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Research Article

On the Fermionic p-adic Integral Representation of Bernstein Polynomials Associated with Euler Numbers and Polynomials

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The purpose of this paper is to give some properties of several Bernstein type polynomials to represent the fermionic p-adic integral on $\mathbb{Z}_p$. From these properties, we derive some interesting identities on the Euler numbers and polynomials.

1. Introduction

Throughout this paper, let p be an odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$, respectively.

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$. Note that $\mathbb{Z}_p = \{x \mid |x|_p \leq 1\} = \lim_{\overleftarrow{N}} \mathbb{Z}/p^N\mathbb{Z}_p$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assume $|q| < 1$, and if $q \in \mathbb{C}_p$, we always assume $|1-q|_p < 1$.

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and write $f \in UD(\mathbb{Z}_p)$, if the difference quotient $F_f(x,y) = (f(x)-f(y))/(x-y)$ has a limit $f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{1.1}$$

(see [1]). In the special case $q = 1$ in (1.1), the integral

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x), \tag{1.2}$$

is called the fermionic p-adic invariant integral on $\mathbb{Z}_p$ (see [2]). From (1.2), we note

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \tag{1.3}$$

where $f_1(x) = f(x+1)$.

Moreover, for $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then we note that

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \tag{1.4}$$

It is well known that the Euler polynomials are defined by

$$\frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.5}$$

(see [1–15]). In the special case, $x = 0$, and $E_n(0) = E_n$ are called the nth Euler numbers.

Let $f(x) = e^{tx}$. Then, by (1.3), (1.4), and (1.5), we see that

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.6}$$

Let $C[0,1]$ denote the set of continuous functions on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well-known linear positive operator in the field of real numbers $\mathbb{R}$:

$$\mathbb{B}_n(f : x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x), \tag{1.7}$$

where $\binom{n}{k} = (n(n-1)\cdots(n-k+1))/k! = n!/k!(n-k)!$ (see [3, 4, 7, 10, 11, 14]). Here, $\mathbb{B}_n(f : x)$ is called the Bernstein operator of order n for f.

For $k, n \in \mathbb{Z}_+$, the Bernstein polynomial of degree n is defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } x \in [0,1]. \tag{1.8}$$

For example, $B_{0,1}(x) = 1-x$, $B_{1,1}(x) = x$, $B_{0,2}(x) = (1-x)^2$, $B_{1,2}(x) = 2x - 2x^2$, $B_{2,2}(x) = x^2, \ldots$, and $B_{k,n}(x) = 0$ for $n < k$, $B_{k,n}(x) = B_{n-k,n}(1-x)$.

In this paper, we study the properties of Bernstein polynomials in the p-adic number field. For $f \in UD(\mathbb{Z}_p)$, we give some properties of several type Bernstein polynomials to represent the fermionic p-adic invariant integral on $\mathbb{Z}_p$. From those properties, we derive some interesting identities on the Euler polynomials.

2. Fermionic p-adic Integral Representation of Bernstein Polynomials

By (1.5) and (1.6), we see that

$$\frac{2}{e^t+1}e^{(1-x)t}=\sum_{n=0}^{\infty}E_n(1-x)\frac{t^n}{n!}. \tag{2.1}$$

We also have that

$$\frac{2}{e^t+1}e^{(1-x)t}=\frac{2}{1+e^{-t}}e^{-xt}=\sum_{n=0}^{\infty}E_n(x)\frac{(-1)^n}{n!}t^n. \tag{2.2}$$

From (2.1) and (2.2), we note that $E_n(1-x)=(-1)^nE_n(x)$. It is easy to show that

$$E_n(2)=2-\sum_{l=0}^{n}\binom{n}{l}E_l=2+E_n, \quad \text{for } n>0. \tag{2.3}$$

