Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 790730, 11 pages
doi:10.1155/2010/790730

Research Article

Some Applications of Srivastava-Attiya Operator to p-Valent Starlike Functions

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Received 25 March 2010; Accepted 14 July 2010

Academic Editor: Ram N. Mohapatra

We introduce and study some new subclasses of p-valent starlike, convex, close-to-convex, and quasi-convex functions defined by certain Srivastava-Attiya operator. Inclusion relations are established, and integral operator of functions in these subclasses is discussed.

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in N = \{1,2,3,\ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let the Hadamard product or (convolution) of two functions

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{n+p,j} z^{n+p} \quad (j = 1,2) \tag{1.2}$$

be given by $(f_1 * f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p,1} a_{n+p,2} z^{n+p} = (f_2 * f_1)(z)$.

A function $f(z) \in A(p)$ is said to be in the class $S_p^*(\alpha)$ of p-valent functions of order α if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (0 \le \alpha < p, z \in U). \tag{1.3}$$

we write $S_p^*(0) = S_p^*$, the class of p-valent starlike in U.

A function $f \in A(p)$ is said to be in the class $C_p(\alpha)$ of p-valent convex functions of order α if it satisfies

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \quad (0\le\alpha<p, z\in U). \tag{1.4}$$

The class of p-valent convex functions in U is denoted by $C_p = C_p(0)$.

It follows from (1.3) and (1.4) that

$$f(z)\in C_p(\alpha) \text{ iff } \frac{zf'(z)}{p}\in S_p^*(\alpha) \quad (0\le\alpha<p). \tag{1.5}$$

The classes S_p^* and C_p were introduced by Goodman [1]. Furthermore, a function $f(z) \in A(p)$ is said to be p-valent close-to-convex of order β and type γ in U if there exists a function $g(z) \in S_p^*(\gamma)$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right)>\beta \quad (0\le\beta,\gamma<p, z\in U). \tag{1.6}$$

We denote this class by $K_p(\beta,\gamma)$ The class $K_p(\beta,\gamma)$ was studied by Aouf [2]. We note that $K_1(\beta,\gamma) = K(\beta,\gamma)$ was studied by Libera [3].

A function $f \in A(p)$ is called quasi-convex of order β type γ, if there exists a function $g(z) \in C_p(\gamma)$ such that

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\}>\beta, \quad z\in U, \tag{1.7}$$

where $0 \le \beta, \gamma < p$. We denote this class by $K_p^*(\beta,\gamma)$. Clearly $f(z) \in K_p^*(\beta,\gamma) \Leftrightarrow zf'(z)/p \in K_p(\beta,\gamma)$.The generalized Srivastava-Attiya operator $J_{s,b}f(z) : A(p) \to A(p)$ in [4] is introduced by

$$J_{s,b}f(z) = G_{s,b}(z) * f(z) \quad \left(z\in U : b\in\mathbb{C}\setminus\overline{Z}_0=\{0,-1,-2,-3,\ldots\},\ s\in\mathbb{C},\ p\in N\right) \tag{1.8}$$

where

$$G_{s,b}(z) = (1+b)^s\left[\phi(z,s,b)-b^{-s}\right],$$
$$\phi(z,s,b)=\frac{1}{b^s}+\frac{z^p}{(1+b)^s}+\frac{z^{1+p}}{(2+b)^s}+\cdots. \tag{1.9}$$

It is not difficult to see from (1.8) and (1.9) that

$$J_{s,b}f(z) = z^p+\sum_{n=1}^{\infty}\left(\frac{1+b}{n+1+b}\right)^s a_{n+p}z^{n+p}. \tag{1.10}$$

When $p = 1$, the operator $J_{s,b}$ is well-known Srivastava-Attiya operator [5].

Using the operator $J_{s,b}$, we now introduce the following classes:

$$\begin{aligned} S^*_{p,s,b}(\gamma) &= \left\{ f(z) \in A(p) : J_{s,b}f(z) \in S^*_p(\gamma) \right\}, \\ C_{p,s,b}(\gamma) &= \{ f(z) \in A(p) : J_{s,b}f(z) \in C_p(\gamma) \}, \\ K_{p,s,b}(\beta,\gamma) &= \{ f(z) \in A(p) : J_{s,b}f(z) \in K_p(\beta,\gamma) \}, \\ K^*_{p,s,b}(\beta,\gamma) &= \left\{ f(z) \in A(p) : J_{s,b}f(z) \in K^*_p(\beta,\gamma) \right\}. \end{aligned} \tag{1.11}$$

In this paper, we will establish inclusion relation for these classes and investigate Srivastava-Attiya operator for these classes.

