Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 837951, 14 pages
doi:10.1155/2010/837951

Research Article

On Boundedness of Weighted Hardy Operator in $L^{p(\cdot)}$ and Regularity Condition

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Received 22 September 2010; Accepted 26 November 2010

Academic Editor: P. J. Y. Wong

We give a new proof for power-type weighted Hardy inequality in the norms of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Assuming the logarithmic conditions of regularity in a neighborhood of zero and at infinity for the exponents $p(x) \le q(x), \beta(x)$, necessary and sufficient conditions are proved for the boundedness of the Hardy operator $Hf(x) = \int_{|y|\le|x|} f(y)dy$ from $L^{p(\cdot)}_{|x|^{\beta(\cdot)}}(\mathbb{R}^n)$ into $L^{q(\cdot)}_{|x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)}}(\mathbb{R}^N)$. Also a separate statement on the exactness of logarithmic conditions at zero and at infinity is given. This shows that logarithmic regularity conditions for the functions β, p at the origin and infinity are essentially one.

1. Introduction

The object of this investigation is the Hardy-type weighted inequality

$$\left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} Hf \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C \left\| |x|^{\beta(\cdot)} f \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \quad Hf(x) = \int_{|y|\le|x|} f(y)dy \tag{1.1}$$

in the norms of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. This subject was investigated in the papers [1–7]. For the one-dimensional Hardy operator in [1], the necessary and sufficient condition was obtained for the exponents β, p, q. We give a new proof for this result in more general settings for the multidimensional Hardy operator. Also we prove that the logarithmic regularity conditions are essential one for such kind of inequalities to hold. In that proposal, we improve a result sort of [8] (since, there is an estimation by the maximal function $|x|^{-n} Hf(x) \le CMf(x)$).

At the beginning, a one-dimensional Hardy inequality was considered assuming the the local log condition at the finite interval $[0, l]$. Subsequently, the logarithmic condition was assumed in an arbitrarily small neighborhood of zero, where an additional restriction $p(x) \geq p(0)$ was imposed on the exponent. In [3, 9] it was shown that it is sufficient to assume the logarithmic condition only at the zero point. In [10] the case of an entire semiaxis was considered without using the condition $p(x) \geq p(0)$. However, a more rigid condition $\beta^+ < 1 - 1/p^-$ was introduced for a range of exponents. The exact condition was found in [1]. They proved this result by using of interpolation approaches. In this paper, we use other approaches, analogous to those in [10], based on the property of triangles for $p(x)$-norms and binary decomposition near the origin and infinity. We consider the multidimensional case, and the condition $\beta(x) = \text{const}$ is not obligatory, while the necessary and sufficient condition is obtained by a set of exponents p, q, β without imposing any preliminary restrictions on their values (Theorems 3.1 and 3.2). In Theorem 3.3, it has been proved that logarithmic conditions at zero and at infinity are exact for the Hardy inequality to be valid in the case $q = p$.

Problems of the boundedness of classical integral operators such as maximal and singular operators, the Riesz potential, and others in Lebesgue spaces with variable exponent, as well as the investigation of problems of regularity of nonlinear equations with nonstandard growth condition have become of late the arena of an intensive attack of many authors (see [11–18]).

2. Lebesgue Spaces with a Variable Exponent

As to the basic properties of spaces $L^{p(\cdot)}$, we refer to [19]. Throughout this paper, it is assumed that $p(x)$ is a measurable function in Ω, where $\Omega \in \mathbb{R}^n$ is an open domain, taking its values from the interval $[1, \infty)$ with $p^+ = \sup_{x \in \mathbb{R}^n} p < \infty$. The space of functions $L^{p(\cdot)}(\Omega)$ is introduced as the class of measurable functions $f(x)$ in Ω, which have a finite $I_p(f) := \int_\Omega |f(x)|^{p(x)} dx$-modular. A norm in $L^{p(\cdot)}(\Omega)$ is given in the form

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \leq 1\right\}. \tag{2.1}$$

For $p^- > 1$, $p^+ < \infty$ the space $L^{p(\cdot)}(\Omega)$ is a reflexive Banach space.

Denote by Λ a class of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions:

$$\exists m \in \left(0, \frac{1}{2}\right), \quad \exists f(0) \in \mathbb{R}, \qquad \sup_{x \in B(0,m)} \big|f(x) - f(0)\big| \ln \frac{1}{|x|} < \infty, \tag{2.2}$$

$$\exists M > 1, \quad \exists f(\infty) \in \mathbb{R}, \qquad \sup_{x \in R^n \setminus B(0,M)} \big|f(x) - f(\infty)\big| \ln|x| < \infty. \tag{2.3}$$

For the exponential functions $\beta(x), p(x)$, and $q(x)$, we further assume $\beta, p, q \in \Lambda$.

