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Research Article

On a New Hilbert-Hardy-Type Integral Operator and Applications

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By applying the way of weight functions and a Hardy's integral inequality, a Hilbert-Hardy-type integral operator is defined, and the norm of operator is obtained. As applications, a new Hilbert-Hardy-type inequality similar to Hilbert-type integral inequality is given, and two equivalent inequalities with the best constant factors as well as some particular examples are considered.

1. Introduction

In 1934, Hardy published the following theorem (cf. [1, Theorem 319]).

Theorem A. *If $k(x,y)$ (≥ 0) is a homogeneous function of degree -1 in $(0,\infty)\times(0,\infty)$, $p>1$, $1/p+1/q=1$, and $k_p=\int_0^\infty k(u,1)u^{-1/p}du\in(0,\infty)$, then for $f(x),g(y)\geq 0$, $0<\|f\|_p:=\{\int_0^\infty f^p(x)dx\}^{1/p}<\infty$, and $0<\|g\|_q<\infty$, one has*

$$\iint_0^\infty k(x,y)f(x)g(y)\,dx\,dy<k_p\|f\|_p\|g\|_q, \tag{1.1}$$

where the constant factor k_p is the best possible.

Hardy [2] also published the following Hardy's integral inequality.

Theorem B. *If* $p > 1$, $\rho \neq 1$, $f(x) \geq 0$, *and* $F(x) := \int_0^x f(t)dt(\rho > 1)$; $F(x) := \int_x^\infty f(t)dt(\rho < 1)$, $0 < \int_0^\infty x^{p-\rho} f^p(x)dx < \infty$, *then one has*

$$\int_0^\infty x^{-\rho} F^p(x)dx < \left(\frac{p}{|\rho - 1|}\right)^p \int_0^\infty x^{p-\rho} f^p(x)dx, \tag{1.2}$$

where the constant factor $(p/|\rho - 1|)^p$ *is the best possible (cf. [1, Theorem 330]).*

In 2009, Yang [3] published the following theorem.

Theorem C. *If* $p > 1$, $1/p + 1/q = 1$, $\lambda > 0$, $k_\lambda(x,y)$ (≥ 0) *is a homogeneous function of degree* $-\lambda$ *in* $(0,\infty) \times (0,\infty)$, *and for any* $r > 1(1/r + 1/s = 1)$, $0 < k_\lambda(r) := \int_0^\infty k_\lambda(u,1)u^{\lambda/r-1}du < \infty$, *then for* $f(x)$, $g(y) \geq 0$, $\varphi(x) := x^{p(1-\lambda/r)-1}$, $\psi(y) := y^{q(1-\lambda/s)-1}$, $0 < \|f\|_{p,\varphi} := \{\int_0^\infty \varphi(x)|f(x)|^p dx\}^{1/p} < \infty$ *and* $0 < \|g\|_{q,\psi} < \infty$, *we have*

$$\iint_0^\infty k_\lambda(x,y)f(x)g(y)dx\,dy < k_\lambda(r)\|f\|_{p,\varphi}\|g\|_{q,\psi}, \tag{1.3}$$

where the constant factor $k_\lambda(r)$ *is the best possible.*

For $\lambda = 1, r = q$, (1.3) reduces to (1.1). We name of (1.1) and (1.3) Hilbert-type integral inequalities. Inequalities (1.1), (1.2) and (1.3) are important in analysis and its applications (cf. [4–6]).

Setting $k_1(x,y) = (1/(x+y)^\gamma)x^{R-1/q}y^{S-1/p}(R,S > 0, R+S = \gamma)$, $F(x) = \int_0^x f(t)dt, G(y) = \int_0^y g(t)dt$, by applying (1.2) (for $\rho = p > 1$), Das and Sahoo gave a new integral inequality similar to Pachpatte's inequality (cf. [7, 8]) as follows:

$$\iint_0^\infty \frac{x^{R-1/q}y^{S-1/p}}{(x+y)^\gamma}\frac{F(x)}{x}\frac{G(y)}{y}dx\,dy < pqB(R,S)\|f\|_p\|g\|_q, \tag{1.4}$$

where the constant factor $pqB(R,S)$ is the best possible (cf. [9]). Sulaiman [10] also considered a Hilbert-Hardy-type integral inequality similar to (1.4) with the kernel $k(x,y) = (1/(\max\{x,y\})^\lambda)x^{\beta/q+1}y^{\alpha/p+1}(\alpha,\beta > -1, p = \lambda - \alpha - 1 > 1, q = \lambda - \beta - 1 > 1)$. But he cannot show that the constant factor in the new inequality is the best possible.

