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Research Article

Superstability of Generalized Derivations

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We investigate the superstability of the functional equation $f(xy) = xf(y) + g(x)y$, where f and g are the mappings on Banach algebra A. We have also proved the superstability of generalized derivations associated to the linear functional equation $f(\gamma x+\beta y) = \gamma f(x)+\beta f(y)$, where $\gamma, \beta \in \mathbb{C}$.

1. Introduction

The well-known problem of stability of functional equations started with a question of Ulam [1] in 1940. In 1941, Ulam's problem was solved by Hyers [2] for Banach spaces. Aoki [3] provided a generalization of Hyers' theorem for approximately additive mappings. In 1978, Rassias [4] generalized Hyers' theorem by obtaining a unique linear mapping near an approximate additive mapping.

Assume that E_1 *and* E_2 are real normed spaces with E_2 complete, $f : E_1 \to E_2$ is a mapping such that for each fixed $x \in E_1$ the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, and there exist $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

for all $x, y \in E_1$. Then there exists a unique linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{|2-2^p|}\|x\|^p \tag{1.2}$$

for all $x \in E_1$.

In 1994, Găvruţa [5] provided a generalization of Rassias' theorem in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ in (1.1) by a general control function $\phi(x, y)$.

Since then several stability problems for various functional equations have been investigated by many mathematicians (see [6–8]).

The various problems of the stability of derivations and generalized derivations have been studied during the last few years (see, e.g., [9–18]). The purpose of this paper is to prove the superstability of generalized (ring) derivations on Banach algebras.

The following result which is called the superstability of ring homomorphisms was proved by Bourgin [19] in 1949.

Suppose that A and B are Banach algebras and B is with unit. If $f : A \to B$ is surjective mapping and there exist $\epsilon > 0$ and $\delta > 0$ such that

$$\|f(a+b) - f(a) - f(b)\| \le \epsilon, \qquad \|f(ab) - f(a)f(b)\| \le \delta \tag{1.3}$$

for all $a, b \in A$, then f is a ring homomorphism, that is,

$$f(a+b) = f(a) + f(b), \qquad f(ab) = f(a)f(b). \tag{1.4}$$

The first superstability result concerning derivations between operator algebras was obtained by Šemrl in [20]. In [10], Badora proved the superstability of functional equation $f(xy) = xf(y) + f(x)y$, where f is a mapping on normed algebra A with unit. In Section 2, we generalize Badora's result [10, Theorem 5] for functional equations

$$f(xy) = xf(y) + g(x)y, \tag{1.5}$$

$$f(xy) = xf(y) + yg(x) \tag{1.6}$$

where f and g are mappings on algebra A with an approximate identity.

In [21, 22], the superstability of generalized derivations on Banach algebras associated to the following Jensen type functional equation:

$$f\left(\frac{x+y}{k}\right) = \frac{f(x)}{k} + \frac{f(y)}{k}, \tag{1.7}$$

where $k > 1$ is an integer is considered. Several authors have studied the stability of the general linear functional equation

$$f(\gamma x + \beta y) = Af(x) + Bf(y), \tag{1.8}$$

where γ, β, A, and B are constants in the field and f is a mapping between two Banach spaces (see [23, 24]). In Section 3, we investigate the superstability of generalized (ring) derivations associated to the linear functional equation

$$f(\gamma x + \beta \mathrm{y}) = \gamma f(x) + \beta f(y), \tag{1.9}$$

where $\gamma, \beta \in \mathbb{C}$. Our results in this section generalize some results of Moslehian's paper [14]. It has been shown by Moslehian [14, Corollary 2.4] that for an approximate generalized derivation f on a Banach algebra A, there exists a unique generalized derivation μ near f. We show that the approximate generalized derivation f is a generalized derivation (see Corollary 3.6).

Let A be an algebra. An additive map $\delta : A \to A$ is said to be ring derivation on A if $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in A$. Moreover, if $\delta(\lambda x) = \lambda\delta(x)$ for all $\lambda \in \mathbb{C}$, then δ is a derivation. An additive mapping (resp., linear mapping) $\mu : A \to A$ is called a generalized ring derivation (resp., generalized derivation) if there exists a ring derivation (resp., derivation) $\delta : A \to A$ such that $\mu(xy) = x\mu(y) + \delta(x)y$ for all $a, b \in A$.

2. Superstability of (1.5) and (1.6)

Here we show the superstability of the functional equations (1.5) and (1.6). We prove the superstability of (1.6) without any additional conditions on the mapping g.

