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Research Article

A Hybrid Iterative Scheme for Variational Inequality Problems for Finite Families of Relatively Weak Quasi-Nonexpansive Mappings

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We consider a hybrid projection algorithm basing on the shrinking projection method for two families of relatively weak quasi-nonexpansive mappings. We establish strong convergence theorems for approximating the common fixed point of the set of the common fixed points of such two families and the set of solutions of the variational inequality for an inverse-strongly monotone operator in the framework of Banach spaces. At the end of the paper, we apply our results to consider the problem of finding a solution of the complementarity problem. Our results improve and extend the corresponding results announced by recent results.

1. Introduction

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be an operator. The classical variational inequality problem [1] for A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. The set of all solutions of (1.1) is denoted by $VI(A, C)$. Such a problem is connected with the convex minimization problem, the complementarity, the problem of finding a point $x^* \in E$ satisfying $0 = Ax^*$, and so on. First, we recall that a mapping $A : C \to E^*$ is said to be

(i) *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0$, for all $x, y \in C$.

(ii) *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.2}$$

Let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \quad \forall x \in E. \tag{1.3}$$

It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. Some properties of the duality mapping are given in [2–4].

Recall that a mappings $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.4}$$

If C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is a nonexpansive mapping. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [5] recently introduced a generalized projection operator C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 \tag{1.5}$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*. Observe that, in a Hilbert space H, (1.5) reduces to $\phi(y, x) = \|x - y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem:

$$\phi(x^*, x) = \inf_{y \in C} \phi(y, x). \tag{1.6}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J (see, e.g., [3, 5–7]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2$ for all $x, y \in E$,

(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$ for all $x, y, z \in E$,

(3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$ for all $x, y \in E$,

(4) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \quad \text{iff } x = y. \tag{1.7}$$

For more detail see [2, 3]. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed point of T. A point p in C is said to be an *asymptotic fixed point* of T [8] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called relatively nonexpansive [7, 9, 10] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings were studied in [7, 9]. A point p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of strong asymptotic fixed points of S will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is called relatively weak nonexpansive if $\widetilde{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. If E is a smooth strictly convex and reflexive Banach space, and $A \subset E \times E^*$ is a continuous monotone mapping with $A^{-1}0 \neq \emptyset$, then it is proved in [11] that $J_r = (J + rA)^{-1}J$, for $r > 0$ is relatively weak nonexpansive. T is called relatively weak quasi-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 1.1. The class of relatively weak quasi-nonexpansive mappings is more general than the class of relatively weak nonexpansive mappings [7, 9, 12–14] which requires the strong restriction $\widehat{F}(T) = F(T)$.

Remark 1.2. If $T : C \to C$ is relatively weak quasi-nonexpansive, then using the definition of ϕ (i.e., the same argument as in the proof of [12, page 260]) one can show that $F(T)$ is closed and convex. It is obvious that relatively nonexpansive mapping is relatively weak nonexpansive mapping. In fact, for any mapping $T : C \to C$ we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is a relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$.

Iiduka and Takahashi [15] introduced the following algorithm for finding a solution of the variational inequality for an α-inverse-strongly monotone mapping A with $\|Ay\| \le \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C)$ in a 2-uniformly convex and uniformly smooth Banach space E. For an initial point $x_0 = x \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \quad \forall n \ge 0. \tag{1.8}$$

where J is the duality mapping on E, and Π_C is the generalized projection of E onto C. Assume that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$ where $1/c$ is the 2-uniformly convexity constant of E. They proved that if J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to some element z in $VI(A, C)$ where $z = \lim_{n\to\infty} \Pi_{VI(A,C)}(x_n)$.

The problem of finding a common element of the set of the variational inequalities for monotone mappings in the framework of Hilbert spaces and Banach spaces has been intensively studied by many authors; see, for instance, [16–18] and the references cited therein.

On the other hand, in 2001, Xu and Ori [19] introduced the following implicit iterative process for a finite family of nonexpansive mappings $\{T_1, T_2, \ldots, T_N\}$, with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in C$:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1)T_1 x_1,\\
x_2 &= \alpha_2 x_1 + (1 - \alpha_2)T_2 x_2,\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N)T_N x_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1})T_1 x_{N+1},\\
&\vdots
\end{aligned} \tag{1.9}$$

which can be rewritten in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n, \quad \forall n \geq 1, \tag{1.10}$$

where $T_n = T_{n(\bmod N)}$ (here the mod N function takes values in $\{1, 2, \ldots, N\}$). They obtained the following result in a real Hilbert space.

