_k(u))\rangle dx\\ &\quad=\int_{A_{k,t}}\langle\mathcal{A}(x,\nabla u),E(v,u)\rangle dx\\ &\qquad-\int_{A_{k,t}}\langle\mathcal{A}(x,\nabla u),|\nabla(v-u)|^{r-p}\nabla(v-u)\rangle dx.\end{aligned} \tag{2.7}$$

Now we estimate the left-hand side of (2.7). By condition (a) we have

$$\int_{A_{k,t}}\langle\mathcal{A}(x,\nabla u),\left|\phi^r\nabla u\right|^{r-p}\phi^r\nabla u\rangle d\ge\int_{A_{k,\tau}}\langle\mathcal{A}(x,\nabla u),|\nabla u|^{r-p}\nabla u\rangle dx\ge\alpha\int_{A_{k,\tau}}|\nabla u|^r dx. \tag{2.8}$$

Since $u-T_k(u),v\in\mathcal{K}^r_{\psi-T_k(u),\theta-T_k(u)}(\Omega)$, then using the Hodge decomposition (1.4), we get

$$|\nabla(v-u+T_k(u))|^{r-p}\nabla(v-u+T_k(u))=\nabla\phi+h, \tag{2.9}$$

and by (1.6) we have

$$\|h\|_{r/(r-p+1)}\le c_1(p-r)\|\nabla(v-u+T_k(u))\|_r^{r-p+1}. \tag{2.10}$$

Thus we derive, by Definition 1.1, that

$$\begin{aligned}&\int_{\Omega}\langle\mathcal{A}(x,\nabla(u-T_k(u))),|\nabla(v-u+T_k(u))|^{r-p}\nabla(v-u+T_k(u))\rangle dx\\&\quad\geq\int_{\Omega}\langle\mathcal{A}(x,\nabla(u-T_k(u))),h\rangle dx.\end{aligned}\tag{2.11}$$

This means, by condition (c), that

$$\int_{A_{k,t}}\langle\mathcal{A}(x,\nabla u),|\nabla(v-u)|^{r-p}\nabla(v-u)\rangle dx\geq\int_{A_{k,t}}\langle\mathcal{A}(x,\nabla u),h\rangle dx.\tag{2.12}$$

Combining the inequalities (2.7), (2.8), and (2.12), and using Hölder's inequality and condition (b), we obtain

$$\begin{aligned}\alpha\int_{A_{k,\tau}}|\nabla u|^r dx&\leq\int_{A_{k,t}}\langle\mathcal{A}(x,\nabla u),E(v,u)\rangle dx-\int_{A_{k,t}}\langle\mathcal{A}(x,\nabla u),h\rangle dx\\&\leq\beta\frac{2^{p-r}(p-r+1)}{r-p+1}\int_{A_{k,t}}|\nabla u|^{p-1}\left|\phi^r\nabla\varphi_k-r\phi^{r-1}\nabla\phi(u-\varphi_k)\right|^{r-p+1}dx\\&\quad+\beta\int_{A_{k,t}}|\nabla u|^{p-1}|h|dx\\&\leq\beta\frac{2^{p-r}(p-r+1)}{r-p+1}\int_{A_{k,t}}|\nabla u|^{p-1}\left|\phi^r\nabla\varphi_k\right|^{r-p+1}dx\\&\quad+\beta\frac{2^{p-r}(p-r+1)}{r-p+1}\int_{A_{k,t}}|\nabla u|^{p-1}\left|r\phi^{r-1}\nabla\phi(u-\varphi_k)\right|^{r-p+1}dx\\&\quad+\beta\int_{A_{k,t}}|\nabla u|^{p-1}|h|dx\\&\leq\beta\frac{2^{p-r}(p-r+1)}{r-p+1}\left(\int_{A_{k,t}}|\nabla u|^r dx\right)^{(p-1)/r}\left(\int_{A_{k,t}}\left|\nabla\varphi_k\right|^r dx\right)^{(r-p+1)/r}\\&\quad+\beta\frac{2^{p-r}(p-r+1)}{r-p+1}\left(\int_{A_{k,t}}|\nabla u|^r dx\right)^{(p-1)/r}\\&\quad\times\left(\int_{A_{k,t}}\left|r\phi^{r-1}\nabla\phi(u-\varphi_k)\right|^r dx\right)^{(r-p+1)/r}\\&\quad+\beta\left(\int_{A_{k,t}}|\nabla u|^r dx\right)^{(p-1)/r}\left(\int_{A_{k,t}}|h|^{r/(r-p+1)}dx\right)^{(r-p+1)/r}.\end{aligned}\tag{2.13}$$

