Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 856495, 14 pages
doi:10.1155/2010/856495

*Research Article*

# Almost Sure Convergence for the Maximum and the Sum of Nonstationary Guassian Sequences

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Received 18 December 2009; Accepted 5 April 2010

Academic Editor: Jewgeni Dshalalow



Let $(X_n, n \geq 1)$ be a standardized nonstationary Gaussian sequence. Let $M_n = \max\{X_k, 1 \leq k \leq n\}$ denote the partial maximum and $S_n = \sum_{k-1}^{n} X_k$ for the partial sum with $\sigma_n = (\text{Var } S_n)^{1/2}$. In this paper, the almost sure convergence of $(M_n, S_n/\sigma_n)$ is derived under some mild conditions.

## 1. Introduction

There have been more researches on the almost sure convergence of extremes and partial sums since the pioneer work of Fahrner and Stadtmüller [1] and Cheng et al. [2]. For more related work on almost sure convergence of extremes and partial sums, see Berkes and Csáki [3], Peng et al. [4, 5], Tan and Peng [6], and references therein. For the almost sure convergence of extremes for dependent Gaussian sequence, Csáki and Gonchigdanzan [7] and Lin [8] proved

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}\left( \frac{M_k - b_k}{a_k} \leq x \right) = \int_{-\infty}^{\infty} \exp\left( -e^{-x-\rho+\sqrt{2\rho}z} \right) \phi(z)\, dz \quad \text{a.s.} \tag{1.1}$$

provided

$$\left| r_n \log n - \rho \right| (\log \log n)^{1+\varepsilon} = O(1), \tag{1.2}$$

where $\mathbb{I}$ denotes an indicator function, $\Phi(x)$ is the standard normal distribution function, and $\phi(x) = (1/\sqrt{2\pi}) e^{-x^2/2} = \Phi'(x)$. $M_n$ is the partial maximum of a standard stationary Gaussian sequence $\{X_n,\ n \geq 1\}$ with correlation $r_n = \mathrm{E}X_1X_{n+1},\ n \geq 0$. The norming constants $a_n$ and $b_n$ are defined by

$$a_n = (2\log n)^{-1/2}, \qquad b_n = (2\log n)^{1/2} - \frac{\log\log n + \log 4\pi}{2(2\log n)^{1/2}}. \tag{1.3}$$

For some extensions of (1.1), see Chen and Lin [9] and Peng and Nadarajah [10].

Sometimes, in practice, one would like to know how partial sums and maxima behave simultaneously in the limit; see Anderson and Turkman [11] for a discussion of an application involving extreme wind gusts and average wind speeds. Peng et al. [12] studied the almost sure limiting behavior for partial sums and maxima of i.i.d. random variables. Dudziński [13, 14] proved the almost sure limit theorems in the joint version for the maxima and the partial sums of stationary Gaussian sequences, that is, let $X_1, X_1, \ldots$ be stationary Gaussian sequences and $M_k = \max_{i\leq k} X_i$, $S_n = \sum_{i=1}^n X_i$, $\sigma_n = \sqrt{\mathrm{Var}(S_n)}$, for all $x, y \in (-\infty, \infty)$

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}\left( \frac{M_k - b_k}{a_k} \leq x, \frac{S_k}{\sigma_k} \leq y \right) = \exp(-e^{-x})\Phi(y) \quad \text{a.s.} \tag{1.4}$$

if

(C1) $\sup_{s\geq n} \sum_{t=s-n}^{s-1} |r_t| \ll (\log n)^{1/2} / (\log\log n)^{1+\varepsilon}$ for some $\varepsilon > 0$,

(C2) $\sum_{t=1}^{n} (n-t) r_t \geq 0$ for all $n \geq 1$,

(C3) $\lim_{n\to\infty} r_n \log n = 0$.

Or

$$r_n = \frac{L(n)}{n^{\alpha}}, \quad n \geq 1 \tag{1.5}$$

for some $\alpha > 0$. $L(x)$ is a positive slowly varying function at infinity. Here $a \ll b$ means $a = O(b)$.

This paper focuses on extending (1.4) to nonstationary Gaussian sequences $\{X_n,\ n \geq 1\}$ under some mild conditions similar to (C1)–(C3). The paper is organized as follows: in Section 2, we give the main results, and related proofs are provided in Section 3.

## 2. The Main Results

Let $r_{ij} = \mathrm{E}(X_i X_j),\ i, j \geq 1$, denote the correlations of standard nonstationary Gaussian sequence $\{X_n,\ n \geq 1\}$. $M_n$, $S_n$, and $\sigma_n$ are defined as before. The main results are the following.