By (1.5), (1.6), (2.1), (2.2), and (2.3), we see that for $n>0$,

$$\begin{aligned}\int_{\mathbb{Z}_p}(1-x)^n d\mu_{-1}(x)&=(-1)^n\int_{\mathbb{Z}_p}(x-1)^n d\mu_{-1}(x)=\int_{\mathbb{Z}_p}(x+2)^n d\mu_{-1}(x)\\&=2+\int_{\mathbb{Z}_p}x^n d\mu_{-1}(x).\end{aligned} \tag{2.4}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n\in\mathbb{N}$, one has*

$$\int_{\mathbb{Z}_p}(1-x)^n d\mu_{-1}(x)=2+\int_{\mathbb{Z}_p}x^n d\mu_{-1}(x). \tag{2.5}$$

Theorem 2.1 is important to derive our main result in this paper.

Taking the fermionic p-adic integral on $\mathbb{Z}_p$ for one Bernstein polynomial in (1.8), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_{-1}(x) &= \int_{\mathbb{Z}_p}\binom{n}{k}x^k(1-x)^{n-k}d\mu_{-1}(x)\\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{n-k-j}\int_{\mathbb{Z}_p}x^{n-j}d\mu_{-1}(x)\\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{n-k-j}E_{n-j}\\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{j}E_{k+j}.\end{aligned} \tag{2.6}$$

Therefore, we obtain the following proposition.

Proposition 2.2. *For $k, n \in \mathbb{Z}_+$, one is*

$$\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_{-1}(x) = \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{j}E_{k+j}. \tag{2.7}$$

It is known that $B_{k,n}(x) = B_{n-k,n}(1-x)$. Thus, one has

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x)d\mu_{-1}(x)\\ &= \binom{n}{n-k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\int_{\mathbb{Z}_p}(1-x)^{n-j}d\mu_{-1}(x).\end{aligned} \tag{2.8}$$

By (2.8) and Theorem 2.1, we see that for $n > k$,

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_{-1}(x) &= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(2+\int_{\mathbb{Z}_p}x^{n-j}d\mu_{-1}(x)\right)\\ &= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(2+E_{n-j})\\ &= \begin{cases} 2+E_n & \text{if } k=0,\\ \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E_{n-j} & \text{if } k>0.\end{cases}\end{aligned} \tag{2.9}$$

From (2.9), we obtain the following theorem.

Theorem 2.3. *For $n, k \in \mathbb{Z}_+$ with $n > k$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n}(x)\, d\mu_{-1}(x) = \begin{cases} 2 + E_n & \text{if } k = 0, \\ \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} E_{n-j} & \text{if } k > 0. \end{cases} \tag{2.10}$$

By Proposition 2.2 and Theorem 2.3, we obtain the following corollary.

Corollary 2.4. *For $n, k \in \mathbb{Z}_+$ with $n > k$, we have*

$$\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{k+j} = \begin{cases} 2 + E_n & \text{if } k = 0, \\ \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} E_{n-j} & \text{if } k > 0. \end{cases} \tag{2.11}$$

For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$, fermionic p-adic invariant integral for multiplication of two Bernstein polynomials on $\mathbb{Z}_p$ can be given by the following relation:

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x)\, d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} \binom{n}{k} x^k (1-x)^{n-k} \binom{m}{k} x^k (1-x)^{m-k}\, d\mu_{-1}(x) \\ &= \binom{n}{k}\binom{m}{k} \int_{\mathbb{Z}_p} x^{2k} (1-x)^{n+m-2k}\, d\mu_{-1}(x) \\ &= \binom{n}{k}\binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \int_{\mathbb{Z}_p} (1-x)^{n+m-j}\, d\mu_{-1}(x) \\ &= \binom{n}{k}\binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \left(2 + \int_{\mathbb{Z}_p} x^{n+m-j}\, d\mu_{-1}(x) \right) \\ &= \begin{cases} 2 + E_{n+m} & \text{if } k = 0, \\ \binom{n}{k}\binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j} & \text{if } k > 0. \end{cases} \end{aligned} \tag{2.12}$$

Therefore, we obtain the following theorem.