Theorem XO

Let H be a real Hilbert space, C a nonempty closed convex subset of H, and let $T : C \to C$ be a finite family of nonexpansive self-mappings on C such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (1.10). If $\{\alpha_n\}$ is chosen so that $\alpha_n \to 0$, as $n \to \infty$, then $\{x_n\}$ converges weakly to a common fixed point of the family of $\{T_i\}_{i=1}^{N}$.

On the other hand, Halpern [20] considered the following explicit iteration:

$$x_0 \in C, \qquad x_{n+1} = \alpha_n u + (1-\alpha_n) T x_n, \quad \forall n \geq 0, \tag{1.11}$$

where T is a nonexpansive mapping and $u \in C$ is a fixed point. He proved the strong convergence of $\{x_n\}$ to a fixed point of T provided that $\alpha_n = n^{-\theta}$, where $\theta \in (0,1)$.

Very recently, Qin et al. [21] proposed the following modification of the Halpern iteration for a single relatively quasi-nonexpansive mapping in a real Banach space. More precisely, they proved the following theorem.

Theorem QCKZ. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E and $T : C \to C$ a closed and quasi-ϕ-nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{aligned}
&x_0 = x \in C \quad \text{chosen arbitrary}, \\
&C_1 = C, \quad x_1 = \Pi_{C_1} x_0, \\
&y_n = J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J T x_n), \\
&C_{n+1} = \{ z \in C_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n) \phi(z, x_n) \}, \quad n \geq 1, \\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \geq 1.
\end{aligned} \tag{1.12}$$

Assume that $\{\alpha_n\}$ satisfies the restriction: $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$.

Motivated and inspired by the above results, Cai and Hu [22] introduced the hybrid projection algorithm to modify the iterative processes (1.10), (1.11), and (1.12) to have strong convergence for a finite family of relatively weak quasi-nonexpansive mappings in Banach spaces. More precisely, they obtained the following theorem.

Theorem CH

Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E and $\{T_1, T_2, \ldots, T_N\}$ be finite family of closed relatively weak quasi-nonexpansive mappings of C into itself with $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Assume that T_i is uniformly continuous for all $i \in \{1, 2, \ldots, N\}$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$
\begin{aligned}
&x_0 = x \in C \quad \text{chosen arbitrary,} \\
&C_1 = C, \quad x_1 = \Pi_{C_1} x_0, \\
&z_n = J^{-1}\left(\beta_n J x_{n-1} + \left(1 - \beta_n\right) J T_n x_n\right), \quad T_n = T_{n(\bmod N)}, \\
&y_n = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J z_n), \\
&C_{n+1} = \left\{z \in C_n : \phi\left(z, y_n\right) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\left[\beta_n \phi(z, x_{n-1}) + \left(1 - \beta_n\right) \phi(z, x_n)\right]\right\}, \\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \geq 1.
\end{aligned}
\tag{1.13}
$$

Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $[0,1]$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x_1$, where Π_F is the generalized projection from C onto F.

Motivated and inspired by Iiduka and Takahashi [15], Xu and Ori [19], Qin et al.[21], and Cai and Hu [22], we introduce a new hybrid projection algorithm basing on the shrinking projection method for two finite families of closed relatively weak quasi-nonexpansive mappings to have strong convergence theorems for approximating the common element of the set of common fixed points of two finite families of such mappings and the set of solutions of the variational inequality for an inverse-strongly monotone operator in the framework of Banach spaces. Our results improve and extend the corresponding results announced by recent results.

2. Preliminaries

A Banach space E is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$
\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}
\tag{2.1}
$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well know that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Some properties of the duality mapping have been given in [3, 23–25]. A Banach space E is said to have Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [3, 23, 25] for more details.