We will many times use the following statement in the proof of main results.

Lemma 2.1. *Let $s \in \Lambda$ be a measurable function such that $-\infty < s^-, s^+ < \infty$. Then the condition* (2.2) *for the function $s(x)$ is equivalent to the estimate*

$$C_3^{-1}|x|^{s(0)} \le |x|^{s(x)} \le C_3|x|^{s(0)} \tag{2.4}$$

when $|x| \le m$ and the condition (2.3) *for $s(x)$ is equivalent to the estimate*

$$C_4^{-1}|x|^{s(\infty)} \le |x|^{s(x)} \le C_4|x|^{s(\infty)} \tag{2.5}$$

when $|x| \ge M$. Where the constants $C_3, C_4 > 1$ depend on $s(0)$, $s(\infty)$, s^-, s^+, $s(0)$, $s(\infty)$, m, M, C_1, C_2.

To prove Lemma 2.1, for example (2.4), it suffices to rewrite the inequality (2.4) in the form

$$C_3^{-1} \le |x|^{s(x)-s(0)} \le C_3 \tag{2.6}$$

and pass to logarithmic in this inequality (see also, [1, 7, 17]).

For $1 < p < \infty$, p' denotes the conjugate number of p, $p' = p/(p-1)$. It is further assumed that $p' = \infty$ for $p = 1$, and $p' = 1$ for $p = \infty$, $1/\infty = 0$, $1/0 = \infty$. We denote by C, C_1, C_2 various positive constants whose values may vary at each appearance. $B(x,r)$ denotes a ball with center at x and radius $r > 0$. We write $u \sim v$ if there exist positive constants C_3, C_4 such that $C_3u(x) \le v(x) \le C_4u(x)$. By χ_E, we denote the characteristic function of the set E.

3. The Main Results

The main results of the paper are contained in the next statements. The theorem below gives a solution of the two-weighted problem for the multidimensional Hardy operator in the case of power-type weights.

Theorem 3.1. *Let $q(x) \ge p(x)$ and $\beta(x)$ be measurable functions taken from the class Λ. Let the following conditions be fulfilled:*

$$0 < p^- \le p(x), \qquad q(x) \le q^+ < \infty, \qquad -\infty < \beta^- \le \beta(x) \le \beta^+ < \infty. \tag{3.1}$$

Then the inequality (1.1) *for any positive measurable function f is fulfilled if and only if*

$$p(0) > 1, \quad p(\infty) > 1, \quad \beta(0) < n\left(1 - \frac{1}{p(0)}\right), \quad \beta(\infty) < n\left(1 - \frac{1}{p(\infty)}\right). \tag{3.2}$$

We have the following analogous result for the conjugate Hardy operator $\overline{H}f(x) = \int_{|y|\ge|x|} f(y)dy$.

Theorem 3.2. *Let $q(x) \geq p(x)$ and $\beta(x)$ be measurable functions taken from the class Λ. Let the conditions* (3.1) *be fulfilled. Then the inequality* (1.1) *for any positive measurable function f and operator $\overline{H}f$ is fulfilled if and only if*

$$p(0) > 1, \quad p(\infty) > 1, \quad \beta(0) > n\left(1 - \frac{1}{p(0)}\right), \quad \beta(\infty) > n\left(1 - \frac{1}{p(\infty)}\right). \tag{3.3}$$

In the next theorem, we prove that the logarithmic conditions near zero and at infinity are essentially one.

Theorem 3.3. *If condition* (2.2) *or* (2.3) *does not hold, then there exists an example of functions p, β, and a sequence f below index k violating the inequality*

$$\left\| |x|^{\beta(\cdot)-n} Hf \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| |x|^{\beta(\cdot)} f \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{3.4}$$

4. Proofs of the Main Results

Proof of Theorem 3.1.

Sufficiency. Let $f(x) \geq 0$ be a measurable function such that

$$\left\| |x|^{\beta(\cdot)} f \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1. \tag{4.1}$$

We will prove that

$$\left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} Hf \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C_5. \tag{4.2}$$

Assume that $0 < \delta < m$ is a sufficiently small number such that $n/p'(x) > n/p'(0) - \varepsilon$ for all $x \in B(0, \delta)$, where $\varepsilon = (n/p'(0) - \beta(0))/2$. Let, furthermore, $M < N < \infty$ be a sufficiently large number such that $n/p'(x) > n/p'(\infty) - \delta_1$ for all $x \in \mathbb{R}^n \setminus B(0, N)$, where $\delta_1 = (n/p'(\infty) - \beta(\infty))/2$.

