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Research Article

Jensen Type Inequalities Involving Homogeneous Polynomials

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By means of algebraic, analytical and majorization theories, and under the proper hypotheses, we establish several Jensen type inequalities involving γth homogeneous polynomials as follows: $\sum_{k=1}^{m} w_k f(X_k)/f(I_n) \leq [f(\sum_{k=1}^{m} w_k X_k^{\gamma})/f(I_n)]^{1/\gamma}$, $\sum_{k=1}^{m} w_k f(X_k)/f(N_n) \leq [f(\sum_{k=1}^{m} w_k X_k^{\gamma})/f(N_n)]^{1/\gamma}$, and $\sum_{k=1}^{m} w_k f_*(X_k) \leq f_*(\sum_{k=1}^{m} w_k X_k)$, and display their applications.

1. Introduction

The following notation and hypotheses in [1–4] will be used throughout the paper:

$$\begin{gathered}
x = (x_1,\ x_2,\dots,x_n)^{\dagger}, \qquad \alpha = (\alpha_1,\alpha_2,\dots,\alpha_n)^{\dagger}, \qquad w = (w_1,w_2,\dots,w_m)^{\dagger},\\
X_k = (x_{k,1},x_{k,2},\dots,x_{k,n})^{\dagger}, \quad \mathbb{N} = \{0,1,2,\dots,n,\dots\}, \quad n \in \mathbb{N},\ n \geq 2,\\
\mathbb{R} =]-\infty,\infty[, \quad \mathbb{R}_{+}^{n} = [0,\infty[^{n}, \quad \mathbb{R}_{++}^{n} =]0,\infty[^{n}, \quad \Omega^{n} = \{x \in \mathbb{R}_{+}^{n} \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n\}.
\end{gathered} \tag{1.1}$$

Also let

$$P_{\gamma}[x] = \left\{ \sum_{(\alpha,\sigma)\in\mathcal{B}_{\gamma}\times S_n} \lambda(\alpha,\sigma)\prod_{j=1}^{n} x_{\sigma(j)}^{\alpha_j} \,\middle|\, \lambda : \mathcal{B}_{\gamma}\times S_n \to \mathbb{R} \right\} \setminus \{0\},$$

$$P_{\gamma}^{+}[x] = \left\{ \sum_{(\alpha,\sigma)\in\mathcal{B}_{\gamma}\times S_n} \lambda(\alpha,\sigma)\prod_{j=1}^{n} x_{\sigma(j)}^{\alpha_j} \,\middle|\, \lambda : \mathcal{B}_{\gamma}\times S_n \longrightarrow [0,\infty[\right\} \setminus \{0\},$$

$$\overline{P}_{\gamma}[x] = \left\{ \sum_{\alpha\in\mathcal{B}_{\gamma}} \frac{\lambda(\alpha)}{n!} \operatorname{per}\left[x_j^{\alpha_i}\right] \middle| \lambda : \mathcal{B}_{\gamma} \longrightarrow \mathbb{R} \right\} \setminus \{0\},$$
$$\overline{P}_{\gamma}^{+}[x] = \left\{ \sum_{\alpha\in\mathcal{B}_{\gamma}} \frac{\lambda(\alpha)}{n!} \operatorname{per}\left[x_j^{\alpha_i}\right] \middle| \lambda : \mathcal{B}_{\gamma} \longrightarrow [0,\infty[\right\} \setminus \{0\}, \tag{1.2}$$

where

$$\left[x_j^{\alpha_i}\right] = \left[x_j^{\alpha_i}\right]_{n\times n} = \begin{bmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & x_3^{\alpha_1} & \cdots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & x_3^{\alpha_2} & \cdots & x_n^{\alpha_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_n} & x_2^{\alpha_n} & x_3^{\alpha_n} & \cdots & x_n^{\alpha_n} \end{bmatrix}_{n\times n}, \tag{1.3}$$

$\mathcal{B}_{\gamma}$ is a nonempty and finite subset of

$$\left\{ \alpha \in \mathbb{R}_+^n \middle| \sum_{i=1}^{n} \alpha_i = \gamma \ , \gamma \in [0,\infty[\right\}, \tag{1.4}$$

and the permanent of $n \times n$ matrix $\mathbb{A} = [a_{i,j}]_{n\times n}$ is given by (see [2, 4])

$$\operatorname{per} \mathbb{A} = \sum_{\sigma\in S_n} \prod_{j=1}^{n} a_{j,\sigma(\mathrm{j})}; \tag{1.5}$$

here, the sum extends over all elements σ of the nth symmetric group S_n.

If $f \in P_{\gamma}[x]$, then f is called γth homogeneous polynomial; if $f \in \overline{P}_{\gamma}[x]$, then f is called γth homogeneous symmetric polynomial (see [3]).

The famous Jensen inequality can be stated as follows: if $f : I \longrightarrow \mathbb{R}$ is a convex function, then for any $x \in I^n$ we have

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) \geq f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right). \tag{1.6}$$

A large number of generalizations and applications of the inequality (1.6) had been obtained in [1] and [5–8]. An interesting generalization of (1.6) was given by Chen et al., in [8]: Let $\mathcal{B}_{\gamma} \subset \mathbb{N}^n$ and $f \in \overline{P}_{\gamma}^{+}[x]$. If $X_k \in \mathbb{R}_+^n$ with $1 \leq k \leq m$ and $0 \leq X_1 \leq X_2 \leq \cdots \leq X_m$, then we have the following Jensen type inequality:

$$\frac{1}{m}\sum_{k=1}^{m} f(X_k) \geq f\left(\frac{1}{m}\sum_{k=1}^{m} X_k\right). \tag{1.7}$$

In this paper, by means of algebraic, analytical, and majorization theories, and under the proper hypotheses, we will establish several Jensen type inequalities involving γth homogeneous polynomials and display their applications.

2. Jensen Type Inequalities Involving Homogeneous Polynomials

In this section, we will use the following notation (see [1, 4, 9]):

$$\mathbb{Q}_{++} = \left\{ \frac{q}{p} \,\middle|\, p \in \mathbb{N} \setminus \{0\},\ q \in \mathbb{N} \setminus \{0\} \right\}, \quad I_n = (1,1,\ldots,1)^\dagger, \quad N_n = (1,2,\ldots,n)^\dagger,$$
$$x^\gamma = \left(x_1^\gamma, x_2^\gamma, \ldots, x_n^\gamma\right)^\dagger, \quad \phi(x) = (\phi(x_1), \phi(x_2), \ldots, \phi(x_n))^\dagger, \quad A(x) = \frac{1}{n}\sum_{i=1}^n x_i, \tag{2.1}$$
$$\Delta x = (\Delta x_1, \Delta x_2, \ldots, \Delta x_n)^\dagger = (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1})^\dagger.$$

2.1. A Jensen Type Inequality Involving Homogeneous Polynomials

We begin a Jensen type inequality involving homogeneous polynomials as follows.

Theorem 2.1. *Let $f \in P_\gamma^+[x]$. If $X_k \in \mathbb{R}_+^n$ with $1 \le k \le m$ and $w \in \mathbb{R}_{++}^m$, then*

$$\frac{\sum_{k=1}^m w_k f(X_k)}{f(I_n)} \le \left[\frac{f\left(\sum_{k=1}^m w_k X_k^\gamma\right)}{f(I_n)}\right]^{1/\gamma}. \tag{2.2}$$

The equality holds in (2.2) *if there exists $t \in [0,\infty[$, such that $X_1 = X_2 = \cdots = X_m = tI_n$.*

Lemma 2.2. *(Hölder's inequality, see [1, 10]). Let $a_{i,k} \in [0,\infty[, q_i \in [0,\infty[$ with $1 \le i \le n$ and $1 \le k \le m$. If $\sum_{i=1}^n q_i \le 1$, then*

$$\frac{1}{m}\sum_{k=1}^m \prod_{i=1}^n a_{i,k}^{q_i} \le \prod_{i=1}^n \left(\frac{1}{m}\sum_{k=1}^m a_{i,k}\right)^{q_i}. \tag{2.3}$$

The equality in (2.3) *holds if $a_{i,1} = a_{i,2} = \cdots = a_{i,m}$ for $1 \le i \le n$.*