Denote $c_3 = c_3(p,r) = 2^{p-r}(p-r+1)/(r-p+1)$. It is obvious that if r is sufficiently close to p, then $c_3(p,r) \le 2$. By (2.10) and Young's inequality

$$ab \le \varepsilon a^{p'} + c_4(\varepsilon,p)b^p, \quad \frac{1}{p}+\frac{1}{p'}=1, \quad a,b\ge 0,\ \varepsilon \ge 0,\ p\ge 1, \tag{2.14}$$

we can derive that

$$\begin{aligned}
\alpha\int_{A_{k,\tau}} |\nabla u|^r dx \le{}& \beta c_3(p,r)\varepsilon\int_{A_{k,t}} |\nabla u|^r dx + \beta c_3(p,r)c_4(\varepsilon,p)\int_{A_{k,t}} \left|\nabla \varphi_k\right|^r dx \\
&+ \beta c_3(p,r)\varepsilon\int_{A_{k,t}} |\nabla u|^r dx + \beta c_3(p,r)c_4(\varepsilon,p)\int_{A_{k,t}} \left|r\phi^{r-1}\nabla\phi(u-\varphi_k)\right|^r dx \\
&+ \beta c_1\varepsilon(p-r)\int_{A_{k,t}} |\nabla u|^r dx + \beta c_1 c_4(\varepsilon,p)(p-r)\int_{\Omega} |\nabla(v-u+T_k(u))|^r dx \\
\le{}& \beta\varepsilon(2c_3(p,r)+c_1(p-r))\int_{A_{k,t}} |\nabla u|^r dx + \beta c_3(p,r)c_4(\varepsilon,p)\int_{A_{k,t}} \left|\nabla\varphi_k\right|^r dx \\
&+ \beta c_3(p,r)c_4(\varepsilon,p)\int_{A_{k,t}} \left|r\phi^{r-1}\nabla\phi(u-\varphi_k)\right|^r dx \\
&+ \beta c_1 c_4(\varepsilon,p)(p-r)\int_{\Omega} |\nabla(v-u+T_k(u))|^r dx.
\end{aligned} \tag{2.15}$$

By the equality

$$\nabla v = \nabla(u-T_k(u)) - \phi^r\nabla(u-\varphi_k) - r\phi^{r-1}\nabla\phi(u-\varphi_k), \tag{2.16}$$

and $v-u+T_k(u)=0$ for $x\in\Omega\setminus A_{k,t}$, then we have

$$\begin{aligned}
\int_{\Omega} |\nabla(v-u+T_k(u))|^r dx &= \int_{A_{k,t}} \left|\phi^r\nabla(u-\varphi_k) + r\phi^{r-1}\nabla\phi(u-\varphi_k)\right|^r dx \\
&\le 2^{r-1}\left[\int_{A_{k,t}} |\nabla u|^r dx + \int_{A_{k,t}} \left|\nabla\varphi_k\right|^r dx + \int_{A_{k,t}} \left|r\phi^{r-1}\nabla\phi(u-\varphi_k)\right|^r dx\right].
\end{aligned} \tag{2.17}$$