By Minkowski inequality, for $p(x)$-norms, we have

$$\begin{aligned}
\left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} Hf \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq{}& \left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} Hf \right\|_{L^{q(\cdot)}(B(0,\delta))} \\
&+ \left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} Hf \right\|_{L^{q(\cdot)}(B(0,N)\setminus B(0,\delta))} \\
&+ \left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} \int_{\{t:|t|<N\}} f(t)dt \right\|_{L^{q(\cdot)}(\mathbb{R}^n \setminus B(0,N))} \\
&+ \left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} \int_{\{t:N<|t|<|x|\}} f(t)dt \right\|_{L^{q(\cdot)}(\mathbb{R}^N \setminus B(0,N))} \\
:={}& i_1 + i_2 + i_3 + i_4.
\end{aligned} \tag{4.3}$$

The estimate near zero (i_1).

By Minkowski inequality, we have the inequalities

$$\begin{aligned} i_1 &\le \left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} \sum_{k=0}^{\infty} \int\int_{\{t:2^{-k-1}|x|<|t|<2^{-k}|x|\}} f(t)dt \right\|_{L^{q(\cdot)}(B(0,\delta)} \\ &\le \sum_{k=0}^{\infty} \left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} \int_{\{t:2^{-k-1}|x|<|t|<2^{-k}|x|\}} f(t)dt \right\|_{L^{q(\cdot)}(B(0,\delta))}. \end{aligned} \tag{4.4}$$

Denote $B_{x,k} = \{y \in \mathbb{R}^n : 2^{-k-1}|x| < |y| < 2^{-k}|x|\}$ and $p^-_{x,k} = \min(p(x), \inf_{y\in B_{x,k}} p(y))$. By (2.2) and Lemma 2.1, for $x \in B(0,\delta)$, $t \in B_{x,k}$, we have $|x|^{\beta(x)} \sim 2^{k\beta(0)} t^{\beta(t)}$. To prove this equivalence, we use that $|t| \sim |x|2^{-k}$, $|x|^{\beta(x)} \sim |x|^{\beta(0)}$ and $|t|^{\beta(t)} \sim |t|^{\beta(0)}$. Therefore, and due to Holder's inequality, for $x \in B(0,\delta)$, we get

$$\begin{aligned} &|x|^{\beta(x)-n/p'(x)-n/q(x)} \int_{B_{x,k}} f(t)dt \\ &\le C_6 2^{k\beta(0)} |x|^{-n/p'(x)-n/q(x)} \int_{B_{x,k}} |t|^{\beta(t)} f(t)dt \\ &\le C_6 2^{k\beta(0)} |x|^{-n/p'(0)-n/q(x)} \left(\int_{B_{x,k}} \left(|t|^{\beta(t)} f(t)\right)^{p^-_{x,k}} dt \right)^{1/p^-_{x,k}} \left(2^{-k}|x|\right)^{n/(p^-_{x,k})'}. \end{aligned} \tag{4.5}$$

(a) If $p^-_{x,k} \ne p(x)$, then by (2.2) and Lemma 2.1,

$$\left(2^{-k}|x|\right)^{n/(p^-_{x,k})'} \sim t^{n/p'(t)} \sim t^{n/p'(0)} \sim 2^{-kn/p'(0)}|x|^{n/p'(0)} \sim 2^{-kn/p'(0)}|x|^{n/p'(x)}. \tag{4.6}$$

Demonstrate details in proof of (4.6). For $t \in B_{x,k}$ and $x \in B(0,\delta)$, we have $2^{-k-1}|x| < |t| \le 2^{-k}|x|$. Then

$$\left(2^{-k}|x|\right)^{n/(p^-_{x,k})'} \sim |t|^{n/(p^-_{x,k})'}. \tag{4.7}$$

By hypothesis (a), $p^-_{x,k}$ attains in the interval $B_{x,k}$, because there exists a point $y \in B_{x,k}$ where $p^-_{x,k} \sim p(y)$. Obviously, the point y depends on x, k. Then $|t|^{n/(p^-_{x,k})'} \sim |t|^{n/p'(y)}$. By virtue of $2^{-k-1}|x| < |y| \le 2^{-k-1}|x|$, we have $|t|/2 < |y| \le 2|t|$. Hence, $|t|^{n/p'(y)} \sim |y|^{n/p'(y)}$, by Lemma 2.1, $|y|^{n/p'(y)} \sim |y|^{n/p'(0)} \sim |t|^{n/p'(0)}$.