In this paper, by applying the way of weight functions and inequality (1.2) for $\rho < 1$, a Hilbert-Hardy-type integral operator is defined, and the norm of operator is obtained. As applications, a new Hilbert-Hardy-type inequality similar to (1.3) is given, and two equivalent inequalities with a best constant factor as well as some particular examples are considered.

2. A Lemma and Two Equivalent Inequalities

Lemma 2.1. *If $\lambda < 2$, $k_\lambda(x,y)$ is a nonnegative homogeneous function of degree $-\lambda$ in $(0,\infty)\times(0,\infty)$ with $k_\lambda(ux,uy) = u^{-\lambda}k(x,y)(u,x,y>0)$, and for any $\alpha \in (\lambda-1,1)$, $0 < k(\alpha) := \int_0^\infty k_\lambda(1,u)u^{\alpha-1}du < \infty$, then $\int_0^\infty k_\lambda(u,1)u^{\lambda-\alpha-1}du = k(\alpha)$ and*

$$0 < \int_0^\infty k_\lambda(1,u)u^{\alpha-1}|\ln u|du = \int_0^\infty k_\lambda(u,1)u^{\lambda-\alpha-1}|\ln u|du < \infty. \tag{2.1}$$

Proof. Setting $v = 1/u$, we find

$$\int_0^\infty k_\lambda(u,1)u^{\lambda-\alpha-1}du = \int_0^\infty k_\lambda(1,v)v^{\alpha-1}dv = k(\alpha). \tag{2.2}$$

There exists $\beta > 0$, satisfying $\alpha \pm \beta \in (\lambda-1,1)$ and $0 < k(\alpha\pm\beta) < \infty$. Since we find

$$\lim_{u\to 0^+}\frac{\ln u}{u^\beta + u^{-\beta}} = \lim_{u\to\infty}\frac{\ln u}{u^\beta + u^{-\beta}} = 0, \tag{2.3}$$

there exists $M > 0$, such that $|\ln u| \le M(u^\beta + u^{-\beta})$ $(u \in (0,\infty))$, and then

$$\begin{aligned}
0 &< \int_0^\infty k_\lambda(u,1)u^{\lambda-\alpha-1}|\ln u|du = \int_0^\infty k_\lambda(1,u)u^{\alpha-1}|\ln u|du \\
&\le M\int_0^\infty k_\lambda(1,u)u^{\alpha-1}\left(u^\beta + u^{-\beta}\right)du \\
&= M\left[k(\alpha+\beta) + k(\alpha-\beta)\right] < \infty.
\end{aligned} \tag{2.4}$$

The lemma is proved. □

Theorem 2.2. *If $p > 1$, $1/p + 1/q = 1$, $\lambda_1 + \lambda_2 = \lambda < 2$, $k_\lambda(x,y)(\ge 0)$ is a homogeneous function of degree $-\lambda$ in $(0,\infty)\times(0,\infty)$, and for any $\lambda_1 \in (\lambda-1,1)$, $0 < k(\lambda_1) = \int_0^\infty k(u,1)u^{\lambda_1-1}du < \infty$, then for $f(x), g(y) \ge 0$, $\tilde{\varphi}(x) := x^{p(2-\lambda-\lambda_1)-1}$, $\psi(y) := y^{q(1-\lambda_2)-1}$,*

$$\widetilde{F}_\lambda(x) := \int_x^\infty \frac{1}{t^\lambda}f(t)dt, \qquad \widetilde{G}_\lambda(y) := \int_y^\infty \frac{1}{t^\lambda}g(t)dt, \tag{2.5}$$