Lemma 2.3. *(Power mean inequality, see [1, 10–11]). Let $x \in \mathbb{R}_+^n, \mu \in \mathbb{R}_{++}^n$ and $\sum_{i=1}^n \mu_i = 1$. If $\gamma \in [1,\infty[$, then*

$$\sum_{i=1}^n \mu_i x_i^\gamma \ge \left(\sum_{i=1}^n \mu_i x_i\right)^\gamma. \tag{2.4}$$

The inequality is reversed for $\gamma \in (0,1)$. The equality in (2.4) *holds if and only if $\gamma = 1$, or $x_1 = x_2 = \cdots = x_n$.*

Lemma 2.4. *Let* $g(x,\alpha)=\prod_{j=1}^{n}x_{\sigma(j)}^{\alpha_j}$ *and* $\sigma\in S_n$. *If* $\alpha\in\mathcal{B}_\gamma$ *and* $X_k\in\mathbb{R}_+^n$ *with* $1\le k\le m$, *then*

$$g\left(\sum_{k=1}^{m}X_k^{\gamma},\alpha\right)\ge\left[\sum_{k=1}^{m}g(X_k,\alpha)\right]^{\gamma}. \tag{2.5}$$

The equality in (2.5) *holds if* $\alpha=(1,0,\ldots,0)^{\dagger}$, *or there exists* $t\in[0,\infty[$, *such that* $X_1=X_2=\cdots=X_m=tI_n$.

Proof. According to $\alpha\in\mathcal{B}_\gamma$, $\sum_{j=1}^{n}(\alpha_j/\gamma)\ =1\le 1$ and Lemmas 2.2-2.3, we get that

$$\begin{aligned}
g\left(\frac{1}{m}\sum_{k=1}^{m}X_k^{\gamma},\alpha\right)&=\prod_{j=1}^{n}\left(\frac{1}{m}\sum_{k=1}^{m}x_{k,\sigma(j)}^{\gamma}\right)^{\alpha_j}\\
&=\left[\prod_{j=1}^{n}\left(\frac{1}{m}\sum_{k=1}^{m}x_{k,\sigma(j)}^{\gamma}\right)^{\alpha_j/\gamma}\right]^{\gamma}\\
&\ge\left[\frac{1}{m}\sum_{k=1}^{m}\prod_{j=1}^{n}x_{k,\sigma(j)}^{\alpha_j}\right]^{\gamma}\\
&=\left[\frac{1}{m}\sum_{k=1}^{m}g(X_k,\alpha)\right]^{\gamma}.
\end{aligned} \tag{2.6}$$

From

$$g\left(\frac{1}{m}\sum_{k=1}^{m}X_k^{\gamma},\alpha\right)=\frac{1}{m^{\gamma}}g\left(\sum_{k=1}^{m}X_k^{\gamma},\alpha\right), \tag{2.7}$$

we deduce to the inequality (2.5). Lemma 2.4 is proved. □

Proof of Theorem 2.1. First of all, we assume that $w=I_m$. According to $\gamma\in[1,\infty[$, $f(I_n)=\sum_{(\alpha,\sigma)\in\mathcal{B}_\gamma\times S_n}\lambda(\alpha,\sigma)$ and Lemmas 2.3-2.4, we find that

$$\begin{aligned}
\frac{f\left(\sum_{k=1}^{m}X_k^{\gamma}\right)}{f(I_n)}&=\sum_{(\alpha,\sigma)\in\mathcal{B}_\gamma\times S_n}\frac{\lambda(\alpha,\sigma)}{f(I_n)}g\left(\sum_{k=1}^{m}X_k^{\gamma},\alpha\right)\\
&\ge\sum_{(\alpha,\sigma)\in\mathcal{B}_\gamma\times S_n}\frac{\lambda(\alpha,\sigma)}{f(I_n)}\left[\sum_{k=1}^{m}g(X_k,\alpha)\right]^{\gamma}\\
&\ge\left[\sum_{(\alpha,\sigma)\in\mathcal{B}_\gamma\times S_n}\frac{\lambda(\alpha,\sigma)}{f(I_n)}\sum_{k=1}^{m}g(X_k,\alpha)\right]^{\gamma}\\
&=\left[\frac{\sum_{k=1}^{m}f(X_k)}{f(I_n)}\right]^{\gamma}.
\end{aligned} \tag{2.8}$$

That is, the inequality (2.2) holds.

Secondly, for some of w_k with $1 \le k \le m$ satisfing $w_k \neq 1$, we have the following cases.

(1) If $w \in \mathbb{N}^m$, then the inequality (2.2) holds from the above proof.

(2) If $w \in \mathbb{Q}^m_{++}$, then there exists $N \in \mathbb{N} \setminus \{0\}$ that satisfies $Nw \in \mathbb{N}^m$. By the result in (1), we obtain that

$$\begin{aligned} \frac{\sum_{k=1}^{m} N w_k f(X_k)}{f(I_n)} &\le \left[\frac{f\left(\sum_{k=1}^{m} N w_k X_k^{\gamma}\right)}{f(I_n)}\right]^{1/\gamma} \\ \Longleftrightarrow \frac{\sum_{k=1}^{m} w_k f(X_k)}{f(I_n)} &\le \left[\frac{f\left(\sum_{k=1}^{m} w_k X_k^{\gamma}\right)}{f(I_n)}\right]^{1/\gamma}, \end{aligned} \tag{2.9}$$

which implies that inequality (2.2) is also true.

(3) If $w \in \mathbb{R}^m_{++}$, then there exist sequences $\{w_k^{(i)}\}_{i=1}^{\infty}$, such that

$$w_k^{(i)} \in \mathbb{Q}_{++} \quad (1 \le i < \infty), \quad \lim_{i \to \infty} w_k^{(i)} = w_k \quad (1 \le k \le m). \tag{2.10}$$

We get by the case in (2) that

$$\frac{\sum_{k=1}^{m} w_k^{(i)} f(X_k)}{f(I_n)} \le \left[\frac{f\left(\sum_{k=1}^{m} w_k^{(i)} X_k^{\gamma}\right)}{f(I_n)}\right]^{1/\gamma}, \tag{2.11}$$

and taking $i \to \infty$ in (2.11), we can get the inequality (2.2). The proof of Theorem 2.1 is thus completed. □

2.2. Jensen Type Inequalities Involving Difference Substitution

Exchange the ith row and jth row in nth unit matrix $\mathbb{E}$, then this matrix, written $\mathbb{E}(i,j)$, is called nth exchange matrix. If $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_p$ are nth exchange matrixes, then the $n \times n$ matrix $\mathbb{D}_n = \mathbb{E}_p \mathbb{E}_{p-1} \cdots \mathbb{E}_1 \mathbb{E}_0 \Delta_n$ is called nth difference matrix, and the substitution $x = \mathbb{D}_n y$ is difference substitution, where $p \in \mathbb{N}$, $\mathbb{E}_0 = \mathbb{E}$, and

$$\Delta_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n \times n}. \tag{2.12}$$

Let $f \in P_\gamma[x]$. If $f(\mathbb{D}_n y) \in P_\gamma^+[y]$ is true for any difference matrix $\mathbb{D}_n$, then $f(x) \geq 0$ for any $x \in \mathbb{R}_+^n$ (see [11]), and the homogeneous polynomial f is called positive semidefinite with difference substitution.

If we let

$$\begin{gathered} \mathfrak{D}_n = \{\mathbb{D}_\mathrm{n} \mid \mathbb{D}_n \text{ be } n\text{th difference matrix}\}, \\ P_\gamma^*[x] = \left\{ f(x) \in P_\gamma[x] \mid \mathcal{B}_\gamma \subset \mathbb{N}^n,\ f(\mathbb{D}_n y) \in P_\gamma^+[y],\ \forall \mathbb{D}_n \in \mathfrak{D}_n \right\}, \end{gathered} \tag{2.13}$$

then $\mathfrak{D}_n$ is a finite set and the count of elements of $\mathfrak{D}_n$ is $|\mathfrak{D}_n| = n!$, and $\gamma \in \mathbb{N}$.

We have the following Jensen type inequality involving homogeneous polynomials and difference substitution.