(b) If $p^-_{x,k} = p(x)$, then by choice of δ,

$$\left(2^{-k}|x|\right)^{n/(p^-_{x,k})'} \sim 2^{-kn/p'(x)}|x|^{n/p'(x)} \le 2^{-kn/p'(0)+\varepsilon k}|x|^{n/p'(x)}; \quad x \in B(0,\delta). \tag{4.8}$$

Applying estimate (4.8) to both hypotheses (a) and (b), by choosing of ε and δ, the right-hand part of (4.5) is less than

$$C_7|x|^{-n/q(x)}2^{-k\varepsilon}\left(\int_{B_{x,k}}\left(|t|^{\beta(t)}f(t)\right)^{p^-_{x,k}}dt\right)^{1/p^-_{x,k}}. \tag{4.9}$$

Simultaneously,

$$\begin{aligned}&\int_{B_{x,k}}\left(|t|^{\beta(t)}f(t)\right)^{p^-_{x,k}}dt\\&\quad\le\int_{B_{x,k}\cap\{t\in\mathbb{R}^n:|t|^{\beta(t)}f(t)\ge1\}}\left(|t|^{\beta(t)}f(t)\right)^{p(t)}dt+\int_{B_{x,k}}dt\le1+2^{-kn}\delta^n=C_8.\end{aligned} \tag{4.9$'$}$$

By (4.5) and (4.9′), we have

$$\begin{aligned}&I_{q;B(0,\delta)}\left(|x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)}\int_{B_{x,k}}f(t)dt\right)\\&\quad\le C_9 2^{-k\varepsilon q^-}\int_{B(0,\delta)}|x|^{-n}\left(\int_{B_{x,k}}\left(|t|^{\beta(t)}f(t)\right)^{p^-_{x,k}}dt\right)^{q(x)/p^-_{x,k}}dx\\&\quad\le C_9C_8^{q^+/p^--1}2^{-k\varepsilon q^-}\int_{B(0,\delta)}\left(\int_{B_{x,k}}\left(\left(|t|^{\beta(t)}f(t)\right)^{p(t)}+1\right)dt\right)|x|^{-n}dx\end{aligned} \tag{4.10}$$

which, due to Fubini's theorem, yields

$$\begin{aligned}&\le C_9C_8^{q^+/p^--1}2^{-k\varepsilon q^-}\int_{\{t:|t|<2^{-k}\delta\}}\left(f(t)|t|^{\beta(t)}\right)^{p(t)}\left(\int_{B\left(0,2^{k+1}|t|\right)\setminus B\left(0,2^k|t|\right)}|x|^{-n}dx\right)dt\\&=C_{10}2^{-k\varepsilon q^-}\ln 2\int_{\{t:|t|<2^{-k}\delta\}}\left(\left(f(t)|t|^{\beta(t)}\right)^{p(t)}+1\right)dt\le C_{11}2^{-k\varepsilon q^-}.\end{aligned} \tag{4.11}$$

Therefore,

$$\left\||x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)}\int_{B_{x,k}}f(t)dt\right\|_{L^{q(\cdot)}(B(0,\delta))}\le C_{12}2^{-k\varepsilon q^-/q^+}. \tag{4.12}$$

By (4.12) and (4.4), we get

$$i_1\le C_{12}\sum_{k=0}^{\infty}2^{-k\varepsilon q^-/q^+}=C_{13}<\infty. \tag{4.13}$$

The estimate at infinity (i_4).

Put $f_N(t) = f(t)\chi_{|t|>N}$. Analogously to the case of (4.4), we have

$$i_4 \le \sum_{k=0}^{\infty} \left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} \int_{\{t:2^{-k-1}|x|<|t|<2^{-k}|x|\}} f_N(t)dt \right\|_{L^{q(\cdot)}(\mathbb{R}^n \setminus B(0,N))}. \tag{4.14}$$

By $|t| \sim |x|2^{-k}$, condition (2.3) and Lemma 2.1 for $x \in \mathbb{R}^n \setminus B(0,N)$, $t \in B_{x,k}$, we have

$$|x|^{\beta(x)} \sim |x|^{\beta(\infty)} \sim 2^{k\beta(\infty)} t^{\beta(\infty)} \sim 2^{k\beta(\infty)} t^{\beta(t)}. \tag{4.15}$$