Theorem 2.1. *Let A be a normed algebra with a central approximate identity $(e_\lambda)_{\lambda\in\Lambda}$ and $\alpha \in \mathbb{C}\backslash\{0\}$. Suppose that $f : A \to A$ and $g : A \to A$ are mappings for which there exists $\phi : A \times A \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \alpha^{-n}\phi(\alpha^n a, b) = \lim_{n\to\infty} \alpha^{-n}\phi(a, \alpha^n b) = 0, \tag{2.1}$$

$$\|f(ab) - af(b) - bg(a)\| \le \phi(a, b) \tag{2.2}$$

for all $a, b \in A$. Then $f(ab) = af(b) + bg(a)$ for all $a, b \in A$.

Proof. Replacing a by $\alpha^n a$ in (2.2), we get

$$\|f(\alpha^n ab) - \alpha^n af(b) - bg(\alpha^n a)\| \le \phi(\alpha^n a, b), \tag{2.3}$$

and so

$$\left\|\frac{f(\alpha^n ab)}{\alpha^n} - af(b) - \frac{bg(\alpha^n a)}{\alpha^n}\right\| \le \frac{1}{|\alpha|^n}\phi(\alpha^n a, b) \tag{2.4}$$

for all $a, b \in A$ and $n \in \mathbb{N}$. By taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty}\left(\frac{f(\alpha^n ab)}{\alpha^n} - \frac{bg(\alpha^n a)}{\alpha^n}\right) = af(b) \tag{2.5}$$

for all $a, b \in A$. Similarly, we have

$$\lim_{n\to\infty}\left(\frac{f(\alpha^n ab)}{\alpha^n} - \frac{af(\alpha^n b)}{\alpha^n}\right) = bg(a) \tag{2.6}$$

for all $a, b \in A$.

Let $a, b \in A$ and $\lambda \in \Lambda$. Then we have

$$
\begin{aligned}
&\|f(ab) - af(b) - bg(a)\| \\
&\quad \le \left\| f(ab) - \frac{f(\alpha^n e_\lambda ab)}{\alpha^n} + ab\frac{g(\alpha^n e_\lambda)}{\alpha^n} \right\| \\
&\qquad + \left\| \frac{f(\alpha^n e_\lambda ab)}{\alpha^n} - ab\frac{g(\alpha^n e_\lambda)}{\alpha^n} - af(b) - bg(a) \right\| \\
&\quad \le \left\| f(ab) - \frac{f(\alpha^n e_\lambda ab)}{\alpha^n} + ab\frac{g(\alpha^n e_\lambda)}{\alpha^n} \right\| \\
&\qquad + \left\| \frac{f(\alpha^n e_\lambda ab)}{\alpha^n} + a\frac{f(\alpha^n e_\lambda b)}{\alpha^n} - ab\frac{g(\alpha^n e_\lambda)}{\alpha^n} - a\frac{f(\alpha^n e_\lambda b)}{\alpha^n} - af(b) - bg(a) \right\| \\
&\quad \le \left\| f(ab) - \frac{f(\alpha^n e_\lambda ab)}{\alpha^n} + ab\frac{g(\alpha^n e_\lambda)}{\alpha^n} \right\| \\
&\qquad + \left\| a\left(\frac{f(\alpha^n e_\lambda b)}{\alpha^n} - b\frac{g(\alpha^n e_\lambda)}{\alpha^n} \right) - af(b) \right\| \\
&\qquad + \left\| \frac{f(\alpha^n e_\lambda ab)}{\alpha^n} - a\frac{f(\alpha^n e_\lambda b)}{\alpha^n} - bg(a) \right\|.
\end{aligned}
\tag{2.7}
$$

Since $e_\lambda \in \mathcal{Z}(A)$, we get

$$
\begin{aligned}
\|f(ab) - af(b) - bg(a)\| \le{}& \left\| f(ab) - \frac{f(2^n e_\lambda ab)}{\alpha^n} + ab\frac{g(\alpha^n e_\lambda)}{\alpha^n} \right\| \\
&+ \left\| a\left(\frac{f(\alpha^n e_\lambda b)}{\alpha^n} - b\frac{g(\alpha^n e_\lambda)}{\alpha^n} \right) - af(b) \right\| \\
&+ \left\| \frac{f(\alpha^n a e_\lambda b)}{\alpha^n} - a\frac{f(\alpha^n e_\lambda b)}{\alpha^n} - bg(a) \right\|.
\end{aligned}
\tag{2.8}
$$