Theorem 2.5. *For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$, one has*

$$\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) d\mu_{-1}(x) = \begin{cases} 2 + E_{n+m} & \text{if } k = 0, \\ \binom{n}{k}\binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j} & \text{if } k > 0. \end{cases} \tag{2.13}$$

For $m, n, k \in \mathbb{Z}_+$, one has

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) d\mu_{-1}(x) &= \binom{n}{k}\binom{m}{k} \int_{\mathbb{Z}_p} x^{2k} (1-x)^{n+m-2k} d\mu_{-1}(x) \\ &= \binom{n}{k}\binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j \int_{\mathbb{Z}_p} x^{j+2k} d\mu_{-1}(x) \\ &= \binom{n}{k}\binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k}. \end{aligned} \tag{2.14}$$

Thus, we obtain the following proposition.

Proposition 2.6. *For $m, n, k \in \mathbb{Z}_+$, one has*

$$\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) d\mu_{-1}(x) = \binom{n}{k}\binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k}. \tag{2.15}$$

By Theorem 2.5 and Proposition 2.6, we obtain the following corollary.

Corollary 2.7. *For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$, one has*

$$\sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k} = \begin{cases} 2 + E_{n+m} & \text{if } k = 0, \\ \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j} & \text{if } k > 0. \end{cases} \tag{2.16}$$

In the same manner, multiplication of three Bernstein polynomials can be given by the following relation:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)B_{k,s}(x)d\mu_{-1}(x)\\
&= \binom{n}{k}\binom{m}{k}\binom{s}{k}\sum_{j=0}^{n+m+s-3k}\binom{n+m+s-3k}{j}(-1)^j\int_{\mathbb{Z}_p} x^{j+3k}d\mu_{-1}(x)\\
&= \binom{n}{k}\binom{m}{k}\binom{s}{k}\sum_{j=0}^{n+m+s-3k}\binom{n+m+s-3k}{j}(-1)^j E_{j+3k},
\end{aligned} \tag{2.17}$$

where $m, n, s, k \in \mathbb{Z}_+$ with $m + n + s > 3k$.

For $m, n, s, k \in \mathbb{Z}_+$ with $m + n + s > 3k$, by the symmetry of Bernstein polynomals, we see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)B_{k,s}(x)d\mu_{-1}(x)\\
&= \binom{n}{k}\binom{m}{k}\binom{s}{k}\sum_{j=0}^{3k}\binom{3k}{j}(-1)^{3k-j}\int_{\mathbb{Z}_p}(1-x)^{n+m+s-j}d\mu_{-1}(x)\\
&= \binom{n}{k}\binom{m}{k}\binom{s}{k}\sum_{j=0}^{3k}\binom{3k}{j}(-1)^{3k-j}\left(2+\int_{\mathbb{Z}_p} x^{n+m+s-j}\mu_{-1}(x)\right)\\
&= \begin{cases} 2+E_{n+m+s} & \text{if } k=0,\\ \binom{n}{k}\binom{m}{k}\binom{s}{k}\sum_{j=0}^{3k}\binom{3k}{j}(-1)^{3k-j}E_{n+m+s-j} & \text{if } k>0. \end{cases}
\end{aligned} \tag{2.18}$$

Therefore, we obtain the following theorem.

Theorem 2.8. *For $m, n, s, k \in \mathbb{Z}_+$ with $m + n + s > 3k$, one has*

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)B_{k,s}(x)d\mu_{-1}(x)\\
&= \begin{cases} 2+E_{n+m+s} & \textit{if } k=0,\\ \binom{n}{k}\binom{m}{k}\binom{s}{k}\sum_{j=0}^{3k}\binom{3k}{j}(-1)^{3k-j}E_{n+m+s-j} & \textit{if } k>0. \end{cases}
\end{aligned} \tag{2.19}$$

By (2.17) and Theorem 2.8, we obtain the following corollary.