$0 < \|f\|_{p,\tilde{\varphi}} < \infty$, and $0 < \|\widetilde{G}_\lambda\|_{q,\psi} < \infty$, one has the following equivalent inequalities:

$$I := \iint_0^\infty k_\lambda(x,y)\widetilde{F}_\lambda(x)\widetilde{G}_\lambda(y)dx\,dy < \frac{k(\lambda_1)}{1-\lambda_1}\|f\|_{p,\tilde{\varphi}}\left\|\widetilde{G}_\lambda\right\|_{q,\psi}, \tag{2.6}$$

$$J := \left\{\int_0^\infty \psi^{1-p}(y)\left[\int_0^\infty k_\lambda(x,y)\widetilde{F}_\lambda(x)dx\right]^p dy\right\}^{1/p} < \frac{k(\lambda_1)}{1-\lambda_1}\|f\|_{p,\tilde{\varphi}}. \tag{2.7}$$

Proof. Setting the weight functions $\omega(\lambda_1, y)$ and $\varpi(\lambda_2, x)$ as follows:

$$\omega(\lambda_1, y) := \int_0^\infty k_\lambda(x, y)\frac{y^{\lambda_2}dx}{x^{1-\lambda_1}}, \qquad \varpi(\lambda_2, x) := \int_0^\infty k_\lambda(x, y)\frac{x^{\lambda_1}dy}{y^{1-\lambda_2}}, \tag{2.8}$$

then by Lemma 2.1, we find

$$\begin{aligned}\omega(\lambda_1, y) &\overset{u=x/y}{=} \int_0^\infty k(u, 1)u^{\lambda_1-1}du = k(\lambda_1),\\ \varpi(\lambda_2, x) &\overset{u=y/x}{=} \int_0^\infty k(1, u)u^{\lambda_2-1}du = k(\lambda_1).\end{aligned} \tag{2.9}$$

By Hölder's inequality (cf. [11]) and (2.8), (2.9), we obtain

$$\begin{aligned}\int_0^\infty k_\lambda(x, y)\widetilde{F}_\lambda(x)dx &= \int_0^\infty k_\lambda(x, y)\left[\frac{x^{(1-\lambda_1)/q}}{y^{(1-\lambda_2)/p}}\widetilde{F}_\lambda(x)\right]\left[\frac{y^{(1-\lambda_2)/p}}{x^{(1-\lambda_1)/q}}\right]dx\\ &\leq \left\{\int_0^\infty k_\lambda(x, y)\frac{x^{(1-\lambda_1)(p-1)}}{y^{1-\lambda_2}}\widetilde{F}_\lambda^p(x)dx\right\}^{1/p}\\ &\quad\times\left\{y^{q(1-\lambda_2)-1}\int_0^\infty k_\lambda(x, y)\frac{y^{\lambda_2}dx}{x^{1-\lambda_1}}\right\}^{1/q}\\ &= k^{1/q}(\lambda_1)y^{1/p-\lambda_2}\left\{\int_0^\infty k_\lambda(x, y)\frac{x^{(1-\lambda_1)(p-1)}}{y^{1-\lambda_2}}\widetilde{F}_\lambda^p(x)dx\right\}^{1/p}.\end{aligned} \tag{2.10}$$

Then by Fubini theorem (cf. [12]), it follows:

$$\begin{aligned}J^p &\leq k^{p-1}(\lambda_1)\iint_0^\infty k_\lambda(x, y)\frac{x^{(1-\lambda_1)(p-1)}}{y^{1-\lambda_2}}\widetilde{F}_\lambda^p(x)dx\,dy\\ &= k^{p-1}(\lambda_1)\int_0^\infty\left[\int_0^\infty k_\lambda(x, y)\frac{x^{(1-\lambda_1)(p-1)}}{y^{1-\lambda_2}}dy\right]\widetilde{F}_\lambda^p(x)dx\\ &= k^p(\lambda_1)\int_0^\infty x^{-[p(\lambda_1-1)+1]}\widetilde{F}_\lambda^p(x)dx.\end{aligned} \tag{2.11}$$