We note that

(1) for $s = \sigma,\ b = p$, we get Jung-Kim-Srivastava ([6, 7]);

(2) for $s = 1,\ 1 + b = c + p$, we get the generalized Libera integral operator. [8, 9];

(3) for $s = -k$ being any negative integer, $b = 0$, and $p = 1$, the operator $J_{-k,0} = D^k f(z)$ was studied by Sălăgean [10].

2. Inclusion Relation

In order to prove our main results, we will require the following lemmas.

Lemma 2.1 (see [11]). *Let $w(z)$be regular in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a given point $z_0 \in U$, then $z_0 w'(z_0) = k w(z_0)$, where k is a real number and* $k \geq 1$.

Lemma 2.2 (see [12]). *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$, and let $\psi(u,v)$ be a complex function, $\psi : D \to \mathbb{C}$, $D \subset \mathbb{C} \times \mathbb{C}$. Suppose that ψ satisfies the following conditions:*

(i) *$\psi(u,v)$ is continuous in D,*

(ii) $(1,0) \in D$ *and* $\operatorname{Re}\{\psi(1,0)\} > 0$,

(iii) $\operatorname{Re}\{\psi(iu_2, v_1)\} \leq 0$ *for al$(iu_2,\ v_1) \in D$ such that $v_1 \leq -(1+u_2^2)/2$.*

Let $h(z) = 1 + c_1 z + c_1 z^2 + \cdots$ be analytic in U, such that $(h(z), zh'(z)) \in D$ for $z \in U$. If $\operatorname{Re}\{\psi(h(z), zh'(z))\} > 0, (z \in U)$ *then* $\operatorname{Re} h(z) > 0$ *for $z \in U$.*

Our first inclusion theorem is stated as follows.

Theorem 2.3. *$S^*_{p,s.b}(\gamma) \subset S^*_{p,s+1,b}(\gamma)$ for any complex number s.*

Proof. Let $f(z) \in S^*_{p,s,b}(\gamma)$, and set

$$\frac{z\left(J_{s+1,b}f(z)\right)'}{J_{s+1,b}f(z)} - \gamma = (p-\gamma)h(z), \tag{2.1}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Using the identity

$$z\left(J_{s+1,b}f(z)\right)' = \left(p - (1+b)\right) J_{s+1,b}f(z) + (1+b)J_{s,b}f(z), \tag{2.2}$$

we have

$$\begin{aligned} \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} &= \frac{1}{1+b}\left(\frac{z\left(J_{s+1,b}f(z)\right)'}{J_{s+1,b}f(z)} - p + b + 1\right), \\ \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} &= \frac{1}{b+1}\left(\gamma + (p-\gamma)h(z) - p + b + 1\right). \end{aligned} \tag{2.3}$$

Differentiating (2.3), logarithmically with respect to z, we obtain

$$\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)} - \gamma = (p-\gamma)h(z) + \frac{(p-\gamma)zh'(z)}{(p-\gamma)h(z) + \gamma - p + b + 1}. \tag{2.4}$$

Now, from the function $\psi(u,v)$, by taking $u = h(z),\ v = zh'(z)$ in (2.4) as

$$\psi(u,v) = (p-\gamma)u + \frac{(p-\gamma)v}{(p-\gamma)u + \gamma - p + b + 1}, \tag{2.5}$$

it is easy to see that the function $\psi(u,v)$ satisfies condition (i) and (ii) of Lemma 2.2, in $D = (\mathbb{C} - \{(\gamma - p + b + 1)/(\gamma - p)\}) \times \mathbb{C}$. To verify condition (iii), we calculate as follows:

$$\begin{aligned} \operatorname{Re}\{\psi(iu_2, v_1)\} &= \operatorname{Re}\left\{\frac{(p-\gamma)v_1}{(p-\gamma)iu_2 + \gamma - p + b + 1}\right\} \\ &= \operatorname{Re}\left\{\frac{(p-\gamma)v_1\left[(\gamma - p + b + 1) - i(p-\gamma)u_2\right]}{(p-\gamma)^2 u_2^2 + (1 - p + b + \gamma)^2}\right\} \\ &= \operatorname{Re}\left\{\frac{(p-\gamma)(\gamma - p + b + 1)v_1 - i(p-\gamma)^2 v_1 u_2}{(p-\gamma)^2 u_2^2 + (1 - p + b + \gamma)^2}\right\} \\ &= \frac{(p-\gamma)(\gamma - p + b + 1)v_1}{(p-\gamma)^2 u_2^2 + (1 - p + b + \gamma)^2} \\ &\le -\frac{(p-\gamma)(\gamma - p + b + 1)(1 + u_2^2)}{2\left[(p-\gamma)^2 u_2^2 + (1 - p + b + \gamma)^2\right]} < 0, \end{aligned} \tag{2.6}$$

where $v_1 \le -(1 + u_2^2)/2$ and $(iu_2, v_1) \in D$. Therefore, the function $\psi(u,v)$ satisfies the conditions of Lemma 2.2.

This shows that if $\operatorname{Re}(h(z), zh'(z)) > 0\ (z \in U)$, then

$$\operatorname{Re}(h(z)) > 0, \quad (z \in U). \tag{2.7}$$

if $f \in S_s^*(\gamma)$, then

$$S^*_{p,s,b}(\gamma) \subset S^*_{p,s+1,b}(\gamma). \tag{2.8}$$

This completes the proof of Theorem 2.3. □

Theorem 2.4. *$C_{p,s,b}(\gamma) \subset C_{p,s+1,b}(\gamma)$, for any complex number s.*

Proof. Consider the following:

$$\begin{aligned}
f(z) \in C_{p,s,b}(\gamma) &\Longleftrightarrow J_{s,b}f(z) \in C_p(\gamma) \Longleftrightarrow \frac{z}{p}(J_{s,b}f(z))' \in S_p^*(\gamma) \\
&\Longleftrightarrow J_{s,b}\left(\frac{zf'(z)}{p}\right) \in S_p^*(\gamma) \Longleftrightarrow \frac{zf'(z)}{p} \in S^*_{p,s,b}(\gamma) \\
&\Longrightarrow \frac{zf'(z)}{p} \in S^*_{p,s+1,b}(\gamma) \Longleftrightarrow J_{s+1,b}\left(\frac{zf'(z)}{p}\right) \in S_p^*(\gamma) \\
&\Longleftrightarrow \frac{z}{p}(J_{s+1,b}f(z))' \in S_p^*(\gamma) \Longleftrightarrow J_{s+1,b}f(z) \in C_p(\gamma) \\
&\Longleftrightarrow f(z) \in C_{p,s+1,b}(\gamma),
\end{aligned} \tag{2.9}$$

which evidently proves Theorem 2.4. □

Theorem 2.5. *$K_{p,s,b}(\beta,\gamma) \subset K_{p,s+1,b}(\beta,\gamma)$, for any complex number s.*

Proof. Let $f(z) \in K_{p,s,b}(\beta,\gamma)$. Then, there exists a function $k(z) \in S_p^\star(\gamma)$ such that

$$\operatorname{Re}\left\{\frac{z(J_{s,b}f(z))'}{g(z)}\right\} > \beta \quad (z \in U). \tag{2.10}$$

Taking the function $k(z)$ which satisfies $J_{s,b}k(z) = g(z)$, we have $k(z) \in S_p^\star(\gamma)$ and $\operatorname{Re}\{z(J_{s,b}f(z))'/J_{s,b}k(z)\} > \beta$ $(z \in U)$.