By taking the limit as $n \to \infty$, we get

$$
\|f(ab) - af(b) - bg(a)\| \le \|f(ab) - e_\lambda f(ab)\| + \|ae_\lambda f(b) - af(b)\| + \|e_\lambda bg(a) - bg(a)\|.
\tag{2.9}
$$

Therefore, $f(ab) = af(b) + bg(a)$ for all $a, b \in A$. □

Theorem 2.2. *Let A be a normed algebra with a left approximate identity and $\alpha \in \mathbb{C} \setminus \{0\}$. Let $f : A \to A$ and $g : A \to A$ be the mappings satisfying*

$$
\begin{aligned}
\|f(ab) - af(b) - g(a)b\| &\le \phi(a, b), \\
\|g(ab) - ag(b) - g(a)b\| &\le \phi(a, b)
\end{aligned}
\tag{2.10}
$$

for all $a, b \in A$, where $\phi : A \times A \to [0, \infty)$ is a mapping such that

$$\lim_{n\to\infty} |\alpha|^{-n}\phi(\alpha^n z, w) = \lim_{n\to\infty} |\alpha|^{-n}\phi(z, \alpha^n w) = 0 \tag{2.11}$$

for all $z, w \in A$. Then $f(ab) = af(b) + g(a)b$ for all $a, b \in A$.

Proof. Let $x, y, z \in A$. We have

$$\begin{aligned}
&\|zf(xy) - zxf(y) - zg(x)y\| \\
&\quad \le \|zf(xy) + g(z)xy - f(zxy)\| + \|f(zxy) - g(z)xy - zxf(y) - zg(x)y\| \\
&\quad \le \phi(z, xy) + \|f(zxy) - zxf(y) - g(zx)y\| + \|g(zx)y - g(z)xy - zg(x)y\| \\
&\quad \le \phi(z, xy) + \phi(zx, y) + \phi(z, x)\|y\|.
\end{aligned} \tag{2.12}$$

Replacing $\alpha^n z$ by z, we get

$$\|\alpha^n zf(xy) - \alpha^n zxf(y) - \alpha^n zg(x)y\| \le \phi(\alpha^n z, xy) + \phi(\alpha^n zx, y) + \phi(\alpha^n z, x)\|y\|, \tag{2.13}$$

and so

$$\|zf(xy) - zxf(y) - zg(x)y\| \le |\alpha|^{-n}\phi(\alpha^n z, xy) + |\alpha|^{-n}\phi(\alpha^n zx, y) + |\alpha|^{-n}\phi(\alpha^n z, x)\|y\|. \tag{2.14}$$

By taking the limit as $n \to \infty$, we have $zf(xy) = zxf(y) + zg(x)y$. Since A has a left approximate identity, we have $f(xy) = xf(y) + g(x)y$. □

In the next theorem, we prove the superstability of (1.5) with no additional functional inequality on the mapping g.

Theorem 2.3. *Let A be a Banach algebra with a two-sided approximate identity and $\alpha \in \mathbb{C} \setminus \{0\}$. Let $f : A \to A$ and $g : A \to A$ be mappings such that $F(x) := \lim_{n\to\infty}(f(\alpha^n x)/\alpha^n)$ exists for all $x \in A$ and*

$$\|f(zw) - zf(w) - g(z)w\| \le \phi(z, w) \tag{2.15}$$

for all $z, w \in A$, where $\phi : A \times A \to [0, \infty)$ is a function such that

$$\lim_{n\to\infty} |\alpha|^{-n}\phi(\alpha^n z, w) = \lim_{n\to\infty} |\alpha|^{-n}\phi(z, \alpha^n w) = 0, \tag{2.16}$$

for all $z, w \in A$. Then $F = f$, $f(zw) = zf(w) + g(z)w$, and $g(zw) = zg(w) + g(z)w$.

Proof. Replacing $\alpha^n z$ by z in (2.15), we get

$$\left\|f(\alpha^n zw)-\alpha^n zf(w)-g(\alpha^n z)w\right\|\le\phi(\alpha^n z,w),\tag{2.17}$$

and so

$$\left\|\frac{f(\alpha^n zw)}{\alpha^n}-zf(w)-\frac{g(\alpha^n z)}{\alpha^n}w\right\|\le\frac{1}{|\alpha|^n}\phi(\alpha^n z,w)\tag{2.18}$$

for all $z,w\in A$ and $n\in\mathbb{N}$. By taking the limit as $n\to\infty$, we have

$$F(zw)=zf(w)+\lim_{n\to\infty}\frac{g(\alpha^n z)}{\alpha^n}w\tag{2.19}$$

for all $z,w\in A$.