Theorem 2.5. *Let $f \in P_\gamma^*[x]$. If $w \in \mathbb{R}_{++}^m$, and $X_k \in \Omega^n$ with $1 \leq k \leq m$, then*

$$\frac{\sum_{k=1}^m w_k f(X_k)}{f(N_n)} \leq \left[\frac{f\left(\sum_{k=1}^m w_k X_k^\gamma \right)}{f(N_n)} \right]^{1/\gamma}. \tag{2.14}$$

The equality holds in (2.14) *if there exists $t \in [0, \infty[$, such that $X_1 = X_2 = \cdots = X_m = tI_n$, and $f(I_n) = 0$.*

Lemma 2.6. *(Jensen's inequality, see [12]). For any $x \in \mathbb{R}_+^n$ and $\gamma \in [1, \infty[$, we have*

$$\left(\sum_{k=1}^n x_k \right)^\gamma \geq \sum_{k=1}^n x_k^\gamma. \tag{2.15}$$

The equality in (2.33) *holds if and only if $\gamma = 1$, or at least $n - 1$ numbers equal zero among the set $\{x_1,\ x_2, \ldots, x_n\}$.*

Lemma 2.7. *If $\gamma \in [1, \infty[$ and $x \in \Omega^n$, then for the difference substitution $x = \Delta_n y$, one has the following double inequality:*

$$0 \leq y^\gamma \leq \Delta x^\gamma. \tag{2.16}$$

The equality $y^\gamma = \Delta x^\gamma$ holds if and only if $\gamma = 1$, or $x_1 = x_2 = \cdots = x_{n-1} = 0$, or $x_1 = x_2 = \cdots = x_n$.

Proof. From $x \in \Omega^n$, it is easy to know that $y = \Delta_n^{-1} x = \Delta x \in \mathbb{R}_+^n$. By $\gamma \in [1, \infty[$ and Lemma 2.6, we find that

$$\begin{gathered} 0 \leq y_1^\gamma = x_1^\gamma \leq x_1^\gamma, \\ 0 \leq y_2^\gamma = (x_2 - x_1)^\gamma \leq x_2^\gamma - x_1^\gamma, \\ \vdots \\ 0 \leq y_n^\gamma = (x_n - x_{n-1})^\gamma \leq x_n^\gamma - x_{n-1}^\gamma. \end{gathered} \tag{2.17}$$

This shows that the double inequality (2.16) holds. ◻

Proof of Theorem 2.5. Consider the difference substitution $X_k = \Delta_n Y_k$. Since $X_k \in \Omega^n$, $Y_k = \Delta_n^{-1} X_k = \Delta X_k \in \mathbb{R}_+^n$ with $1 \le k \le m$. From $f \in P_\gamma^*[x]$, we have that $f(\mathbb{D}_n y) \in P_\gamma^+[y]$, for all $\mathbb{D}_n \in \mathfrak{D}_n$. Hence,

$$f(\Delta_n y) \in P_\gamma^+[y]. \tag{2.18}$$

According to Theorem 2.1, we obtain that

$$\frac{\sum_{k=1}^m w_k f(\Delta_n Y_k)}{f(\Delta_n I_n)} \le \left[\frac{f\left(\Delta_n \sum_{k=1}^m w_k Y_k^\gamma\right)}{f(\Delta_n I_n)}\right]^{1/\gamma} = \left[\frac{f\left(\sum_{k=1}^m w_k \Delta_n Y_k^\gamma\right)}{f(N_n)}\right]^{1/\gamma}. \tag{2.19}$$

In view of $Y_k \in \mathbb{R}_+^n$ and with Lemma 2.7, we have

$$0 \le Y_k^\gamma \le \Delta X_k^\gamma, \quad k = 1, 2, \ldots, m. \tag{2.20}$$

By noting that $f(\Delta_n y) \in P_\gamma^+[y]$, it implies that $f(\sum_{k=1}^m w_k \Delta_n Y_k^\gamma)$ is increasing with respect to Y_k^γ. Thus,

$$f\left(\sum_{k=1}^m w_k \Delta_n Y_k^\gamma\right) \le f\left(\sum_{k=1}^m w_k \Delta_n \Delta X_k^\gamma\right) = f\left(\sum_{k=1}^m w_k X_k^\gamma\right). \tag{2.21}$$

Therefore,

$$\begin{aligned}\frac{\sum_{k=1}^m w_k f(X_k)}{f(N_n)} &= \frac{\sum_{k=1}^m w_k f(\Delta_n Y_k)}{f(\Delta_n I_n)} \\ &\le \left[\frac{f\left(\sum_{k=1}^m w_k \Delta_n Y_k^\gamma\right)}{f(N_n)}\right]^{1/\gamma} \\ &\le \left[\frac{f\left(\sum_{k=1}^m w_k X_k^\gamma\right)}{f(N_n)}\right]^{1/\gamma}.\end{aligned} \tag{2.22}$$

This evidently completes the proof of Theorem 2.5.

As an application of Theorem 2.5, we have the following. □

Theorem 2.8. *Let $f(x) = A(x^\gamma) - A^\gamma(x), \gamma \in \mathbb{N}$ and $\gamma \ge 2$. If $w \in \mathbb{R}_{++}^m, X_k \in \Omega^n$ with $1 \le k \le m$, then the inequality* (2.14) *holds. The equality holds in* (2.14) *if there exists $t \in [0, \infty[$, such that $X_1 = X_2 = \cdots = X_m = tI_n$.*

Proof. First of all, we prove that $f \in P_{\gamma}^{*}[x]$. If the function $\phi : I \to \mathbb{R}$ satisfies the condition that $\phi'' : I \to \mathbb{R}$ is continuous, then we have the following identity:

$$A(\phi(x)) - \phi(A(x)) = \frac{1}{n^2} \sum_{1\le i<j\le n} \left\{ \iint_{\nabla} \phi''\left[t_1 x_i + t_2 x_j + (1 - t_1 - t_2)A(x)\right] dt_1 dt_2 \right\} (x_i - x_j)^2, \tag{2.23}$$

where

$$x \in I^n, \qquad \phi''(t) = \frac{d^2\phi}{dt^2}, \qquad \nabla = \left\{ (t_1, t_2)^{\dagger} \in \mathbb{R}_+^2 \mid t_1 + t_2 \le 1 \right\}. \tag{2.24}$$