Now, put $z(J_{s+1,b}f(z))'/(J_{s+1,b}k(z)) - \beta = (p-\beta)h(z)$, where $h(z) = 1 + c_1z + c_2z^2 + \cdots$. Using the identity (2.2) we have

$$\begin{aligned}
\frac{z(J_{s,b}f(z))'}{J_{s,b}k(z)} &= \frac{J_{s,b}\left(zf'(z)\right)}{J_{s,b}k(z)} \\
&= \frac{z(J_{s+1,b}(zf')(z))' - (p-(1+b))J_{s+1,b}(zf')(z)}{z(J_{s+1,b}k(z))' - (p-(1+b))J_{s+1,b}k(z)} \\
&= \frac{z\left(J_{s+1,b}\left(zf'\right)(z)\right)'/J_{s+1,b}k(z) - (p-(1+b))J_{s+1,b}\left(zf'\right)(z)/J_{s+1,b}k(z)}{z(J_{s+1,b}k(z))'/J_{s+1,b}k(z) - (p-(1+b))}.
\end{aligned} \tag{2.11}$$

Since $k(z) \in S^*_{p,s,b}(\gamma)$ and $S^*_{p,s,b}(\gamma) \subset S^*_{p,s+1,b}(\gamma)$, we let $z(J_{s+1,b}k(z))'/J_{s+1,b}k(z) = (p-\gamma)H(z) + \gamma$, where $\operatorname{Re} H(z) > 0$ $(z \in U)$ thus (2.11) can be written as

$$\frac{z(J_{s,b}f(z))'}{J_{s,b}k(z)} = \frac{z(J_{s+1,b}(zf')(z))'/J_{s+1,b}k(z) - (p-(1+b))[\beta + (p-\beta)h(z)]}{(p-\gamma)H(z) + \gamma - [p-(1+b)]}. \tag{2.12}$$

Consider that

$$z(J_{s+1,b}f(z))' = J_{s+1,b}k(z)[\beta + (p-\beta)h(z)]. \tag{2.13}$$

Differentiating both sides of (2.13), and multiplying by z, we have

$$\frac{z(J_{s+1,b}(zf')(z))'}{J_{s+1,b}k(z)} = (p-\beta)zh'(z) + (\beta + (p-\beta)h(z)) \cdot [(p-\gamma)H(z) + \gamma]. \tag{2.14}$$

Using (2.14) and (2.12), we get

$$\frac{z(J_{s,b}f(z))'}{J_{s,b}k(z)} - \beta = (p-\beta)h(z) + \frac{(p-\beta)zh'(z)}{(p-\gamma)H(z) + \gamma - (p-(1+b))}. \tag{2.15}$$

Taking $u = h(z), v = zh'(z)$ in (2.15), we form the function $\psi(u,v)$ as

$$\psi(u,v) = (p-\beta)u + \frac{(p-\beta)v}{(p-\gamma)H(z) + \gamma - [p-(1+b)]}. \tag{2.16}$$

It is not difficult to see that $\psi(u,v)$ satisfies the conditions (i) and (ii) of Lemma 2.2 in $D = \mathbb{C} \times \mathbb{C}$. To verify condition (iii), we proceed as follows:

$$\operatorname{Re}\psi(iu_2, v_1) = \frac{(p-\beta)v_1[(p-\gamma)h_1(x,y) + \gamma - [p-(1+b)]]}{[(p-\gamma)h_1(x,y) + \gamma + (1+b) - p]^2 + [(p-\gamma)h_2(x,y)]^2}, \tag{2.17}$$

where $H(z) = h_1(x,y) + ih_2(x,y)$, $h_1(x,y)$ and $h_2(x,y)$ being the functions of x and y and $\operatorname{Re} H(z) = h_1(x,y) > 0$.

By putting $v_1 \le -(1/2)(1+u_2^2)$, we have

$$\operatorname{Re}\psi(iu_2, v_1) \le -\frac{(p-\beta)(1+u_2^2)[(p-\gamma)h_1(x,y) + \gamma - [p-(1+b)]]}{2\left\{[(p-\gamma)h_1(x,y) + \gamma + (1+b) - p]^2 + [(p-\gamma)h_2(x,y)]^2\right\}} < 0. \tag{2.18}$$

Hence, $\operatorname{Re} h(z) > 0$ $(z \in U)$ and $f(z) \in K_{p,s+1,b}(\beta,\gamma)$. The proof of Theorem 2.5 is complete. □

Theorem 2.6. $K^*_{p,s,b}(\beta,\gamma) \subset K^*_{p,s+1,b}(\beta,\gamma)$ *for any complex number s.*