Fix $m\in\mathbb{N}$. By (2.19), we have

$$\begin{aligned}zf(\alpha^m w)&=F(\alpha^m zw)-\lim_{n\to\infty}\left(\frac{g(\alpha^n z)}{\alpha^n}(\alpha^m w)\right)\\&=\alpha^m zf(w)+\lim_{n\to\infty}\left(\frac{g(\alpha^n\alpha^m z)}{\alpha^n}w\right)-\alpha^m\lim_{n\to\infty}\left(\frac{g(\alpha^n z)}{\alpha^n}w\right)\\&=\alpha^m zf(w)+\alpha^m\lim_{n\to\infty}\left(\frac{g(\alpha^{n+m}z)}{\alpha^{n+m}}w\right)-\alpha^m\lim_{n\to\infty}\left(\frac{g(\alpha^n z)}{\alpha^n}w\right)\\&=\alpha^m zf(w)\end{aligned}\tag{2.20}$$

for all $z,w\in A$. Then $zf(w)=z(f(\alpha^m w)/\alpha^m)$ for all $z,w\in A$ and all $m\in\mathbb{N}$, and so by taking the limit as $m\to\infty$, we have $zf(w)=zF(w)$. Now we obtain $F=f$, since A has an approximate identity.

Replacing $\alpha^n w$ by w in (2.15), we obtain

$$\left\|f(\alpha^n zw)-zf(\alpha^n w)-\alpha^n g(z)w\right\|\le\phi(z,\alpha^n w),\tag{2.21}$$

and hence

$$\left\|\frac{f(\alpha^n zw)}{\alpha^n}-z\frac{f(\alpha^n w)}{\alpha^n}-g(z)w\right\|\le\frac{1}{|\alpha|^n}\phi(z,\alpha^n w),\tag{2.22}$$

for all $z,w\in A$ and all $n\in\mathbb{N}$. Sending n to infinity, we have

$$f(zw)=zf(w)+g(z)w.\tag{2.23}$$

By (2.23), we get

$$\begin{aligned} g(z_1z_2)w &= f(z_1z_2w) - z_1z_2f(w) \\ &= z_1f(z_2w) + g(z_1)z_2w - z_1z_2f(w) \\ &= \left(z_1g(z_2) + g(z_1)z_2\right)w \end{aligned} \tag{2.24}$$

for all $z_1, z_2, w \in A$. Therefore, we have $g(z_1z_2) = z_1g(z_2) + g(z_1)z_2$. □

The following theorem states the conditions on the mapping f under which the sequence $\{f(\alpha^n x)/\alpha^n\}$ converges for all $x \in A$.

Theorem 2.4. *Let A be a Banach space and $\alpha \in \mathbb{C} \setminus \{0\}$. Suppose that $f : A \to A$ is a mapping for which there exists a function $\phi : A \to [0,\infty)$ such that*

$$\begin{gathered} \widetilde{\phi}(x) := \sum_{n=0}^{\infty} |\alpha|^{-n}\phi(\alpha^n x) < \infty, \\ \left\| \alpha^{-1}f(\alpha x) - f(x) \right\| \le \phi(x) \end{gathered} \tag{2.25}$$

for all $x \in A$. Then $F(x) := \lim_{n\to\infty}(f(\alpha^n x)/\alpha^n)$ exists and $F(\alpha x) = \alpha F(x)$ for all $x \in A$.

Proof. See [25, Theorem 1] or [26, Proposition 1]. □

3. Superstability of the Generalized Derivations

Our purpose is to prove the superstability of generalized ring derivations and generalized derivations. Throughout this section, A is a Banach algebra with a two-sided approximate identity.

Theorem 3.1. *Let $\gamma, \beta \in \mathbb{C} \setminus \{0\}$ such that $\alpha := \gamma + \beta \neq 0$. Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist a map $g : A \to A$ and a function $\phi : A^4 \to [0,\infty)$ such that*

$$\lim_{n\to\infty} \alpha^{-n}\phi(\alpha^n x, \alpha^n y, \alpha^n z, w) = \lim_{n\to\infty} \alpha^{-n}\phi(\alpha^n x, \alpha^n y, z, \alpha^n w) = 0, \tag{3.1}$$

$$H(x) := \sum_{n=0}^{\infty} |\alpha|^{-n}\phi(\alpha^n x, \alpha^n x, 0, 0) < \infty, \tag{3.2}$$

$$\| f(\gamma x + \beta y + zw) - \gamma f(x) - \beta f(y) - zf(w) - g(z)w \| \le \phi(x, y, z, w) \tag{3.3}$$

for all $x, y, z, w \in A$. Then f is a generalized ring derivation and g is a ring derivation. Moreover, $f(\alpha x) = \alpha f(x)$ for all $x \in A$.