In fact,

$$\begin{aligned}
&\iint_{\nabla} \phi''\left[t_1 x_i + t_2 x_j + (1 - t_1 - t_2)A(x)\right] dt_1 dt_2 \\
&= \int_0^1 dt_1 \int_0^{1-t_1} \phi''\left[t_1 x_i + t_2 x_j + (1 - t_1 - t_2)A(x)\right] dt_2 \\
&= \frac{1}{x_j - A(x)} \int_0^1 dt_1 \int_0^{1-t_1} \phi''\left[t_1 x_i + t_2 x_j + (1 - t_1 - t_2)A(x)\right] d\left[t_1 x_i + t_2 x_j + (1 - t_1 - t_2)A(x)\right] \\
&= \frac{1}{x_j - A(x)} \int_0^1 dt_1 \phi' \ \left[t_1 x_i + t_2 x_j + (1 - t_1 - t_2)A(x)\right] \Big|_0^{1-t_1} \\
&= \frac{1}{x_j - A(x)} \int_0^1 \{\phi'[t_1 x_i + (1-t_1)x_j] - \phi'[t_1 x_i + (1-t_1)A(x)]\} dt_1 \\
&= \frac{1}{x_j - A(x)} \left\{ \frac{\phi[t_1 x_i + (1-t_1)x_j]}{x_i - x_j} - \frac{\phi[t_1 x_i + (1-t_1)A(x)]}{x_i - A(x)} \right\} \Bigg|_0^1 \\
&= \frac{1}{x_j - A(x)} \left[\frac{\phi(x_i) - \phi(x_j)}{x_i - x_j} - \frac{\phi(x_i) - \phi(A(x))}{x_i - A(x)} \right] \\
&= \frac{1}{(x_i - x_j)(x_j - A(x))(x_i - A(x))} \begin{vmatrix} \phi(A(x)) & A(x) & 1 \\ \phi(x_i) & x_i & 1 \\ \phi(x_j) & x_j & 1 \end{vmatrix},
\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}
&\sum_{1\le i<j\le n} \left\{ \iint_{\nabla} \phi''\left[t_1 x_i + t_2 x_j + (1 - t_1 - t_2)A(x)\right] dt_1 dt_2 \right\} (x_i - x_j)^2 \\
&\quad = \sum_{1\le i<j\le n} \frac{x_i - x_j}{(x_j - A(x))(x_i - A(x))} \begin{vmatrix} \phi(A(x)) & A(x) & 1 \\ \phi(x_i) & x_i & 1 \\ \phi(x_j) & x_j & 1 \end{vmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{1\le i,j\le n}\left(\frac{1}{x_j - A(x)} - \frac{1}{x_i - A(x)}\right)\begin{vmatrix} \phi(A(x)) & A(x) & 1 \\ \phi(x_i) & x_i & 1 \\ \phi(x_j) & x_j & 1 \end{vmatrix} \\
&= \frac{1}{2}\left(\sum_{1\le i,j\le n}\frac{1}{x_j - A(x)}\begin{vmatrix} \phi(A(x)) & A(x) & 1 \\ \phi(x_i) & x_i & 1 \\ \phi(x_j) & x_j & 1 \end{vmatrix} - \sum_{1\le i,j\le n}\frac{1}{x_i - A(x)}\begin{vmatrix} \phi(A(x)) & A(x) & 1 \\ \phi(x_i) & x_i & 1 \\ \phi(x_j) & x_j & 1 \end{vmatrix}\right) \\
&= \frac{1}{2}\left(\sum_{j=1}^{n}\frac{1}{x_j - A(x)}\sum_{i=1}^{n}\begin{vmatrix} \phi(A(x)) & A(x) & 1 \\ \phi(x_i) & x_i & 1 \\ \phi(x_j) & x_j & 1 \end{vmatrix} - \sum_{i=1}^{n}\frac{1}{x_i - A(x)}\sum_{j=1}^{n}\begin{vmatrix} \phi(A(x)) & A(x) & 1 \\ \phi(x_i) & x_i & 1 \\ \phi(x_j) & x_j & 1 \end{vmatrix}\right) \\
&= \frac{1}{2}\left(\sum_{j=1}^{n}\frac{n}{x_j - A(x)}\begin{vmatrix} \phi(A(x)) & A(x) & 1 \\ \frac{1}{n}\sum_{i=1}^{n}\phi(x_i) & A(x) & 1 \\ \phi(x_j) & x_j & 1 \end{vmatrix} - \sum_{i=1}^{n}\frac{n}{x_i - A(x)}\begin{vmatrix} \phi(A(x)) & A(x) & 1 \\ \phi(x_i) & x_i & 1 \\ \frac{1}{n}\sum_{j=1}^{n}\phi(x_j) & A(x) & 1 \end{vmatrix}\right) \\
&= \frac{1}{2}\left(\sum_{j=1}^{n}\frac{n}{x_j - A(x)}\begin{vmatrix} \phi(A(x)) - \frac{1}{n}\sum_{i=1}^{n}\phi(x_i) & 0 & 0 \\ \frac{1}{n}\sum_{i=1}^{n}\phi(x_i) & A(x) & 1 \\ \phi(x_j) & x_j & 1 \end{vmatrix}\right. \\
&\qquad \left. -\sum_{i=1}^{n}\frac{n}{x_i - A(x)}\begin{vmatrix} \phi(A(x)) & A(x) & 1 \\ \phi(x_i) & x_i & 1 \\ \frac{1}{n}\sum_{j=1}^{n}\phi(x_j) - \phi(A(x)) & 0 & 0 \end{vmatrix}\right) \\
&= \frac{n}{2}\left\{\sum_{j=1}^{n}\frac{-\left[A(\phi(x)) - \phi(A(x))\right]\left[A(x) - x_j\right]}{x_j - A(x)} - \sum_{i=1}^{n}\frac{\left[A(\phi(x)) - \phi(A(x))\right]\left[A(x) - x_i\right]}{x_i - A(x)}\right\} \\
&= \frac{n}{2}\left\{\sum_{j=1}^{n}\left[A(\phi(x)) - \phi(A(x))\right] + \sum_{i=1}^{n}\left[A(\phi(x)) - \phi(A(x))\right]\right\} \\
&= n^2\left[A(\phi(x)) - \phi(A(x))\right].
\end{aligned}
\tag{2.26}
$$

That is, the identity (2.23) holds.

Setting

$$\phi : [0,\infty[\longrightarrow \mathbb{R}, \quad \phi(t) = t^{\gamma} \tag{2.27}$$

in (2.23), we have that

$$f(x) = \frac{1}{n^2} \sum_{1\le i<j\le n} \left\{ \int\int_{\nabla} \gamma(\gamma-1) \left[t_1 x_i + t_2 x_j + (1-t_1-t_2) A(x) \right]^{\gamma-2} dt_1 dt_2 \right\} (x_i - x_j)^2. \tag{2.28}$$

Since $f \in \overline{P}_{\gamma}[x]$, $f \in P_{\gamma}^{*}[x]$ if and only if $f(\Delta_n y) \in P_{\gamma}^{+}[y]$. Consider the difference substitution $x = \Delta_n y$. From

$$(x_i - x_j)^2 = \left(\sum_{k=i+1}^{j} y_k \right)^2 \in P_2^{+}[y] \Longrightarrow (x_i - x_j)^2 \in P_2^{*}[x] \tag{2.29}$$

for arbitrary $i, j : 1 \le i < j \le n$, it is easy to see that $f \in P_2^{*}[x]$ if $\gamma = 2$. If $\gamma \ge 3$, then

$$\begin{aligned}
&(x_i - x_j)^2 \in P_2^{*}[x], \\
&\left[t_1 x_i + t_2 x_j + (1-t_1-t_2) A(x) \right]^{\gamma-2} \in P_{\gamma-2}^{+}[x] \\
&\quad \Longrightarrow \int\int_{\nabla} \gamma(\gamma-1) \left[t_1 x_i + t_2 x_j + (1-t_1-t_2) A(x) \right]^{\gamma-2} dt_1 dt_2 \in P_{\gamma-2}^{+}[x] \\
&\quad \Longrightarrow \int\int_{\nabla} \gamma(\gamma-1) \left[t_1 x_i + t_2 x_j + (1-t_1-t_2) A(x) \right]^{\gamma-2} dt_1 dt_2 \in P_{\gamma-2}^{*}[x]
\end{aligned} \tag{2.30}$$

for arbitrary $i, j : 1 \le i < j \le n$. Therefore, we get that $f \in P_{\gamma}^{*}[x]$. It follows that the inequality (2.14) holds by using Theorem 2.5. Since $f(I_n) = 0$, the equality holds in (2.14) if there exists $t \in [0,\infty[$, such that $X_1 = X_2 = \cdots = X_m = tI_n$.