Therefore, by virtue of Holder's inequality,

$$\begin{aligned} &|x|^{\beta(x)-n/p'(x)-n/q(x)} \int_{B_{x,k}} f_N(t)dt \\ &\quad\le C_{14} 2^{k\beta(\infty)} |x|^{-n/p'(x)-n/q(x)} \int_{B_{x,k}} |t|^{\beta(t)} f_N(t)dt \\ &\quad\le C_{14} 2^{k\beta(\infty)} |x|^{-n/p'(x)-n/q(x)} \left(\int_{B_{x,k}} \left(|t|^{\beta(t)} f_N(t) \right)^{p^-_{x,k}} dt \right)^{1/p^-_{x,k}} \left(2^{-k}|x| \right)^{n/(p^-_{x,k})'}. \end{aligned} \tag{4.16}$$

(i) If $p^-_{x,k} \ne p(x)$ and $t \in B_{x,k}$, by (2.3) and Lemma 2.1, we have

$$\left(2^{-k}|x| \right)^{n/(p^-_{x,k})'} \sim t^{n/p'(t)} \sim t^{n/p'(\infty)} \sim 2^{-kn/p'(\infty)} |x|^{n/p'(\infty)} \sim 2^{-kn/p'(\infty)} |x|^{n/p'(x)}. \tag{4.17}$$

(ii) If $p^-_{x,k} = p(x)$, then by choice of δ_1,

$$\left(2^{-k}|x| \right)^{n/(p^-_{x,k})'} \sim 2^{-kn/p'(x)} |x|^{n/p'(x)} \le 2^{-kn/p'(\infty)+\delta_1 k} |x|^{n/p'(x)}. \tag{4.18}$$

In both hypotheses (i) and (ii) by choosing of δ_1, we have

$$|x|^{\beta(x)-n/p'(x)-n/q(x)} \int_{B_{x,k}} f_N(t)dt \le C_{15} |x|^{-n/q(x)} 2^{-k\delta_1} \left(\int_{B_{x,k}} \left(|t|^{\beta(t)} f_N(t) \right)^{p^-_{x,k}} dt \right)^{1/p^-_{x,k}}. \tag{4.19}$$

On the other hand,

$$\int_{B_{x,k}} \left(|t|^{\beta(t)} f(t) \right)^{p^-_{x,k}} dt \le \int_{B_{x,k} \cap \{t \in \mathbb{R}^n : |t|^{\beta(t)} f(t) \ge G(t)\}} \left(\frac{|t|^{\beta(t)} f(t)}{G(t)} \right)^{p^-_{x,k}} G(t)^{p^-_{x,k}} dt + \int_{B_{x,k}} G(t)^{p^-} dt, \tag{4.20}$$

where $G(t) = 1/(1+t^2)$. Hence,

$$\leq \int_{B_{x,k}} \left(f_N(t)|t|^{\beta(t)}\right)^{p(t)} G(t)^{p^-_{x,k}-p(t)} + \int_{B_{x,k}} G(t)dt. \tag{4.21}$$

By (2.3), for $t \in B_{x,k}$, we have

$$G(t)^{p^-_{x,k}-p(t)} \leq \left(1+t^2\right)^{p(t)-p^-_{x,k}} \leq C_{16}. \tag{4.22}$$

Then (4.21) implies

$$\int_{B_{x,k}} \left(|t|^{\beta(t)} f_N(t)\right)^{p^-_{x,k}} dt \leq C_{17}. \tag{4.23}$$

Therefore,

$$\begin{aligned} &I_{q;\mathbb{R}^n\setminus B(0,N)}\left(|x|^{\beta(x)-n/p'(x)-n/q(x)} \int_{B_{x,k}} f_N(t)dt\right) \\ &\quad\leq C_{17}^{q^+/p^-} 2^{-k\delta_1 q^-} \int_{\mathbb{R}^n\setminus B(0,N)} |x|^{-n}\left(\int_{B_{x,k}} \left(|t|^{\beta(t)} f_N(t)\right)^{p(t)} dt\right)dx, \end{aligned} \tag{4.24}$$

by Fubini's theorem,

$$\leq C_{17}^{q^+/p^- -1} 2^{-k\delta_1 q^-} \ln 2 \int_{\{t:|t|>2^{-k}N\}} \left(f_N(t)|t|^{\beta(t)}\right)^{p(t)} dt \leq C_{18} 2^{-k\delta_1 q^-}. \tag{4.25}$$

From (4.25) and expansion (4.14), we get

$$i_4 \leq C_{18} \sum_{k=0}^{\infty} 2^{-kq^-\delta_1/q^+} = C_{19}. \tag{4.26}$$