Proof. Put $x = y$ and $z = w = 0$ in (3.3). We have $\|f(\alpha x) - \alpha f(x)\| \le \phi(x, x, 0, 0)$, and so$\|\alpha^{-1}f(\alpha x) - f(x)\| \le |\alpha|^{-1}\phi(x, x, 0, 0)$ for all $x \in A$.

Then by (3.2) and applying Theorem 2.4, we have $F(x) := \lim_{n\to\infty}(f(\alpha^n x)/\alpha^n)$ and $F(\alpha x) = \alpha F(x)$ for all $x \in A$.

Put $x = y = 0$ in (3.3). We get

$$\|f(zw) - zf(w) - g(z)w\| \leq \phi(0,0,z,w) \tag{3.4}$$

for all $z, w \in A$. It follows from (3.1) and Theorem 2.3 that $F = f$, $f(zw) = zf(w) + g(z)w$, and $g(zw) = zg(w) + g(z)w$ for all $z, w \in A$.

It suffices to show that f and g are additive.

Replacing x by $\alpha^n x$ and y by $\alpha^n y$ and putting $z = w = 0$ in (3.3), we obtain

$$\|f(\alpha^n(\gamma x + \beta y)) - \gamma f(\alpha^n x) - \beta f(\alpha^n y)\| \leq \phi(\alpha^n x, \alpha^n y, 0, 0), \tag{3.5}$$

and so

$$\left\| \frac{f(\alpha^n(\gamma x + \beta y))}{\alpha^n} - \gamma \frac{f(\alpha^n x)}{\alpha^n} - \beta \frac{f(\alpha^n y)}{\alpha^n} \right\| \leq \frac{1}{|\alpha|^n} \phi(\alpha^n x, \alpha^n y, 0, 0) \tag{3.6}$$

for all $x, y \in A$ and $n \in \mathbb{N}$.

By taking the limit as $n \to \infty$, we get $F(\gamma x + \beta y) = \gamma F(x) + \beta F(y)$, and so

$$f(\gamma x + \beta y) = \gamma f(x) + \beta f(y). \tag{3.7}$$

Putting $y = 0$ and replacing x by $\gamma^{-1}x$ in (3.7), we have $f(\gamma^{-1}x) = \gamma^{-1}f(x)$. Similarly, $f(\beta^{-1}x) = \beta^{-1}f(x)$.

Replacing x by $\gamma^{-1}x$ and y by $\beta^{-1}y$ in (3.7), we obtain $f(x + y) = f(x) + f(y)$ for all $x, y \in A$. Therefore f is an additive mapping.

Since $f(zw) = zf(w) + g(z)w$, f is additive, and A has an approximate identity, g is additive. □

Theorem 3.2. *Let $\gamma, \beta \in \mathbb{C} \setminus \{0\}$ such that $\alpha := \gamma + \beta \neq 0$. Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist a map $g : A \to A$ and a function $\phi : A^4 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \alpha^n \phi(\alpha^{-n}x, \alpha^{-n}y, \alpha^{-n}z, w) = \lim_{n\to\infty} \alpha^n \phi(\alpha^{-n}x, \alpha^{-n}y, z, \alpha^{-n}w) = 0, \tag{3.8}$$

$$H(x) := \sum_{n=0}^{\infty} |\alpha|^n \phi(\alpha^{-n}x, \alpha^{-n}x, 0, 0) < \infty, \tag{3.9}$$

$$\|f(\gamma x + \beta y + zw) - \gamma f(x) - \beta f(y) - zf(w) - g(z)w\| \leq \phi(x, y, z, w) \tag{3.10}$$

for all $x, y, z, w \in A$. Then f is a generalized ring derivation and g is a ring derivation. Moreover, $f(\alpha x) = \alpha f(x)$ for all $x \in A$.