We define the function $\delta : [0,2] \to [0,1]$ which is called the modulus of convexity of E as following

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in C, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\right\}. \tag{2.2}$$

Then E is said to be 2-uniformly convex if there exists a constant $c > 0$ such that constant $\delta(\varepsilon) > c\varepsilon^2$ for all $\varepsilon \in (0,2]$. Constant $1/c$ is called the 2-uniformly convexity constant of E. A 2-uniformly convex Banach space is uniformly convex, see [26, 27] for more details. We know the following lemma of 2-uniformly convex Banach spaces.

Lemma 2.1 (see [28, 29]). *Let E be a 2-uniformly convex Banach, then for all x, y from any bounded set of E and $jx \in Jx$, $jy \in Jy$,*

$$\langle x-y, jx-jy\rangle \geq \frac{c^2}{2}\|x-y\|^2 \tag{2.3}$$

where $1/c$ is the 2-uniformly convexity constant of E.

Now we present some definitions and lemmas which will be applied in the proof of the main result in the next section.

Lemma 2.2 (Kamimura and Takahashi [30]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If $\lim_{n\to\infty}\phi(y_n, z_n) = 0$, then $\lim_{n\to\infty}\|y_n - z_n\| = 0$.*

Lemma 2.3 (Alber [5]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for any $y \in C$.*

Lemma 2.4 (Alber [5]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x) \tag{2.4}$$

for all $y \in C$.

Let E be a reflexive strictly convex, smooth and uniformly Banach space and the duality mapping J from E to E^*. Then J^{-1} is also single-valued, one to one, surjective, and it is the duality mapping from E^* to E. We need the following mapping V which studied in Alber [5]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x\|^2 \tag{2.5}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \phi(x, J^{-1}(x^*))$. We know the following lemma.

Lemma 2.5 (Kamimura and Takahashi [30]). *Let E be a reflexive, strictly convex and smooth Banach space, and let V be as in* (2.5). *Then*

$$V(x,x^*)+2\left\langle J^{-1}(x^*)-x,y^*\right\rangle \leq V(x,x^*+y^*) \tag{2.6}$$

for all $x\in E$ and $x^*,y^*\in E^*$.

Lemma 2.6 ([31, Lemma 1.4]). *Let E be a uniformly convex Banach space and $B_r(0)=\{x\in E:\|x\|\leqslant r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g:[0,\infty)\to[0,\infty)$ with $g(0)=0$ such that*

$$\|\lambda x+\mu y+\gamma z\|^2\leqslant\lambda\|x\|^2+\mu\|y\|^2+\gamma\|z\|^2-\lambda\mu g(\|x-y\|), \tag{2.7}$$

for all $x,y,z\in B_r(0)$ and $\lambda,\mu,\gamma\in[0,1]$ with $\lambda+\mu+\gamma=1$.

An operator A of C into E^* is said to be hemicontinuous if for all $x,y\in C$, the mapping F of $[0,1)$ into E^* defined by $F(t)=A(tx+(1-t)y)$ is continuous with respect to the weak* topology of E^*. We denote by $N_C(v)$ the normal cone for C at a point $v\in C$, that is

$$N_C(v)=\{x^*\in E^*:\langle v-y,x^*\rangle\geq 0,\ \forall y\in C\}. \tag{2.8}$$

Lemma 2.7 (see [32]). *Let C be a nonempty closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T\subset E\times E^*$ be an operator defined as follows:*

$$Tv=\begin{cases}Av+N_C(v), & v\in C,\\ \emptyset, & v\notin C.\end{cases} \tag{2.9}$$

Then T is maximal monotone and $T^{-1}0=VI(A,C)$.

3. Main Results

In this section, we prove strong convergence theorem which is our main result.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E, let A be an α-inverse-strongly monotone mapping of C into E^* with $\|Ay\|\leq\|Ay-Aq\|$ for all $y\in C$ and $q\in F$. Let $\{T_1,T_2,\ldots,T_N\}$ and $\{S_1,S_2,\ldots,S_N\}$ be two finite families of closed relatively weak quasi-nonexpansive mappings from C into itself with $F\neq\emptyset$, where $F:=\bigcap_{i=1}^N F(T_i)\cap\bigcap_{i=1}^N F(S_i)\cap VI(A,C)$. Assume that T_i and S_i are uniformly continuous for all*