The estimate in the middle (i_2, i_3).
We have

$$\begin{aligned} i_2 &= \left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} \int_{\{t\in\mathbb{R}^n:|t|<|x|\}} f(t)dt \right\|_{L^{q(\cdot)}(B(0,N)\setminus B(0,\delta))} \\ &\leq \left(\int_{B(0,N)} f(t)dt\right) \left\| |x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} \right\|_{L^{q(\cdot)}(B(0,N)\setminus B(0,\delta))} \\ &\leq C_{20} \int_{B(0,N)} f(t)dt, \end{aligned} \tag{4.27}$$

from which, by virtue of Holder's inequality, for $p(x)$-norms, we obtain the estimate

$$\int_{B(0,N)} f(t)dt \leq \left\| |t|^{\beta(\cdot)} f(t) \right\|_{L^{p(\cdot)}(B(0,N))} \left\| |t|^{-\beta(\cdot)} \right\|_{L^{p'(\cdot)}(B(0,N)}. \tag{4.27'}$$

Using $t^{-\beta(t)p'(t)} \sim t^{-\beta(0)p'(0)}$ by Lemma 2.1 for $t \in B(0,N)$ and taking the condition $\beta(0) < n/p'(0)$ into account, we find

$$I_{p';B(0,N)}\left(|t|^{-\beta(\cdot)}\right) = \int_{B(0,N)} |t|^{-\beta(t)p'(t)} dt \leq C_{21} \int_{B(0,N)} |t|^{-\beta(0)p'(0)} dt = C_{22}. \tag{4.28}$$

From (4.27′) and (4.28), it follows that

$$i_2 \leq C_{23}. \tag{4.29}$$

Furthermore, we have

$$i_3 \leq \left(\int_{B(0,N)} f(t)dt\right) \left\| |x|^{\beta(x)-n/p'(x)-n/q(x)} \right\|_{L^{q(\cdot)}(\mathbb{R}^n \setminus B(0,\delta))}. \tag{4.30}$$

The boundedness of the first term follows by (4.27′). Due to (2.3) and Lemma 2.1, for $x \in \mathbb{R}^n \setminus B(0,N)$, we have

$$|x|^{(\beta(x)-n/p'(x))q(x)-n} \sim |x|^{(\beta(\infty)-n/p'(\infty))q(x)-n}. \tag{4.31}$$

Applying condition (4.31), we get

$$I_{q;\mathbb{R}^n/B(0,N)}\left(|x|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)}\right) \leq C_{24} \int_{\mathbb{R}^n \setminus B(0,N)} |x|^{-n-2\delta_1} dx = C_{25}. \tag{4.32}$$

Then

$$i_3 \leq C_{25}^{1/p^-}. \tag{4.33}$$

Necessity. Let $\beta(0) > n/p'(0)$. Fix a sufficiently large $\tau > 0$ and apply inequality (1.1) by the test function

$$f_\tau(t) = t^{-n/p(t)-\beta(t)} \chi_{B(0,\delta/\tau) \setminus B(0,\delta/2\tau)}(t). \tag{4.34}$$

We come to a contradiction

$$
\begin{aligned}
&I_p\left(|t|^{\beta(\cdot)} f_\tau\right) = \int_{B(0,\delta/\tau)\setminus B(0,\delta/(2\tau))} |x|^{-n} dx = C_0 \ln 2 < \infty,\\
&I_q\left(|t|^{\beta(\cdot)-n/p'(\cdot)-n/q(\cdot)} \int_{B(0,t)} f_\tau(y)dy\right)\\
&\quad\geq \int_{B(0,1)\setminus B(0,\delta/\tau)} |t|^{(\beta(t)-n/p'(t)-n/q(t))q(t)} \left(\int_{B(0,\delta/\tau)\setminus B(0,\delta/(2\tau))} |y|^{-n/p(0)-\beta(0)} dy\right)^{q(t)} dt\\
&\quad\geq \left(\frac{\delta}{2\tau}\right)^{(n/p'(0)-\beta(0))q^-} \int_{B(0,1)\setminus B(0,\delta/\tau)} |t|^{(\beta(0)-n/p'(0))q(t)-n} dt \longrightarrow \infty
\end{aligned}
\tag{4.35}
$$

as $\tau \to \infty$.

If $0 < p(0) \leq 1$, then by virtue of inequalities (4.35) and (3.2) we obtain

$$
I_q\left(|t|^{\beta(t)-n/p'(t)-n/q(t)} \int_{B(0,t)} f_\tau(y)dy\right) \longrightarrow \infty, \quad \text{as } \tau \longrightarrow \infty. \tag{4.36}
$$

Also,

$$
I_p\left(|t|^{\beta(t)} f_\tau(t)\right) = C_0 \ln 2, \tag{4.37}
$$

and we come to a contradiction.