Proof. Replacing x, y by $\alpha^{-1}x$ and putting $z = w = 0$ in (3.10), we get

$$\left\| f(x) - \alpha f\left(\alpha^{-1}x\right) \right\| \le \phi\left(\alpha^{-1}x, \alpha^{-1}x, 0, 0\right) \tag{3.11}$$

for all $x \in A$. Since

$$\begin{aligned}\sum_{n=0}^{\infty} \left|\alpha^{-1}\right|^{-n} \phi\left(\alpha^{-1}\alpha^{-n}x, \alpha^{-1}\alpha^{-n}x, 0, 0\right) \\ = \sum_{n=0}^{\infty} |\alpha|^{n} \phi\left(\alpha^{-n}\left(\alpha^{-1}x\right), \alpha^{-n}\left(\alpha^{-1}x\right), 0, 0\right) = H\left(\alpha^{-1}x\right) < \infty,\end{aligned} \tag{3.12}$$

it follows from Theorem 2.4 that $F(x) := \lim_{n\to\infty} \alpha^n f(\alpha^{-n}x)$ exists for all $x \in A$. By (3.8), we have

$$\lim_{n\to\infty} \left|\alpha^{-1}\right|^{-n} \phi\left(0, 0, \left(\alpha^{-1}\right)^n z, w\right) = \lim_{n\to\infty} \left|\alpha^{-1}\right|^{-n} \phi\left(0, 0, z, \left(\alpha^{-1}\right)^n w\right) = 0, \tag{3.13}$$

for all $z, w \in A$. Putting $x = y = 0$ in (3.10), it follows from Theorem 2.3 that $f(zw) = zf(w) + g(z)w$ and $g(zw) = zg(w) + g(z)w$ for all $z, w \in A$ and $F(x) = f(x)$ for all $x \in A$.

Replacing x by $\alpha^{-n}x$ and y by $\alpha^{-n}y$, putting $z = w = 0$ in (3.10), and multiplying both sides of the inequality by $|\alpha|^n$, we obtain

$$\left\| \alpha^n f\left(\alpha^{-n}(\gamma x + \beta y)\right) - \alpha^n \gamma f\left(\alpha^{-n}x\right) - \alpha^n \beta f\left(\alpha^{-n}y\right) \right\| \le |\alpha|^n \phi\left(\alpha^{-n}x, \alpha^{-n}y, 0, 0\right) \tag{3.14}$$

for all $x, y \in A$ and $n \in \mathbb{N}$. By taking the limit as $n \to \infty$, we get

$$f(\gamma x + \beta y) = \gamma f(x) + \beta f(y) \tag{3.15}$$

for all $x, y \in A$. Hence, by the same reasoning as in the proof of Theorem 3.1, f and g are additive mappings. Therefore, f is a generalized ring derivation and g is a ring derivation. □

Remark 3.3. We note that Theorems 3.1 and 3.2 and all that following results are obtained with no special conditions on the mapping g (see [21, Theorems 2.1 and 2.5]).

Corollary 3.4. *Let $p, q, s, t \in (-\infty, 1)$ or $p, q, s, t \in (1, \infty)$, $\gamma, \beta \in \mathbb{C} \setminus \{0\}$, and $\alpha = \gamma + \beta$ with $|\alpha| \notin \{0, 1\}$. Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist a map $g : A \to A$ and $\epsilon > 0$ such that*

$$\left\| f(\gamma x + \beta y + zw) - \gamma f(x) - \beta f(y) - zf(w) - g(z)w \right\| \le \epsilon\left(\|x\|^p + \|y\|^q + \|z\|^s \|w\|^t\right) \tag{3.16}$$

for all $x, y, z, w \in A$. Then f is a generalized ring derivation and g is a ring derivation.

Proof. Let $\phi(x,y,z,w) = \epsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t)$. For $0 < |\alpha| < 1$, if $p,q,s,t \in (1,\infty)$, then ϕ satisfies (3.1), (3.2), and we apply Theorem 3.1, and if $p,q,s,t \in (-\infty,1)$, then we apply Theorem 3.2 since ϕ has conditions (3.8), (3.9) in this case.

For $|\alpha| > 1$, apply Theorem 3.2 if $p,q,s,t \in (1,\infty)$ and apply Theorem 3.1 if $p,q,s,t \in (-\infty,1)$. □

Theorem 3.5. *Let $\alpha \in \mathbb{C} \setminus \{0\}$ and let $\phi : A^4 \to [0,\infty)$ be a function satisfying either* (3.1), (3.2) *or* (3.8), (3.9). *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a map $g : A \to A$ such that*

$$\|f(\lambda x + \lambda y + zw) - \lambda f(x) - \lambda f(y) - zf(w) - g(z)w\| \leq \phi(x,y,z,w) \tag{3.17}$$

for all $x,y,z,w \in A$ and all $\lambda \in S(0;|\alpha|/2) = \{\lambda \in \mathbb{C} : |\lambda| = |\alpha|/2\}$. Then f is a generalized derivation and g is a derivation.