Since $\lambda_1 < 1$, $\rho = p(\lambda_1 - 1) + 1 < 1$, then by (1.2) (for $\rho < 1$), we have

$$\begin{aligned}\int_0^\infty x^{-[p(\lambda_1-1)+1]}\widetilde{F}_\lambda^p(x)dx &< \left(\frac{1}{1-\lambda_1}\right)^p\int_0^\infty x^{p-[p(\lambda_1-1)+1]}\left(\frac{f(x)}{x^\lambda}\right)^p dx\\ &= \left(\frac{1}{1-\lambda_1}\right)^p\int_0^\infty x^{p(2-\lambda-\lambda_1)-1}f^p(x)dx.\end{aligned} \tag{2.12}$$

Hence by (2.11), we have (2.7). Still by Hölder's inequality, we find

$$I = \int_0^\infty \left[\psi^{-1/q}(y)\int_0^\infty k_\lambda(x,y)\widetilde{F}_\lambda(x)dx\right]\left[\psi^{1/q}(y)\widetilde{G}_\lambda(y)\right]dy \le J\left\|\widetilde{G}_\lambda\right\|_{q,\psi}. \tag{2.13}$$

Then by (2.7), we have (2.6).

On the other-hand, supposing that (2.6) is valid, by (2.11) and (1.2) (for $\rho < 1$), it follows $J < \infty$. If $J = 0$, then (2.7) is naturally valid; if $0 < J < \infty$, setting

$$\widetilde{G}_\lambda(y) = \psi^{1-p}(y)\left[\int_0^\infty k_\lambda(x,y)\widetilde{F}_\lambda(x)dx\right]^{p-1}, \tag{2.14}$$

then by (2.6), we find

$$\begin{gathered}\left\|\widetilde{G}_\lambda\right\|_{q,\psi}^q = J^p = I < \frac{k(\lambda_1)}{1-\lambda_1}\|f\|_{p,\widetilde{\varphi}}\left\|\widetilde{G}_\lambda\right\|_{q,\psi}, \\ \left\|\widetilde{G}_\lambda\right\|_{q,\psi}^{q-1} = J < \frac{k(\lambda_1)}{1-\lambda_1}\|f\|_{p,\widetilde{\varphi}}.\end{gathered} \tag{2.15}$$

Hence, we have (2.7), which is equivalent to (2.6). □

3. A Hilbert-Hardy-Type Integral Operator and Applications

Setting a real function space as follows:

$$L^p_{\widetilde{\varphi}}(0,\infty) := \left\{f; \|f\|_{p,\widetilde{\varphi}} = \left\{\int_0^\infty \widetilde{\varphi}(x)|f(x)|^p dx\right\}^{1/p} < \infty\right\}, \tag{3.1}$$

for $f(\ge 0) \in L^p_{\widetilde{\varphi}}(0,\infty)$, $\widetilde{F}_\lambda(x) = \int_x^\infty (f(t)/t^\lambda)dt$, define an integral operator $T : L^p_{\widetilde{\varphi}}(0,\infty) \to L^p_{\psi^{1-p}}(0,\infty)$ as follows:

$$Tf(y) := \int_0^\infty k_\lambda(x,y)\widetilde{F}_\lambda(x)dx, y \in (0,\infty). \tag{3.2}$$

Then, by (2.7), $Tf \in L^p_{\psi^{1-p}}(0,\infty)$, and T is bounded with

$$\|T\| = \sup_{f(\ne\theta)\in L^p_{\widetilde{\varphi}}(0,\infty)} \frac{\|Tf\|_{p,\psi^{1-p}}}{\|f\|_{p,\widetilde{\varphi}}} \le \frac{k(\lambda_1)}{1-\lambda_1}. \tag{3.3}$$

Theorem 3.1. *Let the assumptions of Theorem 2.2 be fulfilled, and additionally setting* $\widetilde{\varphi}(y) := y^{q(2-\lambda-\lambda_2)-1}$. *Then one has*

$$\iint_0^\infty k_\lambda(x,y)\widetilde{F}_\lambda(x)\widetilde{G}_\lambda(y)dx\,dy < \frac{k(\lambda_1)}{(1-\lambda_1)(1-\lambda_2)}\|f\|_{p,\widetilde{\varphi}}\|g\|_{q,\widetilde{\psi}}, \tag{3.4}$$

where the constant factor $k(\lambda_1)/(1-\lambda_1)(1-\lambda_2)$ *is the best possible. Moreover the constant factor in* (2.6) *and* (2.7) *is the best possible and then*