**Theorem 2.1.** *Let $\{X_n,\ n \geq 1\}$ be a standardized nonstationary Gaussian sequence. Suppose that there exists numerical sequence $\{u_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ such that $\sum_{i=1}^{n}(1 - \Phi(u_{ni})) \to \tau$ for some $0 < \tau < \infty$ and $n(1 - \Phi(\lambda_n))$ is bounded, where $\lambda_n = \min_{1\leq i\leq n} u_{ni}$. If*

$$\sup_{i\neq j} |r_{ij}| < \delta < 1, \tag{2.1}$$

$$\sum_{j=2}^{n}\sum_{i=1}^{j-1}|r_{ij}| = o(n), \tag{2.2}$$

$$\sup_{i\geq 1}\sum_{j=1}^{n}|r_{ij}| \ll \frac{(\log n)^{1/2}}{(\log\log n)^{1+\varepsilon}} \quad \textit{for some } \varepsilon > 0, \tag{2.3}$$

*then*

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left(\bigcap_{i=1}^{k}(X_i \leq u_{ki}), \frac{S_k}{\sigma_k} \leq y\right) = e^{-\tau}\Phi(y) \quad a.s. \tag{2.4}$$

*for all* $y \in (-\infty, \infty)$*.*

**Theorem 2.2.** *For the nonstationary Gaussian sequence* $\{X_n,\ n \geq 1\}$*, under the conditions* (2.1)–(2.3)*, we have*

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left(M_k \leq a_k x + b_k, \frac{S_k}{\sigma_k} \leq y\right) = \exp(-e^{-x})\Phi(y) \quad a.s. \tag{2.5}$$

*for all* $x, y \in (-\infty, \infty)$*, where* $a_n$ *and* $b_n$ *are defined as in* (1.3)*.*

## 3. Proof of the Main Results

To prove the main results, we need some auxiliary lemmas.

**Lemma 3.1.** *Suppose that the standardized nonstationary Gaussian sequences* $\{X_n,\ n \geq 1\}$ *satisfy the conditions* (2.1)–(2.3)*. Assume that* $n(1-\Phi(\lambda_n))$ *is bounded. Then for* $< l$*,*

$$\mathrm{E}\left|\mathbb{I}\left(\bigcap_{i=1}^{l}(X_i \leq u_{li}), \frac{S_l}{\sigma_l} \leq y\right) - \mathbb{I}\left(\bigcap_{i=k+1}^{l}(X_i \leq u_{li}), \frac{S_l}{\sigma_l} \leq y\right)\right| \ll \frac{1}{(\log\,\log l)^{1+\varepsilon}} + \frac{k}{l}. \tag{3.1}$$

*Proof.* We will start with the following observations. For all $1 \leq i \leq l$,

$$\left|\mathrm{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| = \frac{1}{\sigma_l}|\mathrm{Cov}(X_i, S_l)| \leq \frac{1}{\sigma_l}\sum_{j=1}^{l}|r_{ij}|. \tag{3.2}$$

Clearly,

$$\sigma_l = \left(l + 2\sum_{j=2}^{l}\sum_{i=1}^{j-1}r_{ij}\right)^{1/2}. \tag{3.3}$$

By (2.2), for large $l$ there exists $c_1 > 0$ such that

$$\sigma_l \geq c_1 l^{1/2}. \tag{3.4}$$

By (2.3) and (3.4), we have

$$\sup_{1\leq i\leq l}\left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| \ll \frac{(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} \tag{3.5}$$

for large $l$. Obviously,

$$\lim_{l\to\infty}\frac{(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} = 0, \tag{3.6}$$

which implies that there exist $\mu > 0$ and $l_0$ such that

$$\sup_{1\leq i\leq l}\left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| < \mu < 1 \quad \forall l > l_0. \tag{3.7}$$