Proof. Let $\lambda = \alpha/2$ in (3.17). We have

$$\left\|f\left(\frac{\alpha}{2}x + \frac{\alpha}{2}y + zw\right) - \frac{\alpha}{2}f(x) - \frac{\alpha}{2}f(y) - zf(w) - g(z)w\right\| \leq \phi(x,y,z,w) \tag{3.18}$$

for all $x,y,z,w \in A$.

Suppose that ϕ satisfies (3.1), (3.2). By Theorem 3.1, f is a generalized ring derivation and g is a ring derivation. Moreover, $f(\alpha x) = \alpha f(x)$ for all $x \in A$.

Replacing x by $\alpha^n x$ and putting $y = z = w = 0$ in (3.17), we get

$$\|f(\lambda\alpha^n x) - \lambda f(\alpha^n x)\| \leq \phi(\alpha^n x, 0, 0, 0), \tag{3.19}$$

for all $x \in A$, $n \in \mathbb{N}$, and $\lambda \in S(0;|\alpha|/2)$. Since $f(\alpha x) = \alpha f(x)$, we obtain

$$\|f(\lambda x) - \lambda f(x)\| \leq |\alpha|^{-n}\phi(\alpha^n x, 0, 0, 0). \tag{3.20}$$

Hence, by taking the limit as $n \to \infty$, we get $f(\lambda x) = \lambda f(x)$ for all $x \in A$ and $\lambda \in S(0;|\alpha|/2)$.

Let $\beta \in \mathbb{C}$ with $|\beta| = 1$. Then $\beta(\alpha/2) \in S(0;|\alpha|/2)$, and so

$$f(\beta x) = f\left(\beta\frac{\alpha}{2}\frac{2}{\alpha}x\right) = \beta\frac{\alpha}{2}f\left(\frac{2}{\alpha}x\right) = \beta\frac{\alpha}{2}f\left(\alpha^{-1}x + \alpha^{-1}x\right) = \beta f(x) \tag{3.21}$$

for all $x \in A$. Now by [21, Lemma 2.4], f is a linear mapping and hence g is a linear mapping. □

The following result generalizes Corollary 2.4 and Theorem 2.7 of [14].

Corollary 3.6. *Let $p,q,s,t \in (-\infty,1)$ and $\alpha \in \mathbb{C}$ with $|\alpha| \notin \{0,1\}$. Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist a map $g : A \to A$ and $\epsilon > 0$ such that*

$$\|f(\lambda x + \lambda y + zw) - \lambda f(x) - \lambda f(y) - zf(w) - g(z)w\| \leq \epsilon\left(\|x\|^p + \|y\|^q + \|z\|^s + \|w\|^t\right) \tag{3.22}$$

for all $x,y,z,w \in A$ and all $\lambda \in S(0;|\alpha|/2)$. Then f is a generalized derivation and g is a derivation.

Proof. Define $\phi(x,y,z,w) = \epsilon(\|x\|^p + \|y\|^q + \|z\|^s + \|w\|^t)$ and apply Theorem 3.5. □

Theorem 3.7. *Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$ and let $\phi : A^4 \to [0,\infty)$ be a function satisfying either* (3.1), (3.2) *or* (3.8), (3.9). *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a map $g : A \to A$ such that*

$$\left\|f\left(\frac{\alpha}{2}x + \frac{\alpha}{2}y + zw\right) - \frac{\alpha}{2}f(x) - \frac{\alpha}{2}f(y) - zf(w) - g(z)w\right\| \leq \phi(x,y,z,w) \tag{3.23}$$

for all $x,y,z,w \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then f is a generalized derivation and g is a derivation.

Proof. Suppose that ϕ satisfies (3.1), (3.2). By Theorem 3.1, f is a generalized ring derivation, g is a ring derivation, and $f(\alpha x) = \alpha f(x)$ for all $x \in A$.

Let $x \in A$. The mapping $h : \mathbb{R} \to A$, defined by $h(t) = f(tx)$, is continuous in $t \in \mathbb{R}$. Also, the mapping h is additive, since f is additive. Hence h is $\mathbb{R}$-linear, and so

$$f(tx) = h(t) = th(1) = tf(x) \tag{3.24}$$

for all $t \in \mathbb{R}$. Therefore, f is $\mathbb{R}$-linear.