Proof. Consider the following:

$$
\begin{aligned}
f(z) \in K^{*}_{p,s,b}(\beta,\gamma) &\Longleftrightarrow J_{s,b}f(z) \in K^{*}_{p}(\beta,\gamma) \\
&\Longleftrightarrow \frac{z}{p}\left(J_{s,b}f(z)\right)' \in K_{p}(\beta,\gamma) \\
&\Longleftrightarrow J_{s,b}\left(\frac{zf'(z)}{p}\right) \in K_{p}(\beta,\gamma) \Longrightarrow \frac{zf'(z)}{p} \in K_{p,s,b}(\beta,\gamma) \\
&\Longrightarrow \frac{zf'(z)}{p} \in K_{p,s+1,b}(\beta,\gamma) \Longleftrightarrow J_{s+1,b}\left(\frac{zf'(z)}{p}\right) \in K_{p}(\beta,\gamma) \\
&\Longleftrightarrow \frac{z}{p}\left(J_{s+1,b}f(z)\right)' \in K_{p}(\beta,\gamma) \\
&\Longleftrightarrow J_{s+1,b}f(z) \in K^{*}_{p}(\beta,\gamma) \Longrightarrow f(z) \in K^{*}_{p,s+1,b}(\beta,\gamma).
\end{aligned}
\tag{2.19}
$$

The proof of Theorem 2.6 is complete. □

3. Integral Operator

For $c > -1$ and $f(z) \in A(p)$, we recall here the generalized Bernardi-Libera-Livingston integral operator $L_c f(z)$ as follows

$$
\begin{aligned}
L_c f(z) &= \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \\
&= z^p + \sum_{n=1}^{\infty}\left(\frac{c+p}{c+p+n}\right) a_{n+p} z^{n+p}.
\end{aligned}
\tag{3.1}
$$

The operator $L_c(f(z))$ when $c \in N = \{1,2,3,\ldots\}$ was studied by Bernardi [13], for $c = 1$, $L_1(f(z))$ was investigated earlier by Libera [14]. Now, we have

$$
J_{s,b}(L_c f(z)) = z^p \sum_{n=1}^{\infty}\left(\frac{1+b}{1+b+n}\right)^s \left(\frac{c+p}{c+p+n}\right) a_{n+p} z^{n+p}, \tag{3.2}
$$

so we get the identity

$$
z\left(J_{s,b}(L_c f(z))\right)' = (c+p) J_{s,b} f(z) - c\left(L_c f(z)\right). \tag{3.3}
$$

The following theorems deal with the generalized Bernard-Libera-Livingston integral operator $L_c(f(z))$ defined by (3.1).

Theorem 3.1. *Let $c > -\gamma$, $0 \le \gamma < p$. If $f(z) \in S^{*}_{p,s,b}(\gamma)$, then $L_c f(z) \in S^{*}_{p,s,b}(\gamma)$.*

Proof. From (3.3), we have

$$\frac{z\left(J_{s,b}\left(L_c f(z)\right)\right)'}{J_{s,b}L_c f(z)} = \frac{(c+p)J_{s,b}f(z)}{J_{s,b}\left(L_c f(z)\right)} - c = \frac{1+(1-2\gamma)\omega(z)}{1-\omega(z)}, \tag{3.4}$$

where $w(z)$ is analytic in U, $w(0)=0$. Using (3.3) and (3.4) we get

$$\frac{J_{s,b}f(z)}{J_{s,b}L_c f(z)} = \frac{(c+p)+w(z)(1-c-2\gamma)\omega(z)}{(c+p)(1-\omega(z))}. \tag{3.5}$$

Differentiating (3.5), we obtain

$$\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)} = \frac{1+(1-2\gamma)w(z)}{1-w(z)} - \frac{zw'(z)}{1-w(z)} + \frac{(1-c-2\gamma)zw'(z)}{p+c+(1-c-2\gamma)w(z)}. \tag{3.6}$$