The proof of Theorem 2.8 is thus completed. □

Remark 2.9. Theorem 2.8 has significance in the theory of matrices. Let $\mathbb{A} = [a_{i,j}]_{n\times n}$ be an $n \times n$ positive definite Hermitian matrix and $\lambda_1, \ldots, \lambda_n$ its eigenvalues, let $\operatorname{diag}(x)$ be the diagonal matrix with the components of $x = (x_1, x_2, \ldots, x_n)^{\dagger}$ as its diagonal elements, and also let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{\dagger}$. Then $\mathbb{A} = U \operatorname{diag}(\lambda) U^{*}$ for some unitary matrix U (where U^{*} is the conjugate transpose of U and $U^{*}U = \mathbb{E}$, see [9, 13]). If $\gamma \in \mathbb{R}$, then

$$\begin{gathered}
\mathbb{A}^{\gamma} = U \operatorname{diag}\,(\lambda^{\gamma}) U^{*}, \\
\operatorname{tr}\,\mathbb{A} = \sum_{i=1}^{n} a_{i,i} = \sum_{i=1}^{n} \lambda_i, \quad \operatorname{tr}\,\mathbb{A}^{\gamma} = \sum_{i=1}^{n} \lambda_i^{\gamma}.
\end{gathered} \tag{2.31}$$

Write

$$D_\gamma(\mathbb{A}) = \frac{1}{n}\operatorname{tr}\mathbb{A}^\gamma - \left(\frac{1}{n}\operatorname{tr}\mathbb{A}\right)^\gamma = A(\lambda^\gamma) - A^\gamma(\lambda) = f(\lambda), \tag{2.32}$$

then Theorem 2.8 can be rewritten as follows, let $w \in \mathbb{R}^m_{++}, \gamma \in \mathbb{N}$ and $\gamma \geq 2$. If $\mathbb{A}_k$ are $n \times n$ positive definite Hermitian matrix, $\lambda_{\mathbb{A}_k} \in \Omega^n$, $\mathbb{A}_i\mathbb{A}_j = \mathbb{A}_j\mathbb{A}_i$ with $1 \leq i, j, k \leq n$, then

$$\frac{\sum_{k=1}^m w_k D_\gamma(\mathbb{A}_k)}{D_\gamma(\operatorname{diag}(N_n))} \leq \left[\frac{D_\gamma\left(\sum_{k=1}^m w_k \mathbb{A}_k^\gamma\right)}{D_\gamma(\operatorname{diag}(N_n))}\right]^{1/\gamma}. \tag{2.33}$$

In fact, if $\mathbb{A}, \mathbb{B}$ are $n \times n$ positive definite Hermitian matrix and $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$, there exists a unitary matrix U such that (see [13])

$$\mathbb{A} = U\operatorname{diag}(\lambda_{\mathbb{A}})U^*, \qquad \mathbb{B} = U\operatorname{diag}(\lambda_{\mathbb{B}})U^*. \tag{2.34}$$

Thus,

$$D_\gamma(\mathbb{A}+\mathbb{B}) = D_\gamma\left(U\operatorname{diag}(\lambda_{\mathbb{A}}+\lambda_{\mathbb{B}})U^*\right) = D_\gamma\left(\operatorname{diag}(\lambda_{\mathbb{A}}+\lambda_{\mathbb{B}})\right) = f(\lambda_{\mathbb{A}}+\lambda_{\mathbb{B}}). \tag{2.35}$$

From $\mathbb{A}_i\mathbb{A}_j = \mathbb{A}_j\mathbb{A}_i$ with $1 \leq i, j \leq n$, we get that

$$D_\gamma(\mathbb{A}_k) = f(\lambda_{\mathbb{A}_k}), \quad D_\gamma\left(\sum_{k=1}^m w_k \mathbb{A}_k^\gamma\right) = f\left(\sum_{k=1}^m w_k \lambda_{\mathbb{A}_k}^\gamma\right), \quad D_\gamma\left(\operatorname{diag}(N_n)\right) = f(N_n). \tag{2.36}$$

According to Theorem 2.8, the inequality (2.33) holds.

Remark 2.10. Theorem 2.8 has also significance in statistics. By using the same proving method of Theorems 2.1–2.8, we can prove the following: under the hypotheses of the Theorem 2.8, if $f(x) = A(x^\gamma, p) - A^\gamma(x, p)$, then $f \in P^*_\gamma[x]$ and the inequality (2.14) also holds, where

$$p \in \mathbb{R}^n_{++}, \quad A(x,p) = \sum_{i=1}^n p_i x_i, \quad \sum_{i=1}^n p_i = 1. \tag{2.37}$$

Let ξ be a random variable, $x \in \Omega^n$, let $P(\xi = x_i) = p_i$ be the probability of random events $\xi = x_i$ with $1 \leq i \leq n$. If $\gamma = 2$, then

$$D_\gamma[\xi] = \frac{2}{\gamma(\gamma-1)}\{E[\xi^\gamma] - E^\gamma[\xi]\} = \frac{2}{\gamma(\gamma-1)}\left[A(x^\gamma, p) - A^\gamma(x,p)\right] = D_\gamma(x,p) \tag{2.38}$$

is the variance of random variable ξ. The $D_\gamma[\xi]$ is called γth variance of random variable ξ and $D_\gamma[\xi] \geq 0$ for arbitrary $\gamma \in \mathbb{R}$, where

$$
\begin{aligned}
D_0[\xi] &= \lim_{\gamma \to 0} D_\gamma[\xi] = 2\left[\log A(x,p) - A(\log x, p)\right], \\
D_1[\xi] &= \lim_{\gamma \to 1} D_\gamma[\xi] = 2\left[A(x\log x, p) - A(x,p)\log A(x,p)\right].
\end{aligned}
\tag{2.39}
$$

Let ξ_0 be also a random variable, $P(\xi_0 = i) = p_i$ with $1 \leq i \leq n$, and let the function $f_k : [0,\infty[\to [0,\infty[$ be increasing with $1 \leq k \leq m$. Then the inequality (2.14) can be rewritten as follows:

$$
\frac{\sum_{k=1}^m w_k D_\gamma[f_k(\xi)]}{D_\gamma[\xi_0]} \leq \left\{ \frac{D_\gamma\left[\sum_{k=1}^m w_k f_k^\gamma(\xi)\right]}{D_\gamma[\xi_0]} \right\}^{1/\gamma}, \tag{2.40}
$$

where $w \in \mathbb{R}_{++}^m, \gamma \in \mathbb{N}, \gamma \geq 2$.

2.3. Applications of Jensen Type Inequalities

By (1.7) and the same proving method of Theorem 2.1, we can obtain the following result.

Corollary 2.11. *Let $\mathcal{B}_\gamma \subset \mathbb{N}^n$, $f \in \overline{P}_\gamma^+[x]$. If $w \in \mathbb{R}_{++}^m$, $\sum_{k=1}^m w_k = 1$, $X_k \in \mathbb{R}_+^n$ with $1 \leq k \leq m$ and $0 \leq X_1 \leq X_2 \leq \cdots \leq X_m$, then*

$$
\sum_{k=1}^m w_k f(X_k) \geq f\left(\sum_{k=1}^m w_k X_k\right). \tag{2.41}
$$

One gives several integral analogues of (2.2) *and* (2.41) *as follows.*

Corollary 2.12. *Let E be bounded closed region in $\mathbb{R}^s$, and let the functions $w : E \to]0,\infty[$ and $g : E \to \mathbb{R}_+^n$ be continuous, and $\int_E w\, dt = 1$. If $f \in P_\gamma^+[x]$ and $f(I_n) = 1$, then*

$$
\int_E w f(g) dt \leq \left[f\left(\int_E w g^\gamma dt\right)\right]^{1/\gamma}. \tag{2.42}
$$

If $\mathcal{B}_\gamma \subset \mathbb{N}^n$, $f \in \overline{P}_\gamma^+[x]$, and $g(E)$ is an ordered set, that is,

$$
g(t_1) \leq g(t_2) \quad \text{or} \quad g(t_2) \leq g(t_1) \tag{2.43}
$$

for arbitrary $t_1, t_2 : t_1 \in E$ and $t_2 \in E$, then

$$\int_E wf(g)\,dt \geq f\left(\int_E wg\,dt\right). \tag{2.44}$$

As an application of the proof of Theorem 2.8, one has the following.

Corollary 2.13. *Let $w \in \mathbb{R}^m_{++}$ and $\sum_{k=1}^m w_k = 1, p \in \mathbb{R}^n_{++}$ and $\sum_{k=1}^n p_k = 1, \gamma \in (1,2]$. If $X_k \in \mathbb{R}^n_{++}$, with $1 \leq k \leq m$, then one has the following Jensen type inequality:*

$$\sum_{k=1}^m w_k D_\gamma(X_k, p) \geq D_\gamma\left(\sum_{k=1}^m w_k X_k, p\right), \tag{2.45}$$

where $D_\gamma(x,p) = (2/\gamma(\gamma-1))[A(x^\gamma, p) - A^\gamma(x,p)]$.