Now let $\lambda \in \mathbb{C}$. Since $\alpha \notin \mathbb{R}$, there exist $s,r \in \mathbb{R}$ such that $\lambda = s + r\alpha$. So

$$f(\lambda x) = f(sx + r\alpha x) = sf(x) + rf(\alpha x) = sf(x) + r\alpha f(x) = \lambda f(x) \tag{3.25}$$

for all $x \in A$. Therefore, the mapping f is linear and it follows that g is linear. □

Corollary 3.8. *Let $p,q,s,t \in (1,\infty)$ or $p,q,s,t \in (-\infty,1)$. Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a map $g : A \to A$ such that*

$$\|f(ix + iy + zw) - if(x) - if(y) - zf(w) - g(z)w\| \leq \epsilon\left(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t\right) \tag{3.26}$$

for all $x,y,z,w \in A$. Suppose that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$. Then f is a generalized derivation and g is a derivation.

Proof. Let $\alpha = 2i$, define $\phi(x,y,z,w) = \epsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t)$, and apply Theorem 3.7. □

Theorem 3.9. *Let $f : A \to A$ be a mapping with $f(0) = 0$ for which there exist a map $g : A \to A$ and a function $\phi : A^4 \to [0,\infty)$ such that*

$$\lim_{n\to\infty} 2^{-n}\phi(2^n x, 2^n y, 2^n z, w) = \lim_{n\to\infty} 2^{-n}\phi(2^n x, 2^n y, z, 2^n w) = 0, \tag{3.27}$$

$$\sum_{n=0}^{\infty} 2^{-n}\phi(2^n x, 2^n x, 0, 0) < \infty, \tag{3.28}$$

$$\|f(\lambda x + \lambda y + zw) - \lambda f(x) - \lambda f(y) - zf(w) - g(z)w\| \leq \phi(x,y,z,w) \tag{3.29}$$

for $\lambda = i$ and all $x, y, z, w \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then f is a generalized derivation and g is a derivation.

Proof. Let $\alpha = 2i$. By Theorem 3.7, it suffices to prove that ϕ satisfies (3.1), (3.2).

Let $a_n = |2i|^{-n}\phi((2i)^n x, (2i)^n y, (2i)^n z, w)$. We have

$$\begin{aligned} a_{4n-3} &= 2^{-(4n-3)}\phi\left(2^{4n-3}(ix), 2^{4n-3}(iy), 2^{4n-3}(iz), w\right), \\ a_{4n-2} &= 2^{-(4n-2)}\phi\left(2^{4n-2}(-x), 2^{4n-2}(-y), 2^{4n-2}(-z), w\right), \\ a_{4n-1} &= 2^{-(4n-1)}\phi\left(2^{4n-1}(-ix), 2^{4n-1}(-iy), 2^{4n-1}(-iz), w\right), \\ a_{4n} &= 2^{-4n}\phi\left(2^{4n}x, 2^{4n}y, 2^{4n}z, w\right). \end{aligned} \tag{3.30}$$

Then $\lim_{n\to\infty} a_{4n-3} = \lim_{n\to\infty} a_{4n-2} = \lim_{n\to\infty} a_{4n-1} = \lim_{n\to\infty} a_{4n} = 0$, and so $\lim_{n\to\infty} a_n = 0$. Hence ϕ satisfies (3.1).

Let $b_n = |2i|^{-n}\phi((2i)^n x, (2i)^n x, 0, 0)$. By (3.28), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{4n+1} &= \sum_{n=0}^{\infty} 2^{-(4n+1)}\phi\left(2^{4n+1}(ix), 2^{4n+1}(ix), 0, 0\right) < \infty, \\ \sum_{n=0}^{\infty} b_{4n+2} &= \sum_{n=0}^{\infty} 2^{-(4n+2)}\phi\left(2^{4n+2}(-x), 2^{4n+2}(-x), 0, 0\right) < \infty, \\ \sum_{n=0}^{\infty} b_{4n+3} &= \sum_{n=0}^{\infty} 2^{-(4n+3)}\phi\left(2^{4n+3}(-ix), 2^{4n+3}(-ix), 0, 0\right) < \infty, \\ \sum_{n=0}^{\infty} b_{4n} &= \sum_{n=0}^{\infty} 2^{-4n}\phi\left(2^{4n}x, 2^{4n}x, 0, 0\right) < \infty. \end{aligned} \tag{3.31}$$

Hence

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (b_{4n} + b_{4n+1} + b_{4n+2} + b_{4n+3}) < \infty, \tag{3.32}$$

and so ϕ satisfies (3.2). □

The theorems similar to Theorem 3.9 have been proved by the assumption that the relations similar to (3.29) are true for $\lambda = 1, i$ (see, e.g., [9, 14]). We proved Theorem 3.9, under condition that inequality (3.29) is true for $\lambda = i$.

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