Notice,

$$\begin{aligned}
&\mathrm{E}\left|\mathbb{I}\left(\bigcap_{i=1}^{l}(X_i \leq u_{li}), \frac{S_l}{\sigma_l} \leq y\right) - \mathbb{I}\left(\bigcap_{i=k+1}^{l}(X_i \leq u_{li}), \frac{S_l}{\sigma_l} \leq y\right)\right| \\
&\quad = \mathrm{P}\left(\bigcap_{i=k+1}^{l}(X_i \leq u_{li}), \frac{S_l}{\sigma_l} \leq y\right) - \mathrm{P}\left(\bigcap_{i=1}^{l}(X_i \leq u_{li}), \frac{S_l}{\sigma_l} \leq y\right) \\
&\quad \leq \left|\mathrm{P}\left(\bigcap_{i=1}^{l}(X_i \leq u_{li}), \frac{S_l}{\sigma_l} \leq y\right) - \mathrm{P}\left(\bigcap_{i=1}^{l}(X_i \leq u_{li})\right)\mathrm{P}\left(\frac{S_l}{\sigma_l} \leq y\right)\right| \\
&\qquad + \left|\mathrm{P}\left(\bigcap_{i=k+1}^{l}(X_i \leq u_{li}), \frac{S_l}{\sigma_l} \leq y\right) - \mathrm{P}\left(\bigcap_{i=k+1}^{l}(X_i \leq u_{li})\right)\mathrm{P}\left(\frac{S_l}{\sigma_l} \leq y\right)\right| \\
&\qquad + \mathrm{P}\left(\frac{S_l}{\sigma_l} \leq y\right)\left(\mathrm{P}\left(\bigcap_{i=k+1}^{l}(X_i \leq u_{li})\right) - \mathrm{P}\left(\bigcap_{i=1}^{l}(X_i \leq u_{li})\right)\right) \\
&\quad =: A_1(l) + A_2(l) + A_3(l).
\end{aligned} \tag{3.8}$$

By the Normal Comparison Lemma [13, Theorem 4.2.1 ], we get

$$\begin{aligned}
A_1(l) &\ll \sum_{i=1}^{l}\left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| \exp\left(-\frac{u_{li}^2 + y^2}{2(1 + |\operatorname{Cov}(X_i, S_l/\sigma_l)|)}\right) \\
&\leq \sum_{i=1}^{l}\left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| \exp\left(-\frac{\lambda_l^2}{2(1+\mu)}\right).
\end{aligned} \tag{3.9}$$

Since $n(1-\Phi(\lambda_n))$ is bounded, for large $n$ and some absolute positive constant $C$,

$$\exp\left(-\frac{\lambda_n^2}{2}\right) \sim C\frac{\log^{1/2} n}{n}. \tag{3.10}$$

So,

$$A_1(l) \ll \frac{l^{1/2}(\log l)^{1/2}}{(\log\ \log l)^{1/2}}\frac{(\log l)^{1/2(1+\mu)}}{l^{1/(1+\mu)}} = \frac{(\log l)^{1/2+1/2(1+\mu)}}{l^{1/(1+\mu)-1/2}(\log\ \log l)^{1+\varepsilon}} \ll \frac{1}{(\log\ \log l)^{1+\varepsilon}}. \tag{3.11}$$

Similarly,

$$A_2(l) \ll \frac{1}{(\log\ \log l)^{1+\varepsilon}}. \tag{3.12}$$

It remains to estimate $A_3(l)$. It is easy to check that

$$\begin{aligned} A_3(l) &\le \mathrm{P}\left(\bigcap_{i=k+1}^{l} (\mathrm{X}_i \le u_{li})\right) - \mathrm{P}\left(\bigcap_{i=1}^{l} (\mathrm{X}_i \le u_{li})\right) \\ &\le \left|\mathrm{P}\left(\bigcap_{i=1}^{l} (\mathrm{X}_i \le u_{li})\right) - \Phi^l(\lambda_l)\right| + \left|\mathrm{P}\left(\bigcap_{i=k+1}^{l} (\mathrm{X}_i \le u_{li})\right) - \Phi^{l-k}(\lambda_l)\right| + \left(\Phi^{l-k}(\lambda_l) - \Phi^l(\lambda_l)\right) \\ &=: B_1(l) + B_2(l) + B_3(l). \end{aligned} \tag{3.13}$$

By the arguments similar to that of Lemma 2.4 in Csáki and Gonchigdanzan [7], we get

$$B_3(l) \ll \frac{k}{l}. \tag{3.14}$$

By the Normal Comparison Lemma and (3.4), we derive that

$$\begin{aligned} B_1(l) &\ll \sum_{1\le i<j\le l} |r_{ij}| \exp\left(-\frac{u_{li}^2 + \lambda_l^2}{2(1+|r_{ij}|)}\right) \le l \sum_{1\le i\le l} |r_{ij}| \exp\left(-\frac{\lambda_l^2}{1+\delta}\right) \\ &\ll l\frac{(\log l)^{1/2}}{(\log\ \log l)^{1+\varepsilon}}\frac{(\log l)^{1/(1+\delta)}}{l^{2/(1+\delta)}} \\ &\ll \frac{1}{(\log\ \log l)^{1+\varepsilon}}, \\ B_2(l) &\ll \frac{1}{(\log\ \log l)^{1+\varepsilon}}. \end{aligned} \tag{3.15}$$

Combining with above analysis, we have

$$A_3(l) \ll \frac{k}{l} + \frac{1}{(\log\ \log\ l)^{1+\varepsilon}}. \tag{3.16}$$

The proof is complete. □

We also need the following auxiliary result.