Proof. We can suppose that $p = (1/n)I_n, \gamma \in (1,2)$, and

$$\omega_{i,j}(x, t_1, t_2) = t_1 x_i + t_2 x_j + (1 - t_1 - t_2)A(x). \tag{2.46}$$

Since $0 < 2 - \gamma < 1$, from Lemma 2.3, we get that

$$\begin{aligned}
&\left[\sum_{k=1}^m w_k \omega_{i,j}(X_k, t_1, t_2)\right]^{\gamma-2} \left(\sum_{k=1}^m w_k |x_{k,i} - x_{k,j}|\right)^2 \\
&\quad\leq \left[\sum_{k=1}^m w_k \omega_{i,j}^{2-\gamma}(X_k, t_1, t_2)\right]^{-1} \left(\sum_{k=1}^m w_k |x_{k,i} - x_{k,j}|\right)^2 \\
&\quad= \left[\sum_{k=1}^m w_k \omega_{i,j}^{2-\gamma}(X_k, t_1, t_2)\right] \left(\frac{\sum_{k=1}^m w_k \omega_{i,j}^{2-\gamma}(X_k, t_1, t_2)\omega_{i,j}^{\gamma-2}(X_k, t_1, t_2)|x_{k,i} - x_{k,j}|}{\sum_{k=1}^m w_k \omega_{i,j}^{2-\gamma}(X_k, t_1, t_2)}\right)^2 \\
&\quad\leq \left[\sum_{k=1}^m w_k \omega_{i,j}^{2-\gamma}(X_k, t_1, t_2)\right] \frac{\sum_{k=1}^m w_k \omega_{i,j}^{2-\gamma}(X_k, t_1, t_2)\omega_{i,j}^{2\gamma-4}(X_k, t_1, t_2)|x_{k,i} - x_{k,j}|^2}{\sum_{k=1}^m w_k \omega_{i,j}^{2-\gamma}(X_k, t_1, t_2)} \\
&\quad= \sum_{k=1}^m w_k \omega_{i,j}^{\gamma-2}(X_k, t_1, t_2)|x_{k,i} - x_{k,j}|^2.
\end{aligned} \tag{2.47}$$

By using (2.28), we find that

$$
\begin{aligned}
&D_\gamma\left(\sum_{k=1}^{m} w_k X_k, p\right)\\
&= \frac{2}{n^2}\sum_{1\le i<j\le n}\left\{\iint_\nabla \left[\omega_{i,j}\left(\sum_{k=1}^{m} w_k X_k, t_1, t_2\right)\right]^{\gamma-2} dt_1 dt_2\right\}\left[\sum_{k=1}^{m} w_k\left(x_{k,i}-x_{k,j}\right)\right]^2\\
&\le \frac{2}{n^2}\sum_{1\le i<j\le n}\left\{\iint_\nabla \left[\omega_{i,j}\left(\sum_{k=1}^{m} w_k X_k, t_1, t_2\right)\right]^{\gamma-2} dt_1 dt_2\right\}\left(\sum_{k=1}^{m} w_k\left|x_{k,i}-x_{k,j}\right|\right)^2\\
&= \frac{2}{n^2}\sum_{1\le i<j\le n}\left\{\iint_\nabla \left[\omega_{i,j}\left(\sum_{k=1}^{m} w_k X_k, t_1, t_2\right)\right]^{\gamma-2} \left(\sum_{k}^{m} w_k\left|x_{k,i}-x_{k,j}\right|\right)^2 dt_1 dt_2\right\}\\
&= \frac{2}{n^2}\sum_{1\le i<j\le n}\left\{\iint_\nabla \left[\sum_{k=1}^{m} w_k \omega_{i,j}\left(X_k, t_1, t_2\right)\right]^{\gamma-2} \left(\sum_{k=1}^{m} w_k\left|x_{k,i}-x_{k,j}\right|\right)^2 dt_1 dt_2\right\}\\
&\le \frac{2}{n^2}\sum_{1\le i<j\le n}\left\{\iint_\nabla \sum_{k=1}^{m} w_k \omega_{i,j}^{\gamma-2}\left(X_k, t_1, t_2\right)\left|x_{k,i}-x_{k,j}\right|^2 dt_1 dt_2\right\}\\
&= \sum_{k=1}^{m} w_k\left\{\frac{2}{n^2}\sum_{1\le i<j\le n}\iint_\nabla \omega_{i,j}^{\gamma-2}\left(X_k, t_1, t_2\right) dt_1 dt_2 \left(x_{k,i}-x_{k,j}\right)^2\right\}\\
&= \sum_{k=1}^{m} w_k D_\gamma\left(X_k, p\right).
\end{aligned}
\tag{2.48}
$$

The proof of Corollary 2.13 is thus completed. □

Corollary 2.14. *If* $X_k \in \Omega^n$ *with* $1 \le k \le m$, *then*

$$
\sum_{k=1}^{m} \sqrt[n(n-1)/2]{\det\left[\left(X_k\right)_j^{i-1}\right]_{n\times n}} \le \sqrt[n(n-1)/2]{\det\left[\left(\sum_{k=1}^{m} X_k\right)_j^{i-1}\right]_{n\times n}}.
\tag{2.49}
$$

Proof. The nth Vandermonde determinant is wellknown (see [14]):

$$
\det\left[x_j^{i-1}\right]_{n\times n} = \prod_{1\le i<j\le n}\left(x_j - x_i\right).
\tag{2.50}
$$

By Theorem 2.1, for arbitrary $x_{k,i,j} \in [0,\infty[$ with $1 \le i < j \le n$, $1 \le k \le m$, we get that

$$
\sum_{k=1}^{m} \prod_{1\le i<j\le n} x_{k,i,j} \le \sqrt[n(n-1)/2]{\prod_{1\le i<j\le n} \sum_{k=1}^{m} x_{k,i,j}^{n(n-1)/2}}.
\tag{2.51}
$$

Letting

$$w = I_m, \quad x_{k,i,j} = \sqrt[n(n-1)/2]{x_{k,j} - x_{k,i}}, \quad 1 \le i < j \le n, \ 1 \le k \le m \tag{2.52}$$

in inequality (2.51), it implies that the inequality (2.49) holds. The proof is completed. □

Example 2.15. Given N-inscribed-polygon $\Gamma_k = \Gamma_k(A_{k,1}, A_{k,2}, \dots, A_{k,N})$ with $1 \le k \le m$. Defining the summation of them is an N-inscribed-polygon $\Gamma = \sum_{k=1}^{m} \Gamma_k = \Gamma(A_1, A_2, \dots, A_N)$, and its sides lengths are given by $|A_i A_{i+1}| = \sum_{k=1}^{m} |A_{k,i} A_{k,i+1}|$ with $1 \le i \le N$. Also defining $A_i = A_j \Leftrightarrow i \equiv j (\operatorname{mod} N)$, and $A_{k,i} = A_{k,j} \Leftrightarrow i \equiv j (\operatorname{mod} N)$ with $1 \le k \le m$.

Wen and Zhang in [15] raised a conjecture: prove that

$$\sqrt{\left|\sum_{k=1}^{m} \Gamma_k\right|} \ge \sum_{k=1}^{m} \sqrt{|\Gamma_k|}, \tag{2.53}$$

where $|\Gamma| = \operatorname{Area} \Gamma$ is the area of the N-inscribed-polygon Γ.

Now, we prove that the inequality (2.53) holds for $N = 3, 4$ by using Theorem 2.1.

Denote

$$\begin{gathered} a_{k,i} = |A_{k,i} A_{k,i+1}|, \quad p_k = \frac{1}{2} \sum_{i=1}^{N} a_{k,i}, \quad a_i = |A_i A_{i+1}| = \sum_{k=1}^{m} a_{k,i}, \\ p = \frac{1}{2} \sum_{i=1}^{N} a_i = \sum_{k=1}^{m} p_k, \quad 1 \le i \le N, \ 1 \le k \le m. \end{gathered} \tag{2.54}$$

If $N = 3$, we have that

$$\begin{gathered} \sqrt{|\Gamma_k|} = \sqrt[4]{p_k \prod_{i=1}^{3} (p_k - a_{k,i})}, \\ \sqrt{\left|\sum_{k=1}^{m} \Gamma_k\right|} = \sqrt[4]{\left(\sum_{k=1}^{m} p_k\right) \prod_{i=1}^{3} \left[\sum_{k=1}^{m} (p_k - a_{k,i})\right]}. \end{gathered} \tag{2.55}$$

Setting

$$\begin{gathered} f \in P_4[x], \quad f(x) = \prod_{i=1}^{4} x_i, \quad n = 4, \\ w = I_m, \quad x_{k,1} = \sqrt[4]{p_k}, \quad x_{k,i} = \sqrt[4]{p_k - a_{k,i}}, \quad 2 \le i \le 4, \ 1 \le k \le m \end{gathered} \tag{2.56}$$

in Theorem 2.1, then inequality (2.2) is just (2.53).