If $\beta(\infty) \geq n/p'(\infty)$, then, using condition (2.3) and Lemma 2.1 assuming $0 < \tau < 1$, we again obtain

$$
\begin{aligned}
&I_p\left(|t|^{\beta(t)} f_\tau(t)\right) = C_0 \ln 2,\\
&I_q\left(|t|^{\beta(t)-n/p'(t)-n/q(t)} \int_{B(0,t)} f_\tau(t)dy\right)\\
&\quad\geq \int_{\mathbb{R}^n\setminus B(0,\delta/\tau)} |t|^{(\beta(t)-n/p'(t))q(t)-n} \left(\int_{B(0,\delta/\tau)\setminus B(0,\delta/(2\tau))} |y|^{-n/p(\infty)-\beta(\infty)} dy\right) dt\\
&\quad\geq \left(\frac{\delta}{2\tau}\right)^{(n/p'(\infty)-\beta(\infty))q^+} \int_{\mathbb{R}^n\setminus B(0,\delta/\tau)} |t|^{(\beta(\infty)-n/p'(\infty))q(t)-n} dt \longrightarrow \infty
\end{aligned}
\tag{4.38}
$$

as $\tau \to \infty$. If $\beta(\infty) = n/p'(\infty)$, then from (4.38) we have

$$I_q\left(|t|^{\beta(t)-n/p'(t)-n/q(t)} \int_{B(0,t)} f_\tau(t)dy \right) = \infty. \tag{4.39}$$

From (4.38) and (3.2), we derive, as above, the necessity of the condition $p(\infty) > 1$.

This completes the proof of Theorem 3.1. □

The proof of Theorem 3.2 easily follows from Theorem 3.1 by using the equivalence of inequalities

$$\begin{gathered} \left\| |x|^{\beta(x)-n/p'(x)-n/q(x)} \overline{H} f(x) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C \left\| |x|^{\beta(x)} f(x) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \\ \left\| |z|^{n-\overline{\beta}(z)-2n/\overline{q}(z)} H f(x) \right\|_{L^{\overline{q}(\cdot)}(\mathbb{R}^n)} \leq C \left\| |z|^{-\overline{\beta}(z)-2n/\overline{p}(z)} \overline{f}(z) \right\|_{L^{\overline{p}(\cdot)}(\mathbb{R}^n)}, \end{gathered} \tag{4.40}$$

where $\overline{p}(x), \overline{q}(x)$, and $\overline{\beta}(x)$ stand for the functions $p(x/|x|^2), q(x/|x|^2)$, and $\beta(x/|x|^2)$, respectively. The equivalence readily follows from the equality

$$\|g\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \left\| |z|^{-2n/\overline{p}(z)} \overline{g} \right\|_{L^{\overline{p}(\cdot)}(\mathbb{R}^n)} \tag{4.41}$$

for any function $g : \mathbb{R}^n \to \mathbb{R}$, where $\overline{g}(z) = g(z/|z|^2)$, which easily can be proved by changing of variable $x = z/|z|^2$ in the definition of $p(x)$-norm.

5. Exactness of the Logarithmic Conditions

Proof of Theorem 3.3. Assume $\delta_k = 1/4^k$, $k \in \mathbb{N}$, $f_k(x) = |x|^{-n/p(x)-\beta(x)} \chi_{B(0,2\delta_k)\setminus B(0,\delta_k)}(x)$, and $\beta(x) = \beta_0$. Define the function $p : (0,\infty) \to (1,\infty)$ as

$$p(x) = \begin{cases} p_0, & x \in B(0, 2\delta_k) \setminus B(0, \delta_k), \\ p_k, & x \in B(0, 4\delta_k) \setminus B(0, 2\delta_k), \ k \in \mathbb{N} \end{cases} \tag{5.1}$$

where $p_0 > 1$, $p_k = p_0 + \alpha_k$, $\beta_0 \in \mathbb{R}$, and $\{\alpha_k\}$ is an arbitrary sequence of positive numbers satisfying the condition

$$k\alpha_k \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \tag{5.2}$$

Then $\alpha_k \ln(1/\delta_k) \to \infty$, and condition (2.2) does not hold for the function $p(x)$. Since