**Lemma 3.2.** *Suppose that the standardized nonstationary Gaussian sequences* $\{X_n,\ n \geq 1\}$ *satisfy the conditions* (2.1)–(2.3). *Assume that* $n(1-\Phi(\lambda_n))$ *is bounded; then*

$$\left|\operatorname{Cov}\left(\mathbb{I}\left(\bigcap_{i=1}^{k}(X_i \leq u_{ki}), \frac{S_l}{\sigma_l} \leq y\right), \mathbb{I}\left(\bigcap_{i=k+1}^{l}(X_i \leq u_{ki}), \frac{S_l}{\sigma_l} \leq y\right)\right)\right| \ll \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\ \log l)^{1+\varepsilon}} \tag{3.17}$$

*for* $k < \min(\beta^2 l(\log\ \log\ )^{2+2\varepsilon}/c_2^2 \log l, l)$, *where* $0 < \beta < 1,\ c_2 > 0$.

*Proof.* By (2.2) and (2.3), for $i > k+1$, we get

$$\left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| \ll \frac{(\log l)^{1/2}}{l^{1/2}(\log\ \log\ l)^{1+\varepsilon}}. \tag{3.18}$$

Clearly,

$$\frac{(\log l)^{1/2}}{l^{1/2}(\log\ \log\ l)^{1+\varepsilon}} \longrightarrow 0 \quad \text{as } l \longrightarrow \infty, \tag{3.19}$$

which implies that there exist $\varrho > 0$ and $k_0$ such that for $k > k_0$,

$$\sup_{i \geq k+1}\left(\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right) < \varrho < 1. \tag{3.20}$$

For $k < l$, we have

$$\begin{aligned}\left|\operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l}\right)\right| &= \frac{1}{\sigma_k\sigma_l}\left|\sigma_k^2 + \operatorname{Cov}(S_k, S_l)\right| \\ &\leq \frac{\sigma_k}{\sigma_l} + \frac{1}{\sigma_k\sigma_l}\sum_{i=1}^{k}\sum_{j=i+1}^{l}|r_{ij}|.\end{aligned} \tag{3.21}$$

Condition (2.2) implies that there exist positive numbers $c_3$ and $c_4$ such that $c_3 k^{1/2} \le \sigma_k \le c_4 k^{1/2}$ and

$$\left|\operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l}\right)\right| \ll \frac{k^{1/2}}{l^{1/2}} + \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} \ll \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}}. \tag{3.22}$$

So there exists $0 < \nu < 1$ such that

$$\left|\operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l}\right)\right| < \nu < 1 \tag{3.23}$$

for $l_1 < k < \min(\beta^2 l(\log\log l)^{2+2\varepsilon}/c_2^2 \log l, l)$.By applying the inequalities above and the Normal Comparison Lemma, we get

$$\begin{aligned}
&\left|\operatorname{Cov}\left(\mathbb{I}\left(\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} < y\right), \mathbb{I}\left(\bigcap_{i=k+1}^{k}(X_i \le u_{ki}), \frac{S_l}{\sigma_l} < y\right)\right)\right| \\
&\quad = \left|\mathrm{P}\left(X_1 \le u_{k1}, \ldots, X_k \le u_{kk}, \frac{S_k}{\sigma_k} \le y, X_{k+1} \le u_{l(k+1)}, \ldots, X_{ll} \le u_{ll}, \frac{S_l}{\sigma_l} \le y\right)\right. \\
&\qquad \left. - \mathrm{P}\left(X_1 \le u_{k1}, \ldots, X_k \le u_{kk}, \frac{S_k}{\sigma_k} \le y\right)\mathrm{P}\left(X_{k+1} \le u_{l(k+1)}, \ldots, X_{ll} \le u_{ll}, \frac{S_l}{\sigma_l} \le y\right)\right| \\
&\quad \ll \sum_{i=1}^{k}\sum_{j=k+1}^{l} |r_{ij}| \exp\left(-\frac{u_{ki}^2 + u_{lj}^2}{2(1+|r_{ij}|)}\right) \\
&\qquad + \sum_{i=1}^{k}\left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| \exp\left(-\frac{u_{ki}^2 + y^2}{2(1+|\operatorname{Cov}(X_i, S_l/\sigma_l)|)}\right) \\
&\qquad + \sum_{j=k+1}^{l}\left|\operatorname{Cov}\left(X_j, \frac{S_k}{\sigma_k}\right)\right| \exp\left(-\frac{u_{lj}^2 + y^2}{2(1+|\operatorname{Cov}(X_j, S_k/\sigma_k)|)}\right) \\
&\qquad + \left|\operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l}\right)\right| \exp\left(-\frac{1}{1+|\operatorname{Cov}(S_k/\sigma_k, S_l/\sigma_l)|}\right) \\
&\quad =: D_1(l) + D_2(l) + D_3(l) + D_4(l).
\end{aligned} \tag{3.24}$$