Corollary 2.9. *For $m, n, s, k \in \mathbb{Z}_+$ with $m + n + s > 3k$, one has*

$$\sum_{j=0}^{n+m+s-3k} \binom{n+m+s-3k}{j} (-1)^j E_{j+3k} = \begin{cases} 2 + E_{n+m+s} & \text{if } k = 0, \\ \sum_{j=0}^{3k} \binom{3k}{j} (-1)^{3k-j} E_{n+m+s-j} & \text{if } k > 0. \end{cases} \quad (2.20)$$

Using the above theorems and mathematical induction, we obtain the following theorem.

Theorem 2.10. *Let $s \in \mathbb{N}$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk$, the multiplication of the sequence of Bernstein polynomials $B_{k,n_1}(x), \ldots, B_{k,n_s}(x)$ with different degrees under fermionic p-adic invariant integral on $\mathbb{Z}_p$ can be given as*

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}(x) \right) d\mu_{-1}(x) = \begin{cases} 2 + E_{n_1+n_2+\cdots+n_s} & \text{if } k = 0, \\ \left(\prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk-j} E_{n_1+n_2+\cdots+n_s-j} & \text{if } k > 0. \end{cases} \quad (2.21)$$

We also easily see that

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}(x) \right) d\mu_{-1}(x) = \left(\prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{j} (-1)^j E_{j+sk}. \quad (2.22)$$

By Theorem 2.10 and (2.22), we obtain the following corollary.

Corollary 2.11. *Let $s \in \mathbb{N}$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk$, one has*

$$\sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{j} (-1)^j E_{j+sk} = \begin{cases} 2 + E_{n_1+n_2+\cdots+n_s} & \text{if } k = 0, \\ \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk-j} E_{n_1+n_2+\cdots+n_s-j} & \text{if } k > 0. \end{cases} \quad (2.23)$$

Let $m_1, \ldots, m_s, n_1, \ldots, n_s, k \in \mathbb{Z}_+$ with $m_1 n_1 + \cdots + m_s n_s > (m_1 + \cdots + m_s)k$. By the definition of $B_{k,n_s}^{m_s}(x)$, we easily get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}^{m_i}(x) \right) d\mu_{-1}(x) \\
&= \left(\prod_{i=1}^{s} \binom{n_i}{k}^{m_i} \right) \sum_{j=0}^{k \sum_{i=1}^{s} m_i} (-1)^{k \sum_{i=1}^{s} m_i - j} \int_{\mathbb{Z}_p} (1-x)^{\sum_{i=1}^{s} n_i m_i - j} d\mu_{-1}(x) \\
&= \left(\prod_{i=1}^{s} \binom{n_i}{k}^{m_i} \right) \sum_{j=0}^{k \sum_{i=1}^{s} m_i} \binom{k \sum_{i=1}^{s} m_i}{j} (-1)^{k \sum_{i=1}^{s} m_i - j} \left(2 + E_{\sum_{i=1}^{s} n_i m_i - j} \right) \\
&= \begin{cases} 2 + E_{m_1 n_1 + \cdots + m_s n_s} & \text{if } k = 0, \\ \left(\prod_{i=1}^{s} \binom{n_i}{k}^{m_i} \right) \sum_{j=0}^{k \sum_{i=1}^{s} m_i} \binom{k \sum_{i=1}^{s} m_i}{j} (-1)^{k \sum_{i=1}^{s} m_i - j} E_{\sum_{i=1}^{s} n_i m_i - j} & \text{if } k > 0. \end{cases}
\end{aligned} \tag{2.24}$$

Therefore, we obtain the following theorem.