$$\|T\| = \frac{k(\lambda_1)}{1-\lambda_1}. \tag{3.5}$$

Proof. Since $\lambda_2 < 1$, by (1.2), for $\rho = q(\lambda_2 - 1) + 1 < 1$, it follows:

$$\begin{aligned}
\left\|\widetilde{G}_\lambda\right\|_{q,\psi} &= \left\{\int_0^\infty y^{-[q(\lambda_2-1)+1]}\widetilde{G}_\lambda^q(y)dy\right\}^{1/q} \\
&< \frac{q}{1-[q(\lambda_2-1)+1]}\left\{\int_0^\infty y^{q-[q(\lambda_2-1)+1]}\left(\frac{g(y)}{y^\lambda}\right)^q dy\right\}^{1/q} \\
&= \frac{1}{1-\lambda_2}\left\{\int_0^\infty y^{q(2-\lambda-\lambda_2)-1}g^q(y)dy\right\}^{1/q} = \frac{1}{1-\lambda_2}\|g\|_{q,\widetilde{\psi}}.
\end{aligned} \tag{3.6}$$

Then, by (2.6), we have (3.4).

For $T > 2$, setting $\widetilde{f}(x), \widetilde{g}(y)$ as follows:

$$\begin{aligned}
\widetilde{f}(x) &= \begin{cases} x^{\lambda+\lambda_1-2}, & 1 \le x \le T, \\ 0, & 0 < x < 1;\ x > T, \end{cases} \\
\widetilde{g}(y) &= \begin{cases} y^{\lambda+\lambda_2-2}, & 1 \le y \le T, \\ 0, & 0 < y < 1;\ y > T, \end{cases}
\end{aligned} \tag{3.7}$$

then for $1 \leq x, y \leq T$, we find

$$\begin{aligned}
\widetilde{F}_\lambda(x) &= \int_x^\infty \frac{\widetilde{f}(t)}{t^\lambda} dt = \int_x^T t^{\lambda_1 - 2} dt = \frac{1}{1-\lambda_1}\left(x^{\lambda_1 - 1} - T^{\lambda_1 - 1}\right), \\
\widetilde{G}_\lambda(y) &= \int_y^\infty \frac{\widetilde{g}(t)}{t^\lambda} dt = \frac{1}{1-\lambda_2}\left(y^{\lambda_2 - 1} - T^{\lambda_2 - 1}\right), \\
\widetilde{I} &:= \iint_1^T k_\lambda(x,y)\widetilde{F}_\lambda(x)\widetilde{G}_\lambda(y)\,dx\,dy = \frac{1}{(1-\lambda_1)(1-\lambda_2)} \\
&\quad \times \int_1^T \left[\int_1^T k_\lambda(x,y)\left(x^{\lambda_1 - 1} - T^{\lambda_1 - 1}\right)\left(y^{\lambda_2 - 1} - T^{\lambda_2 - 1}\right) dy\right] dx \\
&\geq \frac{1}{(1-\lambda_1)(1-\lambda_2)}[I_1 - I_2 - I_3],
\end{aligned} \tag{3.8}$$

where I_1, I_2, and I_3 are indicated as follows;

$$\begin{aligned}
I_1 &:= \int_1^T \left[\int_1^T k_\lambda(x,y) x^{\lambda_1 - 1} y^{\lambda_2 - 1} dy\right] dx, \\
I_2 &:= T^{\lambda_1 - 1} \int_1^T \left[\int_1^T k_\lambda(x,y) y^{\lambda_2 - 1} dy\right] dx, \\
I_3 &:= T^{\lambda_2 - 1} \int_1^T \left[\int_1^T k_\lambda(x,y) x^{\lambda_1 - 1} dx\right] dy.
\end{aligned} \tag{3.9}$$