By (3.10), we have

$$\begin{aligned} D_1(l) &\le \sum_{i=1}^{k}\sum_{j=k+1}^{l}|r_{ij}|\exp\left(-\frac{\lambda_k^2+\lambda_l^2}{2(1+\delta)}\right) \\ &\ll k\frac{(\log l)^{1/2}}{(\log\ \log\ l)^{1+\varepsilon}}\frac{(\log k)^{1/2(1+\delta)}}{k^{1/(1+\delta)}}\frac{(\log l)^{1/2(1+\delta)}}{l^{1/(1+\delta)}} \\ &\ll \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\ \log\ l)^{1+\varepsilon}}, \\ D_2(l) &< \exp\left(-\frac{u_{ki}^2}{2(1+\mu)}\right)\sum_{i=1}^{k}\left|\operatorname{Cov}\left(X_i,\frac{S_l}{\sigma_l}\right)\right| \\ &\ll \frac{(\log k)^{1/2(1+\mu)}}{k^{1/(1+\mu)}}k\frac{(\log l)^{1/2}}{l^{1/2}(\log\ \log\ l)^{1+\varepsilon}} \\ &\ll \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\ \log\ l)^{1+\varepsilon}}. \end{aligned} \tag{3.25}$$

Similarly,

$$\begin{aligned} D_3(l) &< \exp\left(-\frac{u_{lj}^2}{2(1+\varrho)}\right)\sum_{j=k+1}^{l}\left|\operatorname{Cov}\left(X_j,\frac{S_k}{\sigma_k}\right)\right| \\ &< \exp\left(-\frac{u_{lj}^2}{2(1+\varrho)}\right)\frac{1}{\sigma_k}\sum_{i=1}^{k}\sum_{j=1}^{l}|r_{ij}| \\ &\ll \frac{(\log l)^{1/2(1+\varrho)}}{l^{1/(1+\varrho)}}\frac{k}{k^{1/2}}\frac{(\log l)^{1/2}}{(\log\ \log\ l)^{1+\varepsilon}} \\ &\ll \frac{k^{1/2}}{l^{1/2}(\log\ \log\ l)^{1+\varepsilon}}. \end{aligned} \tag{3.26}$$

While (3.22) implies

$$D_4(l) < \left|\operatorname{Cov}\left(\frac{S_k}{\sigma_k},\frac{S_l}{\sigma_l}\right)\right| \ll \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\ \log\ l)^{1+\varepsilon}}, \tag{3.27}$$

the proof is complete. □

We also need the following auxiliary result.

**Lemma 3.3.** *Let $X_1, X_2, \ldots$ be a standardized nonstationary Gaussian sequences satisfying assumptions* (2.1)–(2.3). *Assume that $\sum_{i=1}^{n}(1-\Phi(u_{ni})) \to \tau$ for some $0<\tau<\infty$ and $n(1-\Phi(\lambda_n))$ is bounded. Then*

$$\lim_{k\to\infty} P\left(\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right) = e^{-\tau}\Phi(y) \tag{3.28}$$

*for all $\in (-\infty, \infty)$.*

*Proof.* By the Normal Comparison Lemma and the proof of Lemma 3.1, we have

$$\left| P\left(\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right) - P\left(\bigcap_{i=1}^{k}(X_i \le u_{ki})\right) P\left(\frac{S_k}{\sigma_k} \le y\right)\right| \ll \frac{1}{(\log\ \log\ k)^{1+\epsilon}}, \tag{3.29}$$

where

$$\lim_{k\to\infty} \frac{1}{(\log\ \log\ k)^{1+\epsilon}} = 0, \tag{3.30}$$

which implies

$$\lim_{k\to\infty} P\left(\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right) = \lim_{k\to\infty} P\left(\bigcap_{i=1}^{k}(X_i \le u_{ki})\right) P\left(\frac{S_k}{\sigma_k} \le y\right). \tag{3.31}$$

By Theorem 6.1.3 of Leadbetter et al. [15], we have

$$\lim_{k\to\infty} P\left(\bigcap_{i=1}^{k}(X_i \le u_{ki})\right) = e^{-\tau}. \tag{3.32}$$

Since $S_k/\sigma_k$ follows the standard normal distribution, we get

$$\lim_{k\to\infty} P\left(\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right) = e^{-\tau}\Phi(y), \tag{3.33}$$

which completes the proof. □

We now only give the proof of Theorem 2.1. Theorem 2.2 is a special case of Theorem 2.1.