Theorem 2.12. *Let $s \in \mathbb{N}$. For $m_1, \ldots, m_s, n_1, \ldots, n_s, k \in \mathbb{Z}_+$ with $m_1 n_1 + \cdots + m_s n_s > (m_1 + \cdots + m_s)k$, one has*

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}^{m_i}(x) \right) d\mu_{-1}(x) \\
&= \begin{cases} 2 + E_{m_1 n_1 + \cdots + m_s n_s} & \text{if } k = 0, \\ \left(\prod_{i=1}^{s} \binom{n_i}{k}^{m_i} \right) \sum_{j=0}^{k \sum_{i=1}^{s} m_i} \binom{k \sum_{i=1}^{s} m_i}{j} (-1)^{k \sum_{i=1}^{s} m_i - j} E_{\sum_{i=1}^{s} n_i m_i - j} & \text{if } k > 0. \end{cases}
\end{aligned} \tag{2.25}$$

By simple calculation, we easily get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}^{m_i}(x) \right) d\mu_{-1}(x) \\
&= \left(\prod_{i=1}^{s} \binom{n_i}{k}^{m_i} \right) \sum_{j=0}^{\sum_{i=1}^{s} n_i m_i - k \sum_{i=1}^{s} m_i} \binom{\sum_{i=1}^{s} n_i m_i - k \sum_{i=1}^{s} m_i}{j} (-1)^j E_{k \sum_{i=1}^{s} m_i - j},
\end{aligned} \tag{2.26}$$

where $m_1, \ldots, m_s, n_1, \ldots, n_s, k \in \mathbb{Z}_+$ for $s \in \mathbb{N}$. By Theorem 2.12 and (2.26), we obtain the following corollary.

Corollary 2.13. *Let $s \in \mathbb{N}$. For $m_1, \ldots, m_s, n_1, \ldots, n_s, k \in \mathbb{Z}_+$ with $m_1n_1 + \cdots + m_sn_s > (m_1 + \cdots + m_s)k$, one has*

$$\sum_{j=0}^{\sum_{i=1}^s n_im_i - k\sum_{i=1}^s m_i} \binom{\sum_{i=1}^s n_im_i - k\sum_{i=1}^s m_i}{j} (-1)^j E_{k\sum_{i=1}^s m_i - j} = \begin{cases} 2 + E_{m_1n_1+\cdots+m_sn_s} & \text{if } k = 0, \\ \sum_{j=0}^{k\sum_{i=1}^s m_i} \binom{k\sum_{i=1}^s m_i}{j} (-1)^{k\sum_{i=1}^s m_i - j} E_{\sum_{i=1}^s n_im_i - j} & \text{if } k > 0. \end{cases} \tag{2.27}$$

The fermionic p-adic invariant integral of multiplication of $(n+1)$ Bernstein polynomials, the nth degree Bernstein polynomials $B_{i,n}(x)$ with $i = 0, 1, \ldots, n$ and with multiplicity $m_0, m_1, \ldots, m_n$ on $\mathbb{Z}_p$, respectively, can be given by

$$\begin{aligned} \int_{\mathbb{Z}_p} \left(\prod_{i=0}^n B_{i,n}^{m_i}(x) \right) d\mu_{-1}(x) &= \left(\prod_{i=0}^n \binom{n}{i}^{m_i} \right) \int_{\mathbb{Z}_p} x^{\sum_{i=1}^n im_i} (1-x)^{n\sum_{i=0}^n m_i - \sum_{i=1}^n im_i} d\mu_{-1}(x) \\ &= \frac{\left(\prod_{i=1}^n \binom{n}{i}^{m_i} \right)}{\binom{n\sum_{i=0}^n m_i}{\sum_{i=1}^n im_i}} \int_{\mathbb{Z}_p} B_{\sum_{i=1}^n im_i,\, n\sum_{i=0}^n m_i}(x) d\mu_{-1}(x), \end{aligned} \tag{2.28}$$

where $m_0, m_1, \ldots, m_n \in \mathbb{Z}_+$ with $n \in \mathbb{Z}_+$.