For $N = 4$, we get that

$$\sqrt{|\Gamma_k|} = \sqrt[4]{\prod_{i=1}^{4}(p_k - a_{k,i})},$$
$$\sqrt{\left|\sum_{k=1}^{m}\Gamma_k\right|} = \sqrt[4]{\prod_{i=1}^{4}\left[\sum_{k=1}^{m}(p_k - a_{k,i})\right]}. \tag{2.57}$$

Taking

$$f \in P_4[x], \quad f(x) = \prod_{i=1}^{4} x_i, \quad n = 4,$$
$$w = I_m, \quad x_{k,i} = \sqrt[4]{p_k - a_{k,i}}, \quad 1 \le i \le 4,\ 1 \le k \le m \tag{2.58}$$

in Theorem 2.1, it is clear to see that inequality (2.2) deduces to (2.53).

Remark 2.16. The following result was obtained in [15]. Let Γ_k with $1 \le k \le m$ and $\sum_{k=1}^{m}\Gamma_k$ all be N-inscribed-polygons. If $p_1 - a_{1,i} \le p_2 - a_{2,i} \le \cdots \le p_m - a_{m,i}$, $i = 1, 2, \ldots, N$, then for $N = 3, 4$, we have

$$\left|\sum_{k=1}^{m}\Gamma_k\right|^2 \le m^3 \sum_{k=1}^{m}|\Gamma_k|^2. \tag{2.59}$$

This inequality can also be deduced from inequality (1.7).

3. Jensen Type Inequalities Involving Homogeneous Symmetric Polynomials

In this section, we will also use the following notation (see [4, 16]):

$$e^x = (e^{x_1}, e^{x_2}, \ldots, e^{x_n})^{\dagger}, \qquad \Omega_*^n = \{x \in \mathbb{R}^n \mid x_1 \le x_2 \le \cdots \le x_n\},$$
$$\alpha^{(l)} = \left(\alpha_1^{(l)}, \alpha_2^{(l)}, \ldots, \alpha_n^{(l)}\right)^{\dagger} \in \mathcal{B}_\gamma, \qquad p = \max_{1\le l\le N}\left\{\alpha_1^{(l)}, \alpha_2^{(l)}, \ldots, \alpha_n^{(l)}\right\},$$
$$\prod_{k=1}^{m} X_k = \left(\prod_{k=1}^{m} x_{k,1}, \prod_{k=1}^{m} x_{k,2}, \ldots, \prod_{k=1}^{m} x_{k,n}\right)^{\dagger},$$
$$\frac{X_1}{X_2} = \left(\frac{x_{1,1}}{x_{2,1}}, \frac{x_{1,2}}{x_{2,2}}, \ldots, \frac{x_{1,n}}{x_{2,n}}\right)^{\dagger}. \tag{3.1}$$

Definition 3.1. (see [17, 18]). $\mathcal{B}_\gamma$ is called the control ordered set if

$$\alpha \prec \beta \quad \text{or} \quad \beta \prec \alpha \tag{3.2}$$

for arbitrary $\alpha, \beta : \alpha \in \mathcal{B}_\gamma$ and $\beta \in \mathcal{B}_\gamma$.

The well-known Chebyshev inequality states: let $a, b \in \mathbb{R}^m$, $w \in \mathbb{R}^m_{++,}$ and $\sum_{k=1}^m w_k = 1$. If $a_1 \le a_2 \le \cdots \le a_m$ and $b_1 \le b_2 \le \cdots \le b_m$, then

$$\sum_{k=1}^m w_k a_k b_k \ge \left(\sum_{k=1}^m w_k a_k\right) \times \left(\sum_{k=1}^m w_k b_k\right). \tag{3.3}$$

The inequality is reversed for $b_1 \ge b_2 \ge \cdots \ge b_m$.

We remark here that Wen and Wang generalized the inequality (3.3) in [4]: if $X_1, X_2 \in \Omega^n$, and $\alpha \in \mathbb{R}^n_+$, then we have the following Chebyshev type inequality:

$$\frac{\operatorname{per}\left[(X_1X_2)_j^{\alpha_i}\right]}{n!} \ge \frac{\operatorname{per}\left[(X_1)_j^{\alpha_i}\right]}{n!} \times \frac{\operatorname{per}\left[(X_2)_j^{\alpha_i}\right]}{n!}. \tag{3.4}$$

3.1. Jensen Type Inequalities Involving Homogeneous Symmetric Polynomials

In this subsection, we first present a Jensen type inequality involving homogeneous symmetric polynomials as follows.

Theorem 3.2. *Let $f \in \overline{P}_\gamma^+[x], f(I_n) = 1, w \in \mathbb{N}^m$, let $\mathcal{B}_\gamma$ be a control ordered set. If $X_k \in \Omega_*^n$ with $1 \le k \le m$, then*

$$\sum_{k=1}^m w_k f_*(X_k) \le f_*\left(\sum_{k=1}^m w_k X_k\right), \tag{3.5}$$

where $f_* : \mathbb{R}^n \to \mathbb{R}, f_*(x) = \log f(e^x)$.

Proof. By using the same proving method of Theorem 2.1, we can suppose that $w = I_m$. If $m = 1$, then inequality (3.5) holds. So we just need to prove the following.

Let $f \in \overline{P}_\gamma^+[x], f(I_n) = 1$ be $\mathcal{B}_\gamma$ is a control ordered set. If $X_k \in \Omega^n$ with $1 \le k \le m$ and $m \ge 2$, then

$$f\left(\prod_{k=1}^m X_k\right) \ge \prod_{k=1}^m f(X_k). \tag{3.6}$$

We will verify inequality (3.6) by induction.

For $m = 2$, we find from the inequality (3.4) that

$$f(X_1X_2) = \sum_{\alpha\in\mathcal{B}_\gamma} \frac{\lambda(\alpha)}{n!} \operatorname{per}\left[(X_1X_2)_j^{\alpha_i}\right]_{n\times n} \ge \sum_{\alpha\in\mathcal{B}_\gamma} \lambda(\alpha) \frac{\operatorname{per}\left[(X_1)_j^{\alpha_i}\right]}{n!} \times \frac{\operatorname{per}\left[(X_2)_j^{\alpha_i}\right]}{n!}. \tag{3.7}$$

Since the control ordered set $\mathcal{B}_\gamma$ is nonempty and finite set by using Definition 3.1, we can suppose that

$$\mathcal{B}_\gamma = \left\{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}\right\}, \quad \alpha^{(1)} \prec \alpha^{(2)} \prec \cdots \prec \alpha^{(N)}. \tag{3.8}$$

From Hardy's inequality (see [17, page 74]), we have that

$$\begin{gathered}\frac{\operatorname{per}\left[(X_1)_j^{\alpha_i^{(1)}}\right]}{n!} \le \frac{\operatorname{per}\left[(X_1)_j^{\alpha_i^{(2)}}\right]}{n!} \le \cdots \le \frac{\operatorname{per}\left[(X_1)_j^{\alpha_i^{(N)}}\right]}{n!}, \\ \frac{\operatorname{per}\left[(X_2)_j^{\alpha_i^{(1)}}\right]}{n!} \le \frac{\operatorname{per}\left[(X_2)_j^{\alpha_i^{(2)}}\right]}{n!} \le \cdots \le \frac{\operatorname{per}\left[(X_2)_j^{\alpha_i^{(N)}}\right]}{n!}.\end{gathered} \tag{3.9}$$

By means of $f(I_n) = \sum_{l=1}^{N} \lambda(\alpha^{(l)}) = 1$, inequality (3.7), and Chebyshev's inequality (3.3), it is easy to obtain that

$$\begin{aligned} f(X_1X_2) &\ge \sum_{l=1}^{N} \lambda\left(\alpha^{(l)}\right) \frac{\operatorname{per}\left[(X_1)_j^{\alpha_i^{(l)}}\right]}{n!} \times \frac{\operatorname{per}\left[(X_2)_j^{\alpha_i^{(l)}}\right]}{n!} \\ &\ge \left(\sum_{l=1}^{N} \lambda\left(\alpha^{(l)}\right) \frac{\operatorname{per}\left[(X_1)_j^{\alpha_i^{(l)}}\right]}{n!}\right) \times \left(\sum_{l=1}^{N} \lambda\left(\alpha^{(l)}\right) \frac{\operatorname{per}\left[(X_2)_j^{\alpha_i^{(l)}}\right]}{n!}\right) \\ &= f(X_1) f(X_2), \end{aligned} \tag{3.10}$$

which implies that the inequality (3.6) holds for $m = 2$.