*Proof of Theorem 2.1.* The idea of this proof is similar to that of Theorem 1.1 in Csáki and Gonchigdanzan [7]. In order to prove Theorem 2.1, it is enough to show that

$$\operatorname{Var}\left(\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left(\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right)\right) \ll \frac{(\log n)^2}{(\log\ \log\ n)^{1+\epsilon}} \tag{3.34}$$

for all fixed $\in (-\infty, \infty)$.

Let $\xi_k = \mathbb{I}(\bigcap_{i=1}^{k}(X_i \le u_{ki}), S_k/\sigma_k \le y) - \mathrm{P}(\bigcap_{i=1}^{k}(X_i \le u_{ki}), S_k/\sigma_k \le y)$, we have

$$\begin{aligned}&\mathrm{Var}\left(\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left(\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right)\right)\\ &\quad\le \sum_{k=1}^{n}\frac{1}{k^2}\mathrm{E}\xi_k^2 + 2\sum_{1\le k<l\le n}\frac{1}{kl}|\mathrm{E}(\xi_k\xi_l)| =: F_1 + F_2.\end{aligned} \tag{3.35}$$

Since $\{\xi_k\}$ are bounded,

$$F_1 \ll \sum_{k=1}^{n}\frac{1}{k^2} < \infty. \tag{3.36}$$

The remainder is to estimate $F_2$. Notice

$$\begin{aligned}|\mathrm{E}(\xi_k\xi_l)| = &\left|\mathrm{Cov}\left(\mathbb{I}\left(\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right), \mathbb{I}\left(\bigcap_{i=1}^{l}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right)\right.\right.\\ &\qquad\left.\left. - \mathbb{I}\left(\bigcap_{i=k+1}^{l}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right)\right|\right)\\ &+ \left|\mathrm{Cov}\left(\mathbb{I}\left(\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right), \mathbb{I}\left(\bigcap_{i=1}^{l}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right)\right)\right|\\ \le{}& \mathrm{E}\left|\mathbb{I}\left(\bigcap_{i=1}^{l}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right) - \mathbb{I}\left(\bigcap_{i=k+1}^{l}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right)\right|\\ &+ \left|\mathrm{Cov}\left(\mathbb{I}\left(\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right), \mathbb{I}\left(\bigcap_{i=1}^{l}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right)\right)\right|.\end{aligned} \tag{3.37}$$

By Lemmas 3.1 and 3.2, we infer that if $k < \beta^2 l(\log\ \log\ l)^{2+2\varepsilon}/(c_2^2\log l)$ and $k < 1$,

$$|E(\xi_k\xi_l)| \ll \frac{1}{(\log\log\ n)^{1+\epsilon}} + \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\ \log\ l)^{1+\varepsilon}} + \frac{k}{l} \tag{3.38}$$

for some $\epsilon > 0$. By the arguments similar to that of Theorem 1 in Dudziński [13], we can get

$$F_2 \ll \frac{(\log n)^2}{(\log\ \log n)^{1+\epsilon}}. \tag{3.39}$$

So by Lemma 3.1 of Csáki and Gonchigdanzan [7] and Lemma 3.3,

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left(\bigcap_{i=1}^{k}(X_i \le u_{ki}), \frac{S_k}{\sigma_k} \le y\right) = e^{-\tau}\Phi(y) \quad \text{a.s.} \tag{3.40}$$