Now we assume that $|w(z)|<1$ $(z\in U)$. Otherwise, there exists a point $z_0 \in U$ such that $\max|w(z)| = |w(z_0)| = 1$. Then by Lemma 2.1, we have $z_0 w'(z_0) = kw(z_0)$, $k \geq 1$. Putting $z=z_0$ and $w(z_0)=e^{i\theta}$ in (3.6), we have

$$\begin{aligned}\operatorname{Re}\left\{\frac{z_0\left(J_{s,b}f(z_0)\right)'}{J_{s,b}f(z_0)} - \gamma\right\} &= \operatorname{Re}\left\{\frac{2(1-\gamma)ke^{i\theta}}{\left(1-e^{i\theta}\right)\left(p+c+(1-c-2\gamma)e^{i\theta}\right)}\right\} \\ &= \frac{-2k(1-\gamma)(c+\gamma)}{(1+c)^2+2(1+c)(1-c-2\gamma)\cos\theta+(1-c-2\gamma)^2} \leq 0,\end{aligned} \tag{3.7}$$

which contradicts the hypothesis that $f(z)\in S^*_{p,s,b}(\gamma)$.

Hence, $|w(z)|<1$, for $z\in U$, and it follows (3.4) that $L_c f \in S^*_{p,s,b}(\gamma)$.

The proof of Theorem 3.1 is complete· □

Theorem 3.2. *Let $c>-\gamma, 0\leq\gamma<p$. If $f\in C_{p,s,b}(\gamma)$, then $L_c f(z)\in C_{p,s,b}(\gamma)$.*

Proof. Consider the following:

$$\begin{aligned} f(z)\in C_{p,s,b}(\gamma) &\Longleftrightarrow \frac{zf'(z)}{p} \in S^*_{p,s,b}(\gamma) \\ &\Longrightarrow L_c\left(\frac{zf'(z)}{p}\right) \in S^*_{p,s,b}(\gamma) \Longleftrightarrow \frac{z}{p}\left(L_c f(z)\right)' \in S^*_{p,s,b}(\gamma) \\ &\Longleftrightarrow L_c f(z) \in C_{p,s,b}(\gamma). \end{aligned} \tag{3.8}$$

This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let $c>-\gamma, 0\leq\gamma<p$.If $f(z)\in K_{p,s,b}(\beta,\gamma)$ then $L_c(f(z))\in K_{p,s,b}(\beta,\gamma)$.*

Proof. Let $f(z) \in K_{p,s,b}(\beta,\gamma)$. Then, by definition, there exists a function $g(z) \in S^*_{p,s,b}(\gamma)$ such that

$$\operatorname{Re}\left\{ \frac{z(J_{s,b}f(z))'}{J_{s,b}g(z)} \right\} > \beta \quad (z \in U). \tag{3.9}$$

Then,

$$\frac{z(J_{s,b}L_c f(z))'}{J_{s,b}L_c g(z)} - \beta = (p-\beta)h(z) \tag{3.10}$$

where $h(z) = c_1 z + c_2 z^2 + \cdots$. From (3.3) and (3.10), we have

$$\begin{aligned} \frac{z(J_{s,b}f(z))'}{J_{s,b}g(z)} &= \frac{J_{s,b}(zf'(z))}{J_{s,b}g(z)} = \frac{z(J_{s,b}L_c(zf'(z)))' + cJ_{s,b}L_c(zf')(z)}{z(J_{s,b}L_c(g(z)))' + cJ_{s,b}L_c g(z)} \\ &= \frac{z(J_{s,b}L_c zf'(z))'/J_{s,b}L_c(g(z)) + cJ_{s,b}L_c(zf'(z))/J_{s,b}L_c(g(z))}{z(J_{s,b}L_c g(z))'/J_{s,b}L_c(g(z)) + c}. \end{aligned} \tag{3.11}$$

Since $g(z) \in S^*_{p,s,b}(\gamma)$, then from Theorem 3.1, we have $L_c(g) \in S^*_{p,s,b}(\gamma)$.