Assume that the inequality (3.6) is true for $m = q \ge 2$, that is,

$$f\left(\prod_{k=1}^{q} X_k\right) \ge \prod_{k=1}^{q} f(X_k). \tag{3.11}$$

For $m = q+1$, from $X_{q+1} \in \Omega^n$ and $X_1, X_2, \ldots, X_q \in \Omega^n$, we have $\prod_{k=1}^{q} X_k \in \Omega^n$. Thus,

$$\begin{aligned} f\left(\prod_{k=1}^{q+1} X_k\right) &= f\left(X_{q+1}\prod_{k=1}^{q} X_k\right) \ge f(X_{q+1}) f\left(\prod_{k=1}^{q} X_k\right) \\ &\ge f(X_{q+1})\left(\prod_{k=1}^{q} f(X_k)\right) = \prod_{k=1}^{q+1} f(X_k). \end{aligned} \tag{3.12}$$

The inequality (3.6) is proved by induction. The proof of Theorem 3.2 is hence completed.

As an application of the inequality (3.6), we have the following. □

Theorem 3.3. *Let $f \in \overline{P}_{\gamma}^{+}[x]$, let $\mathcal{B}_{\gamma}$ be a control ordered set, that is,*

$$\mathcal{B}_{\gamma} = \left\{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}\right\}, \quad \alpha^{(1)} \prec \alpha^{(2)} \prec \cdots \prec \alpha^{(N)}. \tag{3.13}$$

If $X_1/X_2 \in \Omega^n$, $X_2 \in \Omega^n$, and $X_2 \in \mathbb{R}_{++}^n$, then

$$\frac{1}{n!}\operatorname{per}\left[\left(\frac{X_1}{X_2}\right)_j^{\alpha_i^{(1)}}\right] \le \frac{f(X_1)}{f(X_2)} \le \left(\frac{\sum_{i=1}^n x_{1,i}^p}{\sum_{i=1}^n x_{2,i}^p}\right)^{\gamma/p}. \tag{3.14}$$

Proof. The right-hand inequality of (3.14) is proved in [4]. Now, we will give the demonstration of the left-hand inequality in (3.14).

We can suppose that $f(I_n) = 1$. By means of $X_1/X_2 \in \Omega^n$, $X_2 \in \Omega^n$, and $X_2 \in \mathbb{R}_{++}^n$, we find from the inequality (3.6) that

$$f\left(\frac{X_1}{X_2}X_2\right) \ge f\left(\frac{X_1}{X_2}\right)f(X_2) \Longleftrightarrow \frac{f(X_1)}{f(X_2)} \ge f\left(\frac{X_1}{X_2}\right). \tag{3.15}$$

From

$$\mathcal{B}_{\gamma} = \left\{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}\right\}, \quad \alpha^{(1)} \prec \alpha^{(2)} \prec \cdots \prec \alpha^{(N)} \tag{3.16}$$

and Hardy's inequality (see [17, page 74]), we obtain that

$$\operatorname{per}\left[\left(\frac{X_1}{X_2}\right)_j^{\alpha_i}\right] \ge \operatorname{per}\left[\left(\frac{X_1}{X_2}\right)_j^{\alpha_i^{(1)}}\right], \quad \forall \alpha \in \mathcal{B}_{\gamma}. \tag{3.17}$$

Therefore, we deduce that

$$\begin{aligned}
\frac{f(X_1)}{f(X_2)} \ge f\left(\frac{X_1}{X_2}\right) &= \sum_{\alpha \in \mathcal{B}_{\gamma}} \frac{\lambda(\alpha)}{n!}\operatorname{per}\left[\left(\frac{X_1}{X_2}\right)_j^{\alpha_i}\right]_{n\times n} \\
&\ge \sum_{\alpha \in \mathcal{B}_{\gamma}} \frac{\lambda(\alpha)}{n!}\operatorname{per}\left[\left(\frac{X_1}{X_2}\right)_j^{\alpha_i^{(1)}}\right]_{n\times n} \\
&= \frac{1}{n!}\operatorname{per}\left[\left(\frac{X_1}{X_2}\right)_j^{\alpha_i^{(1)}}\right]_{n\times n} f(I_n) \\
&= \frac{1}{n!}\operatorname{per}\left[\left(\frac{X_1}{X_2}\right)_j^{\alpha_i^{(1)}}\right]_{n\times n}.
\end{aligned} \tag{3.18}$$

The proof of Theorem 3.3 is thus completed. □

3.2. Remarks

Remark 3.4. If $\gamma \in]0, \infty[$, then Theorems 3.2 and 3.3 are also true.

Remark 3.5. If $\mathcal{B}_\gamma \subset \mathbb{N}^n$ and $1 \le \gamma \le 5$, then $\mathcal{B}_\gamma$ is a control ordered set.
In fact,

$$\begin{gathered}
(1,0,\ldots,0)^\dagger \prec (1,0,\ldots,0)^\dagger; \\
(1,1,0,\ldots,0)^\dagger \prec (2,0,\ldots,0)^\dagger; \\
(1,1,1,0,\ldots,0)^\dagger \prec (2,1,0,\ldots,0)^\dagger \prec (3,0,\ldots,0)^\dagger; \\
(1,1,1,1,0,\ldots,0)^\dagger \prec (2,1,1,0,\ldots,0)^\dagger \prec (2,2,0,\ldots,0)^\dagger \prec (3,1,0,\ldots,0)^\dagger \prec (4,0,\ldots,0)^\dagger; \\
(1,1,1,1,1,0,\ldots,0)^\dagger \prec (2,1,1,1,0,\ldots,0)^\dagger \prec (2,2,1,0,\ldots,0)^\dagger \prec (3,1,1,0,\ldots,0)^\dagger \\
\prec (3,2,0,\ldots,0)^\dagger \prec (4,1,0,\ldots,0)^\dagger \prec (5,0,\ldots,0)^\dagger.
\end{gathered} \tag{3.19}$$

Remark 3.6. By using the proof of Theorem 3.2 and Remark 3.5, we know the following: if $X_k \in \Omega^n$ with $1 \le k \le m, m \ge 2$ and

$$f(x) = \frac{1}{n^\gamma - n}\left[\left(\sum_{k=1}^n x_k\right)^\gamma - \sum_{k=1}^n x_k^\gamma\right], \quad \gamma \in \mathbb{N},\ 1 < \gamma \le 5, \tag{3.20}$$

then the inequality (3.6) holds.

Remark 3.7. The inequality (3.6) is also a Chebyshev type inequality involving homogeneous symmetric polynomials.

3.3. An Open Problem

According to Theorem 3.3, we pose the following open problem.

Conjecture 3.8. *Under the hypotheses of Theorem 3.3, one has*

$$\frac{1}{n!}\operatorname{per}\left[\left(\frac{X_1}{X_2}\right)_j^{\alpha_i^{(1)}}\right] \le \frac{\operatorname{per}\left[(X_1)_j^{\alpha_i^{(1)}}\right]}{\operatorname{per}\left[(X_2)_j^{\alpha_i^{(1)}}\right]} \le \frac{f(X_1)}{f(X_2)} \le \left(\frac{\sum_{i=1}^n x_{1,i}^p}{\sum_{i=1}^n x_{2,i}^p}\right)^{\gamma/p}. \tag{3.21}$$

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