which completes the proof. □

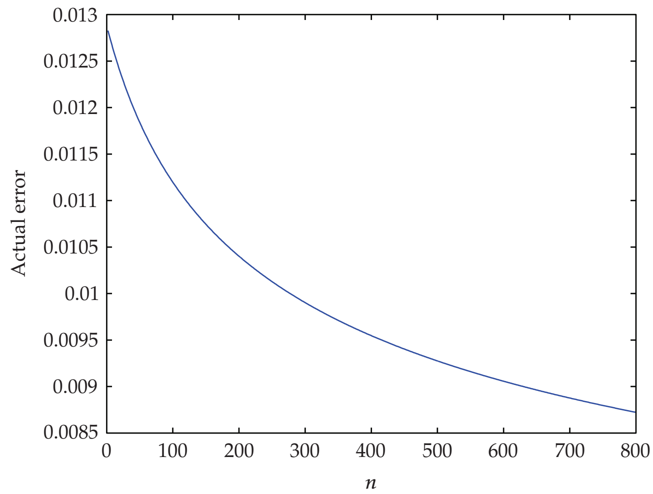


**Figure 1:** T he actual error, $\Delta_n$, for $r_n = 1/[n(\log\ n)^{1/2}(\log\ \log\ n)]$ and $(x, y) = (-1, -1)$.

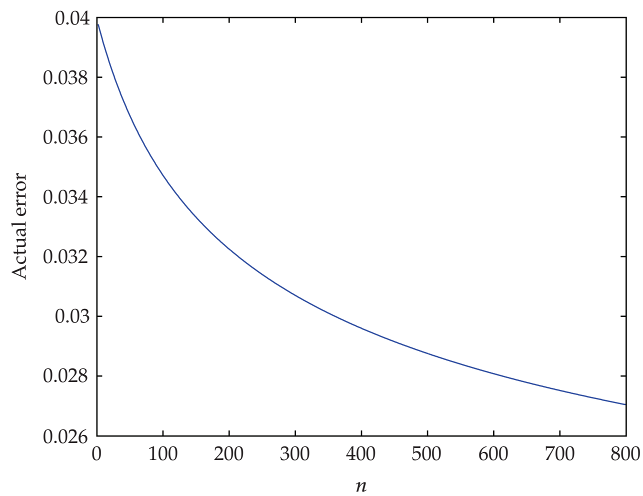


**Figure 2:** T he actual error, $\Delta_n$, for $r_n = 1/[n(\log\ n)^{1/2}(\log\ \log\ n)]$ and $(x, y) = (0, 0)$.

## 4. Numerical Analysis

The aim of this section is to calculate the actual convergence rate of

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left(M_k \leq a_k^{-1}x + b_k, \frac{S_k}{\sigma_k} \leq y\right) \longrightarrow \exp\left(-e^{-x}\right)\Phi(y) \tag{4.1}$$

for finite; that is, calculate

$$\Delta_n(x, y) = \left|\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left(M_k \leq a_k x + b_k, \frac{S_k}{\sigma_k} \leq y\right) - \exp\left(-e^{-x}\right)\Phi(y)\right|, \tag{4.2}$$

where $a_n = (2\log n)^{-1/2}$ and $b_n = (2\log n)^{1/2} - (\log\ \log\ n + \log 4\pi)/2(2\log n)^{1/2}$.

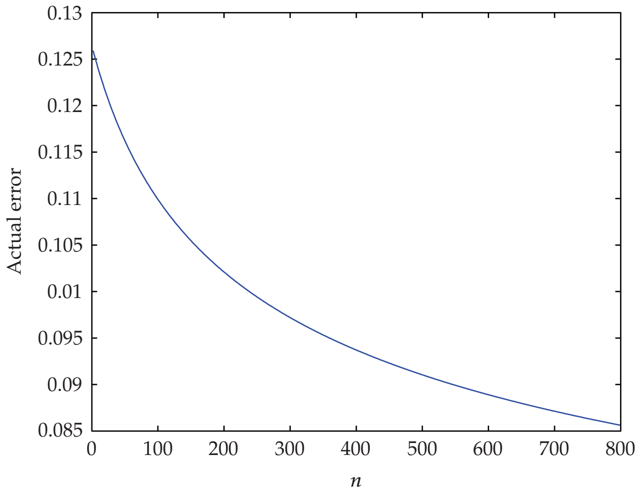


**Figure 3:** T he actual error, $\Delta_n$, for $r_n = 1/[n(\log\ n)^{1/2}(\log\ \log\ n)]$ and $(x, y) = (1, 1)$.

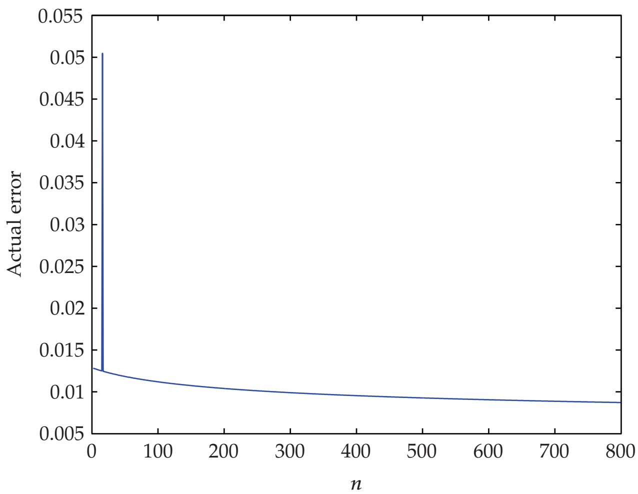


**Figure 4:** T he actual error, $\Delta_n$, for $r_n = 1/[n(\log\ n)^{1/2}(\log\ \log\ n)(\log\ \log\ \log\ n)]$ and $(x, y) = (-1, -1)$.