Finally we obtain that

$$\begin{aligned}
\int_{A_{k,\tau}} |\nabla u|^r dx &\leq \frac{\beta\varepsilon(2c_3(p,r) + c_1(p-r) + 2^{r-1}\beta c_1 c_4(\varepsilon,p)(p-r)}{\alpha} \int_{A_{k,t}} |\nabla u|^r dx \\
&\quad + \frac{\beta c_3(p,r)c_4(\varepsilon,p) + 2^{r-1}\beta c_1 c_4(\varepsilon,p)(p-r)}{\alpha} \int_{A_{k,t}} \left|\nabla \psi_k\right|^r dx \\
&\quad + \frac{\beta c_3(p,r)c_4(\varepsilon,p) + 2^{r-1}\beta c_1 c_4(\varepsilon,p)(p-r)}{\alpha} \int_{A_{k,t}} \left|r\phi^{r-1}\nabla\phi(u-\psi_k)\right|^r dx \\
&\leq \frac{\beta\varepsilon(2c_3(p,r) + c_1(p-r) + 2^{p-1}\beta c_1 c_4(\varepsilon,p)(p-r)}{\alpha} \int_{A_{k,t}} |\nabla u|^r dx \\
&\quad + \frac{\beta c_3(p,r)c_4(\varepsilon,p) + 2^{p-1}\beta c_1 c_4(\varepsilon,p)(p-r)}{\alpha} \int_{A_{k,t}} \left|\nabla \psi\right|^r dx \\
&\quad + \frac{\beta c_3(p,r)c_4(\varepsilon,p) + 2^{p-1}\beta c_1 c_4(\varepsilon,p)(p-r)}{\alpha} \frac{2^p p}{(t-\tau)^r} \int_{A_{k,t}} |u|^r dx.
\end{aligned} \tag{2.18}$$

The last inequality holds since $|u-\psi_k| \leq |u|$ a.e. in $A_{k,t}$. Now we want to eliminate the first term in the right-hand side containing ∇u. Choose ε small enough and r sufficiently close to p such that

$$\theta = \frac{\beta\varepsilon(2c_3(p,r) + c_1(p-r) + 2^{p-1}\beta c_1 c_4(\varepsilon,p)(p-r)}{\alpha} < 1, \tag{2.19}$$

and let ρ, R be arbitrarily fixed with $R_0 \leq \rho < R \leq R_1$. Thus, from (2.18), we deduce that for every τ and t such that $\rho \leq \tau < t \leq R$, we have

$$\int_{A_{k,\tau}} |\nabla u|^r dx \leq \theta \int_{A_{k,t}} |\nabla u|^r dx + \frac{c_5}{\alpha} \int_{A_{k,R}} \left|\nabla\psi\right|^r dx + \frac{c_6}{\alpha(t-\tau)^r} \int_{A_{k,R}} |u|^r dx, \tag{2.20}$$

where $c_5 = \beta c_3(p,r)c_4(\varepsilon,p) + 2^{p-1}\beta c_1 c_4(\varepsilon,p)(p-r)$ with ε and r fixed to satisfy (2.19), and $c_6 = 2^p p c_5$. Applying Lemma 1.9 in (2.20) we conclude that

$$\int_{A_{k,\rho}} |\nabla u|^r dx \leq \frac{c_2 c_5}{\alpha} \int_{A_{k,R}} \left|\nabla\psi\right|^r dx + \frac{c_2 c_6}{\alpha(R-\rho)^r} \int_{A_{k,R}} |u|^r dx, \tag{2.21}$$

where c_2 is the constant given by Lemma 1.9. Thus u satisfies inequality (1.11) with $\varphi_0 = |\nabla\psi|^r$ and $\alpha = r$. Theorem 1.4 follows from Lemma 1.8. □

3. Local Boundedness

Proof of Theorem 1.6. Let u be a very weak solution to the $\mathcal{K}^r_{\psi,\theta}$-obstacle problem. Let $B_{R_1} \subset\subset \Omega$ and $R_1/2 \le \tau < t \le R_1$ be arbitrarily fixed. Fix a cut-off function $\phi \in C_0^\infty(B_{R_1})$ such that

$$\operatorname{supp} \phi \subset B_t, \quad 0 \le \phi \le 1, \ \phi = 1 \text{ in } B_\tau, \left|\nabla \phi\right| \le 2(t-\tau)^{-1}. \tag{3.1}$$

Consider the function

$$v = u - t_k(u) - \phi^r\left(u - \max\{\psi, t_k(u)\}\right), \tag{3.2}$$

where $t_k(u) = \min\{u, k\}$. Now $v \in \mathcal{K}^r_{\psi - t_k(u), \theta - t_k(u)}$; indeed, since $u \in \mathcal{K}^r_{\psi,\theta}(\Omega)$ and $\phi \in C_0^\infty(\Omega)$, then

$$\begin{gathered} v - (\theta - t_k(u)) = u - \theta - \phi^r\left(u - \max\{\psi, t_k(u)\}\right) \in W_0^{1,r}(\Omega), \\ v - (\psi - t_k(u)) = u - \psi - \phi^r\left(u - \max\{\psi, t_k(u)\}\right) \ge (1 - \phi^r)(u - \psi) \ge 0 \end{gathered} \tag{3.3}$$

a.e. in Ω.