Let

$$\frac{z(J_{s,b}L_c(g(z)))'}{J_{s,b}L_c(g(z))} = (p-\gamma)H(z) + \gamma, \tag{3.12}$$

where $\operatorname{Re} H(z) > 0 (z \in U)$. Using (3.11), we have

$$\frac{z(J_{s,b}f(z))'}{J_{s,b}g(z)} = \frac{z(J_{s,b}L_c(zf'(z)))'/J_{s,b}L_c(g) + c((p-\beta)h(z) + \beta)}{(p-\gamma)H(z) + \gamma + c}. \tag{3.13}$$

Also, (3.10) can be written as

$$z(J_{s,b}L_c(f(z)))' = J_{s,b}L_c(g(z))((p-\beta)h(z) + \beta). \tag{3.14}$$

Differentiating both sides, we have

$$z\left\{z(J_{s,b}L_c f(z))'\right\}' = z(J_{s,b}L_c g(z))'((p-\beta)h(z) + \beta) + (p-\beta)zh'(z)J_{s,b}L_c g(z), \tag{3.15}$$

or

$$\begin{aligned} \frac{z\left\{z(J_{s,b}L_c f(z))'\right\}'}{J_{s,b}L_c(g(z))} &= \frac{z(J_{s,b}L_c(zf'(z)))'}{J_{s,b}L_c(g(z))} \\ &= (p-\beta)zh'(z) + ((p-\beta)h(z) + \beta)((1-\gamma)H(z) + \gamma). \end{aligned} \tag{3.16}$$

Now, from (3.13) we have

$$\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}g(z)}-\beta=(p-\beta)h(z)+\frac{(p-\beta)zh'(z)}{(p-\gamma)H(z)+\gamma+c}. \tag{3.17}$$

We form the function $\psi(u,v)$ by taking $u=h(z),\ v=zh'(z)$ in (3.17) as follows

$$\psi(u,v)=(p-\beta)u+\frac{(p-\beta)v}{(p-\gamma)H(z)+\gamma+c}. \tag{3.18}$$

It is clear that the function $\psi(u,v)$ defined in $D=\mathbb{C}\times\mathbb{C}$ by (3.18) satisfies conditions (i) and (ii) of Lemma 2.2. To verify the condition(iii), we proceed as follows:

$$\operatorname{Re}\psi(iu_2,v_1)=\frac{(p-\beta)v_1\left[(p-\gamma)h_1(x,y)+\gamma+c\right]}{\left[(p-\gamma)h_1(x,y)+\gamma+c\right]^2+\left[(p-\gamma)h_2(x,y)\right]^2}, \tag{3.19}$$

where $H(z)=h_1(x,y)+ih_2(x,y), h_1(x,y)$ and $h_2(x,y)$ being the functions of x and y and $\operatorname{Re}H(z)=h_1(x,y)>0$.

By putting $v_1\le-(1/2)(1+u_2^2)$, we have

$$\operatorname{Re}\psi(iu_2,v_1)\le-\frac{(p-\beta)(1+u_2^2)\left[(p-\gamma)h_1(x,y)+\gamma+c\right]}{2\left\{\left[(p-\gamma)h_1(x,y)+\gamma+c\right]^2+\left[(p-\gamma)h_2(x,y)\right]^2\right\}}<0. \tag{3.20}$$

Hence, $\operatorname{Re}h(z)>0(z\in U)$ and $L_cf(z)\in K_{p,s,b}(\beta,\gamma)$. Thus, we have $L_cf(z)\in K_{p,s,b}(\beta,\gamma)$. The proof of Theorem 3.3 is complete. □

Theorem 3.4. *Let $c>-\gamma,\ 0\le\gamma<p$. If $f(z)\in K^*_{p,s,b}(\beta,\gamma)$, then $L_cf(z)\in K^*_{p,s,b}(\beta,\gamma)$.*

Proof. Consider the following:

$$\begin{aligned}f(z)\in K^*_{p,s,b}(\beta,\gamma)&\Longleftrightarrow zf'(z)\in K_{p,s,,b}(\beta,\gamma)\\&\Longrightarrow L_c\left(zf'(z)\right)\in K_{p,s,b}(\beta,\gamma)\\&\Longleftrightarrow z\left(L_cf(z)\right)'\in K_{p,s,b}(\beta,\gamma)\\&\Longleftrightarrow L_cf(z)\in K^*_{p,s,b}(\beta,\gamma),\end{aligned} \tag{3.21}$$

and the proof of Theorem 3.4 is complete. □

Acknowledgement

The authors would like to thank the referees of the paper for their helpful suggestions.

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