Firstly, we will construct a standardized triangular Gaussian array $\{X_{n,j},\ 1 \le j \le n,\ n \ge 1\}$ with equal correlation $r_n$ in $n$ th array for $\ge 1$. Meanwhile, the sequence $r_n$ must satisfy the conditions (2.1), (2.2), and (2.3). By Leadbetter et al. [15], we can construct the Gaussian array by i.i.d Gaussian sequence; that is, let $r_n$ to a convex sequence, $\xi_1, \xi_2, \ldots$ is a standardized i.i.d Gaussian sequence, and $\eta$ is also a standardized normal random variable which is independent of $\xi_k$ $(k \ge 1)$. For each $\ge 1$, let

$$X_{ij} = (1 - r_i)^{1/2}\xi_j + r_i^{1/2}\eta, \tag{4.3}$$

where $= 1, 2, \ldots, i$. Obviously, $X_{ij}$ $(1 \le j \le i)$ is a zero mean normal sequence with equal correlation. By this way, we get the Gaussian array needed.

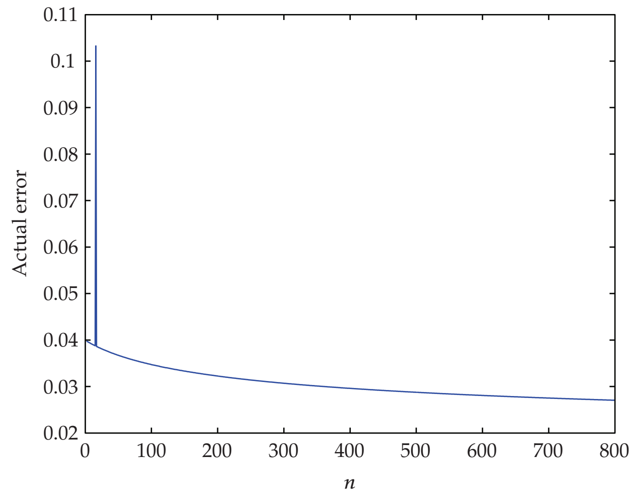


**Figure 5:** T he actual error, $\Delta_n$, for $r_n = 1/[n(\log n)^{1/2}(\log\log n)(\log\log\log n)]$ and $(x, y) = (0, 0)$.

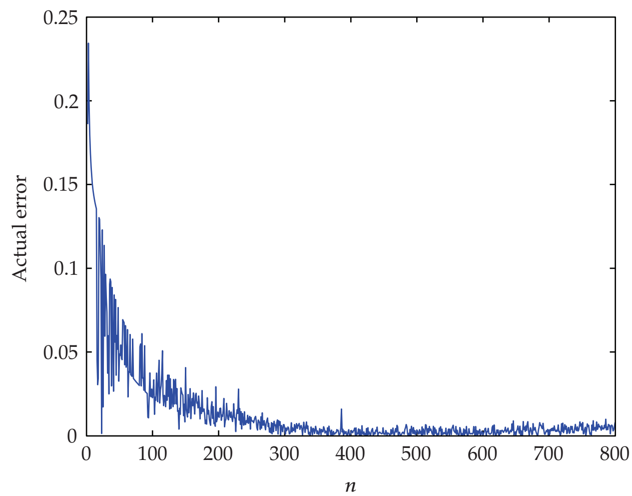


**Figure 6:** T he actual error, $\Delta_n$, for $r_n = 1/[n(\log n)^{1/2}(\log\log n)(\log\log\log n)]$ and $(x, y) = (1, 1)$.

Figures 1 to 3 give the actual error, $\Delta_n$, for $r_n = 1/[n(\log n)^{1/2}(\log\log n)]$ and $(x, y) = (-1, -1), (0, 0), (1, 1)$. In each figure, the actual error shocks tend to zero as $n$ increases. The overall performance of the actual error becomes better as $(x, y) = (0, 0)$.

Figures 4 to 6 give the actual error, $\Delta_n$, for

$$r_n = \frac{1}{\left[n(\log n)^{1/2}(\log\log n)(\log\log\log n)\right]},$$
$$(x, y) = (-1, -1), (0, 0), (1, 1). \tag{4.4}$$

In each figure, the actual error shocks also tend to zero as $n$ increases. Also the overall performance of the actual error becomes better as $(x, y) = (0, 0)$.

## Acknowledgments

The authors wish to thank the referees for some useful comments. The research was partially supported by the National Natural Science Foundation of China (Grant no. 70371061), the National Natural Science Foundation of China (Grant no. 10971227), and the Program for ET of Chongqing Higher Education Institutions (Grant no. 120060-20600204).

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