As in the proof of Theorem 1.4, we obtain

$$\begin{aligned} \int_{A^+_{k,\tau}} |\nabla u|^r dx \le{}& \frac{\beta\varepsilon\left(2c_3(p,r) + c_1(p-r) + 2^{r-1}\beta c_1 c_4(\varepsilon,p)(p-r)\right)}{\alpha} \int_{A^+_{k,t}} |\nabla u|^r dx \\ &+ \frac{\beta c_3 c_4(\varepsilon,p) + 2^{r-1}\beta c_1 c_4(\varepsilon,p)(p-r)}{\alpha} \int_{A^+_{k,t}} \left|\nabla \max\{\psi, t_k(u)\}\right|^r dx \\ &+ \frac{\beta c_3(p,r) c_4(\varepsilon,p) + 2^{r-1}\beta c_1 c_4(\varepsilon,p)(p-r)}{\alpha} \\ &\times \int_{A^+_{k,t}} \left| r\phi^{r-1}\nabla\phi(u - \max\{\psi, t_k(u)\}\right|^r dx \\ \le{}& \frac{\beta\varepsilon\left(2c_3(p,r) + c_1(p-r) + 2^{r-1}\beta c_1 c_4(\varepsilon,p)(p-r)\right)}{\alpha} \int_{A^+_{k,t}} |\nabla u|^r dx \\ &+ \frac{\beta c_3 c_4(\varepsilon,p) + 2^{r-1}\beta c_1 c_4(\varepsilon,p)(p-r)}{\alpha} \int_{A^+_{k,t}} \left|\nabla \psi\right|^r dx \\ &+ \frac{\beta c_3(p,r) c_4(\varepsilon,p) + 2^{r-1}\beta c_1 c_4(\varepsilon,p)(p-r)}{\alpha} \frac{2^p p}{(t-\tau)^r} \int_{A^+_{k,t}} |u-k|^r dx. \end{aligned} \tag{3.4}$$

Choose ε small enough and r_1 sufficiently close to p such that (2.19) holds. Let ρ, R be arbitrarily fixed with $R_1/2 \le \rho < R \le R_1$. Thus from (3.4) we deduce that for every τ and t such that $R_1/2 \le \tau < t \le R_1$, we have

$$\int_{A^+_{k,\tau}} |\nabla u|^r dx \le \theta \int_{A^+_{k,t}} |\nabla u|^r dx + \frac{c_5}{\alpha} \int_{A^+_{k,R}} \left|\nabla \psi\right|^r dx + \frac{c_6}{\alpha(t-\tau)^r} \int_{A^+_{k,R}} |u-k|^r dx. \tag{3.5}$$

Applying Lemma 1.9, we conclude that

$$\begin{aligned}\int_{A^+_{k,\rho}} |\nabla u|^r dx &\le \frac{c_2 c_6}{\alpha (R-\rho)^r}\int_{A^+_{k,R}} |u-k|^r dx + \frac{c_2 c_5}{\alpha}\int_{A^+_{k,R}} \left|\nabla \psi\right|^r dx \\ &\le \frac{c_2 c_6}{\alpha (R-\rho)^r}\int_{A^+_{k,R}} |u-k|^r dx + \frac{c_2 c_5 c_7}{\alpha}\left|A^+_{k,R}\right|,\end{aligned} \tag{3.6}$$

where c_2 is the constant given by Lemma 1.9 and $c_7 = \|\nabla \psi\|^p_{L^\infty(\Omega)}$. Thus u belongs to the class $\mathcal{B}$ with $\gamma = \max\{c_2 c_6/\alpha, c_2 c_5 c_7/\alpha\}$ and $m = r$. Lemma 1.11 yields

$$\max_{B_{R/2}} u(x) \le c. \tag{3.7}$$

This result together with the assumptions $u \ge \psi$ and $\psi \in W^{1,\infty}_{\text{loc}}(\Omega)$ yields the desired result. □

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