If there exists a positive constant $k \leq k(\lambda_1)$, such that (3.4) is still valid as we replace $k(\lambda_1)$ by k, then in particular, we find

$$\begin{aligned}
\widetilde{I} &< \frac{k}{(1-\lambda_1)(1-\lambda_2)} \left\|\widetilde{f}\right\|_{p,\widetilde{\varphi}} \|\widetilde{g}\|_{q,\widetilde{\psi}} \\
&= \frac{k}{(1-\lambda_1)(1-\lambda_2)} \left\{\int_1^T x^{p(2-\lambda-\lambda/r)-1} x^{p(\lambda+\lambda/r-2)} dx\right\}^{1/p} \\
&\quad \times \left\{\int_1^T y^{q(2-\lambda-\lambda/s)-1} y^{q(\lambda+\lambda/s-2)} dy\right\}^{1/q} = \frac{k \ln T}{(1-\lambda_1)(1-\lambda_2)}.
\end{aligned} \tag{3.10}$$

By (3.8) and (3.10), we find

$$\frac{1}{\ln T} I_1 - \frac{1}{\ln T}(I_2 + I_3) < k. \tag{3.11}$$

Since by Fubini theorem, we obtain

$$
\begin{aligned}
I_1 &= \int_1^T \frac{1}{x}\int_{1/x}^{T/x} k_\lambda(1,u)u^{\lambda_2-1}du\,dx \\
&= \int_0^1\left(\int_{1/u}^T \frac{1}{x}dx\right)k_\lambda(1,u)u^{\lambda_2-1}du + \int_1^T\left(\int_1^{T/u}\frac{1}{x}dx\right)k_\lambda(1,u)u^{\lambda_2-1}du \\
&= \ln T\left[\int_0^1 k_\lambda(1,u)u^{\lambda_2-1}du + \frac{1}{\ln T}\int_0^1 k_\lambda(1,u)(\ln u)u^{\lambda_2-1}du\right. \\
&\qquad \left. + \int_1^T k_\lambda(1,u)u^{\lambda_2-1}du - \frac{1}{\ln T}\int_1^T k_\lambda(1,u)(\ln u)u^{\lambda_2-1}du\right], \\
0 \le I_2 &= T^{\lambda_1-1}\int_1^T \frac{1}{x^{\lambda_1}}\int_{1/x}^{T/x} k_\lambda(1,u)u^{\lambda_2-1}du dx \\
&= T^{\lambda_1-1}\left[\int_0^1\left(\int_{1/u}^T \frac{1}{x^{\lambda_1}}dx\right)k_\lambda(1,u)u^{\lambda_2-1}du\right. \\
&\qquad \left. + \int_1^T\left(\int_1^{T/u}\frac{1}{x^{\lambda_1}}dx\right)k_\lambda(1,u)u^{\lambda_2-1}du\right] \\
&= \frac{1}{1-\lambda_1}\left\{\int_0^1\left[1+\left(\frac{u}{T}\right)^{1-\lambda_1}\right]k_\lambda(1,u)u^{\lambda_2-1}du\right. \\
&\qquad \left. + \int_1^T\left[1-\left(\frac{u}{T}\right)^{1-\lambda_1}\right]k_\lambda(1,u)u^{\lambda_2-1}du\right\} \\
&\le \frac{1}{1-\lambda_1}\left\{2\int_0^1 k_\lambda(1,u)u^{\lambda_2-1}du + \int_1^\infty k_\lambda(1,u)u^{\lambda_2-1}du\right\} < \infty, \\
0 \le I_3 &\le \frac{1}{1-\lambda_2}\left\{2\int_0^1 k_\lambda(u,1)u^{\lambda_1-1}du + \int_1^\infty k_\lambda(u,1)u^{\lambda_1-1}du\right\} < \infty,
\end{aligned}
\tag{3.12}
$$

then for $T \to \infty$ in (3.10), by Lemma 2.1, we obtain $k(\lambda_1) = \int_0^\infty k(1,u)u^{\lambda_2-1}du \le k$. Hence $k = k(\lambda_1)$, and then $k(\lambda_1)/(1-\lambda_1)(1-\lambda_2)$ is the best value of (3.4).