Assume that $nm_0 + nm_1 + \cdots + nm_n > m_1 + 2m_2 + \cdots + nm_n$. Then one has

$$\int_{\mathbb{Z}_p} \left(\prod_{i=0}^n B_{i,n}^{m_i}(x) \right) d\mu_{-1}(x) = \begin{cases} 2 + E_{nm_0+nm_1+\cdots+nm_n} & \text{if } \sum_{i=1}^n im_i = 0, \\ \left(\prod_{i=0}^n \binom{n}{i}^{m_i} \right) \sum_{j=0}^{\sum_{i=1}^n m_i} \binom{\sum_{i=1}^n im_i}{j} (-1)^{\sum_{i=1}^n im_i - j} E_{n\sum_{i=0}^n m_i - \sum_{i=1}^n im_i} & \text{if } \sum_{i=1}^n im_i > 0. \end{cases} \tag{2.29}$$

Therefore, we obtain the following theorem.

Theorem 2.14. *Let $n \in \mathbb{Z}_+$.*

(i) *For $m_0, m_1, \ldots, m_n \in \mathbb{Z}_+$ with $n\sum_{i=0}^{n} m_i > \sum_{i=1}^{n} im_i$, one has*

$$\int_{\mathbb{Z}_p}\left(\prod_{i=0}^{n} B_{i,n}^{m_i}(x)\right)d\mu_{-1}(x) = \begin{cases} 2 + E_{nm_0+nm_1+\cdots+nm_n} & \text{if } \sum_{i=1}^{n} im_i = 0, \\ \left(\prod_{i=0}^{n}\binom{n}{i}^{m_i}\right)\sum_{j=0}^{\sum_{i=1}^{n} m_i}\binom{\sum_{i=1}^{n} im_i}{j}(-1)^{\sum_{i=1}^{n} im_i - j}E_{n\sum_{i=0}^{n} m_i - \sum_{i=1}^{n} im_i} & \text{if } \sum_{i=1}^{n} im_i > 0. \end{cases} \tag{2.30}$$

(ii) *For $m_0, m_1, \ldots, m_n \in \mathbb{Z}_+$, one has*

$$\int_{\mathbb{Z}_p}\left(\prod_{i=0}^{n} B_{i,n}^{m_i}(x)\right)d\mu_{-1}(x) = \left(\prod_{i=0}^{n}\binom{n}{i}^{m_i}\right)\sum_{j=0}^{n\sum_{i=0}^{n} m_i - \sum_{i=1}^{n} im_i}\binom{n\sum_{i=0}^{n} m_i - \sum_{i=1}^{n} im_i}{j}(-1)^{j}E_{\sum_{i=1}^{n} im_i + j}. \tag{2.31}$$

By Theorem 2.14, we obtain the following corollary.

Corollary 2.15. *For $n, m_0, m_1, \ldots, m_n \in \mathbb{Z}_+$ with $n\sum_{i=0}^{n} m_i > \sum_{i=1}^{n} im_i$, one has*

$$\sum_{j=0}^{n\sum_{i=0}^{n} m_i - \sum_{i=1}^{n} im_i}\binom{n\sum_{i=0}^{n} m_i - \sum_{i=1}^{n} im_i}{j}(-1)^{j}E_{\sum_{i=1}^{n} im_i + j} = \begin{cases} 2 + E_{nm_0+nm_1+\cdots+nm_n} & \text{if } \sum_{i=1}^{n} im_i = 0, \\ \sum_{j=0}^{\sum_{i=1}^{n} m_i}\binom{\sum_{i=1}^{n} im_i}{j}(-1)^{\sum_{i=1}^{n} im_i - j}E_{n\sum_{i=0}^{n} m_i - \sum_{i=1}^{n} im_i} & \text{if } \sum_{i=1}^{n} im_i > 0. \end{cases} \tag{2.32}$$

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