$$\begin{aligned}
I_p\left(|x|^{\beta(x)} f_k(x)\right) &= \int_{B(0,2\delta_k)\setminus B(0,\delta_k)} \left(|t|^{\beta_0} \cdot |t|^{-n/p_0-\beta_0}\right)^{p_0} dt \\
&= \int_{B(0,2\delta_k)\setminus B(0,\delta_k)} |t|^{-n} dt = C_0 \int_{\delta_k}^{2\delta_k} \frac{dt}{t} = \omega_{n-1} \ln 2, \\
I_p\left(H\left(|\cdot|^{\beta(\cdot)-n} f_k(\cdot)\right)\right) &\geq \int_{B(0,4\delta_k)\setminus B(0,2\delta_k)} \left(\int_{B(0,2\delta_k)\setminus B(0,\delta_k)} |t|^{-n/p(t)-\beta_0} dt\right)^{p_k} |x|^{(\beta_0-n)p_k} dx \\
&\geq C \int_{B(0,3\delta_k)\setminus B(0,2\delta_k)} \delta_k^{(n-n/p_0-\beta_0)p_k} |x|^{(\beta_0-n)(p_0+\alpha_k)} dx \\
&\geq C\delta_k^{-n\alpha_k/p_0} = e^{(n\alpha_k/p_0)\ln(1/\delta_k)} \longrightarrow \infty
\end{aligned} \tag{5.3}$$

as $k \to \infty$, we see that this contradicts inequality (3.4).

The given function $f_k(x)$ and the exponential functions $p(x)$ and $\beta(x)$ are also suitable for proving the necessity of condition (2.3) for the function p. For this we define the numbers δ_k from the equality $\delta_k = 4^k$, $k \in \mathbb{N}$. Let $f_k(x) = |x|^{-n/p(x)-\beta} \chi_{B(0,2\delta_k)\setminus B(0,\delta_k)}(x)$, $\beta(x) = \beta_\infty$, and $x \in \mathbb{R}^n$. We define the function p as

$$p(x) = \begin{cases} p_\infty, & x \in B(0,2\delta_k) \setminus B(0,\delta_k), \\ \overline{p}_k, & x \in B(0,4\delta_k) \setminus B(0,2\delta_k), \ k \in \mathbb{N} \end{cases} \tag{5.4}$$

where $p_\infty > 1$, $\beta_\infty \in \mathbb{R}$, $\overline{p}_k = p_\infty - \alpha_k$, and $\{\alpha_k\}$ is an arbitrary sequence of positive numbers satisfying the condition $k\alpha_k \to \infty$ as $k \to \infty$. Then $\alpha_k \ln \delta_k \to \infty$; hence, condition (2.3) does not hold for the function $p(x)$. Furthermore, we have

$$\begin{aligned}
I_p\left(|x|^{\beta(x)} f_k(x)\right) &= \int_{B(0,2\delta_k)\setminus B(0,\delta_k)} \left(|t|^{\beta_\infty} \cdot |t|^{-n/p_\infty-\beta_\infty}\right)^{p_\infty} dt = \omega_{n-1} \ln 2, \\
I_p\left(|x|^{\beta(x)-n} f_k(x)\right) &\geq \int_{B(0,4\delta_k)\setminus B(0,2\delta_k)} \left(\int_{B(0,2\delta_k)\setminus B(0,\delta_k)} |t|^{-n/p(t)-\beta_\infty} dt\right)^{p_k} |x|^{(\beta_\infty-n)p_k} dx \\
&\geq C \int_{B(0,3\delta_k)\setminus B(0,2\delta_k)} \delta_k^{(n-n/p_\infty-\beta_\infty)p_k} |x|^{(\beta_\infty-n)(p_\infty-\alpha_k)} dx \\
&\geq C\delta_k^{n\alpha_k/p_\infty} = Ce^{(n\alpha_k/p_\infty)\ln \delta_k} \longrightarrow \infty
\end{aligned} \tag{5.5}$$

as $k \to \infty$, which contradicts inequality (3.4).

The same reasoning brings us to the proof of the exactness of conditions (2.2) and (2.3) for the function $\beta(x)$ also. For instance, to show the necessity of condition (2.2), it can be assumed that $p(x) \equiv p_0 > 1$, $x \in \mathbb{R}^n$,

$$\beta(x) = \begin{cases} \beta_0 + \alpha_k, & x \in B(0, 2\delta_k) \setminus B(0, \delta_k), \\ \beta_0, & x \in B(0, 4\delta_k) \setminus B(0, 2\delta_k) \;\; k \in \mathbb{N}. \end{cases} \tag{5.6}$$

Then

$$\begin{gathered} I_p\left(|x|^{\beta(x)-n} f_k(x)\right) \geq C\delta_k^{-p_0\alpha_k} \longrightarrow \infty \quad \text{as } k \longrightarrow \infty, \\ I_p\left(|x|^{\beta(x)} f_k(x)\right) \leq C_0 \ln 2. \end{gathered} \tag{5.7}$$

This completes the proof of Theorem 3.3. □

Acknowledgment

F. I. Mamedov was supported partially by the INTAS Grant for the South-Caucasian Republics, no. 8792.

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