We conclude that the constant factor in (2.6) is the best possible, otherwise we can get a contradiction by (1.2) that the constant factor in (3.4) is not the best possible. By the same way, if the constant factor in (2.7) is not the best possible, then by (2.13), we can get a contradiction that the constant factor in (2.6) is not the best possible. Therefore in view of (3.3), we have (3.5). □

Corollary 3.2. *For $\lambda = 1$, $\lambda_1 = 1/q$, $\lambda_2 = 1/p$, $\widetilde{F}_1(x) := \int_x^\infty (1/t)f(t)dt$, $\widetilde{G}_1(y) := \int_y^\infty (1/t)g(t)dt$, in* (2.6), (2.7) *and* (3.4), *one has the following basic Hilbert-Hardy-type integral inequalities with the best constant factors:*

$$\iint_0^\infty k_1(x,y)\widetilde{F}_1(x)\widetilde{G}_1(y)dx\,dy < pk_p\|f\|_p\left\|\widetilde{G}_1\right\|_q, \tag{3.13}$$

$$\left\{\int_0^\infty \left[\int_0^\infty k_1(x,y)\widetilde{F}_1(x)dx\right]^p dy\right\}^{1/p} < pk_p\|f\|_p, \tag{3.14}$$

$$\iint_0^\infty k_1(x,y)\widetilde{F}_1(x)\widetilde{G}_1(y)dx\,dy < pqk_p\|f\|_p\|g\|_q, \tag{3.15}$$

where $k_p = k(1/q) = \int_0^\infty k_\lambda(u,1)u^{-1/p}du$, and (3.13) *is equivalent to* (3.14).

Example 3.3. For $p > 1$, $r > 1$, $1/p + 1/q = 1/r + 1/s = 1$, $\lambda_1 = \lambda/r$, and $\lambda_2 = \lambda/s$ in(3.4),

(a) if $0 < \lambda < \max\{r,s\}$, $k_\lambda(x,y) = 1/(x+y)^\lambda$, $1/(\max\{x,y\})^\lambda$ and $\ln(x/y)/(x^\lambda - y^\lambda)$, then we obtain the following integral inequalities:

$$\begin{gathered}\iint_0^\infty \frac{\widetilde{F}_\lambda(x)\widetilde{G}_\lambda(y)}{(x+y)^\lambda}dx\,dy < \frac{rsB(\lambda/r,\lambda/s)}{(r-\lambda)(s-\lambda)}\|f\|_{p,\widetilde{\varphi}}\|g\|_{q,\widetilde{\psi}},\\ \iint_0^\infty \frac{\widetilde{F}_\lambda(x)\widetilde{G}_\lambda(y)}{(\max\{x,y\})^\lambda}dx\,dy < \frac{r^2s^2}{\lambda(r-\lambda)(s-\lambda)}\|f\|_{p,\widetilde{\varphi}}\|g\|_{q,\widetilde{\psi}},\\ \iint_0^\infty \frac{\ln(x/y)\widetilde{F}_\lambda(x)\widetilde{G}_\lambda(y)}{x^\lambda - y^\lambda}dx\,dy < \frac{rs[\pi\csc(\pi/r)]^2}{\lambda^2(r-\lambda)(s-\lambda)}\|f\|_{p,\widetilde{\varphi}}\|g\|_{q,\widetilde{\psi}};\end{gathered} \tag{3.16}$$

(b) if $0 < \lambda < 1, k_\lambda(x,y) = 1/|x-y|^\lambda$, then we have

$$\iint_0^\infty \frac{\widetilde{F}_\lambda(x)\widetilde{G}_\lambda(y)}{|x-y|^\lambda}dx\,dy < \frac{rs[B(1-\lambda,\lambda/r) + B(1-\lambda,\lambda/s)]}{(r-\lambda)(s-\lambda)}\|f\|_{p,\widetilde{\varphi}}\|g\|_{q,\widetilde{\psi}}; \tag{3.17}$$

(c) if $\lambda < 0, k_\lambda(x,y) = (\min\{x,y\})^{-\lambda}$, then we find

$$\iint_0^\infty \frac{\widetilde{F}_\lambda(x)\widetilde{G}_\lambda(y)}{(\min\{x,y\})^\lambda}dx\,dy < \frac{-r^2s^2}{\lambda(r-\lambda)(s-\lambda)}\|f\|_{p,\widetilde{\varphi}}\|g\|_{q,\widetilde{\psi}}, \tag{3.18}$$

where the constant factors in the above inequalities are the best possible.

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