$i \in \{1, 2, \ldots, N\}$. *Let* $\{x_n\}$ *be a sequence generated by the following algolithm:*

$$
\begin{aligned}
&x_0 = x \in C, \quad \text{chosen arbitrary,}\\
&C_1 = C, \quad x_1 = \Pi_{C_1} x_0,\\
&w_n = \Pi_C J^{-1}(Jx_n - r_n A x_n),\\
&z_n = J^{-1}(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n),\\
&y_n = J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n),\\
&C_{n+1} = \{u \in C_n : \phi(u, y_n) \le \delta_n \phi(u, x_1) + (1-\delta_n)[\alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n)]\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1,
\end{aligned}
\tag{3.1}
$$

where $T_n = T_{n(\bmod N)}, S_n = S_{n(\bmod N)}$, *and* J *is the normalized duality mapping on* E. *Assume that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ *and* $\{r_n\}$ *are the sequences in* $[0,1]$ *satisfying the restrictions:*

(C1) $\lim_{n\to\infty} \delta_n = 0$;

(C2) $r_n \subset [a, b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$, *where* $1/c$ *is the 2-uniformly convexity constant of* E;

(C3) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

  (a) $\liminf_{n\to\infty} \alpha_n \beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$ *and*
  (b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n \gamma_n > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_F x_1$, *where* Π_F *is the generalized projection from* C *onto* F.

Proof. By the same method as in the proof of Cai and Hu [22], we can show that C_n is closed and convex. Next, we show $F \subset C_n$ for all $n \ge 1$. In fact, $F \subset C_1 = C$ is obvious. Suppose $F \subset C_n$ for some $n \in \mathbb{N}$. Then, for all $q \in F \subset C_n$, we know from Lemma 2.5 that

$$
\begin{aligned}
\phi(q, w_n) &= \phi\left(q, \Pi_C J^{-1}(Jx_n - r_n A x_n)\right)\\
&\le \phi\left(q, J^{-1}(Jx_n - r_n A x_n)\right)\\
&= V(q, Jx_n - r_n A x_n)\\
&\le V(q, (Jx_n - r_n A x_n) + r_n A x_n) - 2\left\langle J^{-1}(Jx_n - r_n A x_n) - q, r_n A x_n\right\rangle\\
&= V(q, Jx_n) - 2r_n\left\langle J^{-1}(Jx_n - r_n A x_n) - q, A x_n\right\rangle\\
&= \phi(q, x_n) - 2r_n\langle x_n - q, A x_n\rangle + 2\left\langle J^{-1}(Jx_n - r_n A x_n) - x_n, -r_n A x_n\right\rangle.
\end{aligned}
\tag{3.2}
$$

Since $q \in VI(A, C)$ and A is α-inverse-strongly monotone, we have

$$
-2r_n\langle x_n - q, A x_n\rangle = -2r_n\langle x_n - q, A x_n - Aq\rangle - 2r_n\langle x_n - q, Aq\rangle \le -2\alpha r_n\|A x_n - Aq\|^2. \tag{3.3}
$$

Therefore, from Lemma 2.1 and the assumption that $\|Ay\| \le \|Ay - Aq\|$ for all $y \in C$ and $q \in F$, we obtain that

$$\begin{aligned}
2\left\langle J^{-1}(Jx_n - r_nAx_n) - x_n, -r_nAx_n\right\rangle &= 2\left\langle J^{-1}(Jx_n - r_nAx_n) - J^{-1}(Jx_n), -r_nAx_n\right\rangle \\
&\le 2\left\|J^{-1}(Jx_n - r_nAx_n) - J^{-1}(Jx_n)\right\|\|r_nAx_n\| \\
&\le \frac{4}{c^2}\left\|JJ^{-1}(Jx_n - r_nAx_n) - JJ^{-1}(Jx_n)\right\|\|r_nAx_n\| \\
&= \frac{4}{c^2}\|(Jx_n - r_nAx_n) - Jx_n\|\|r_nAx_n\| \\
&= \frac{4}{c^2}r_n^2\|Ax_n\|^2 \le \frac{4}{c^2}r_n^2\left\|Ax_n - Aq\right\|^2.
\end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) and using the condition that $r_n < c^2\alpha/2$, we get

$$\phi(q, w_n) \le \phi(q, x_n) + 2r_n\left(\frac{2}{c^2}r_n - \alpha\right)\left\|Ax_n - Aq\right\|^2 \le \phi(q, x_n). \tag{3.5}$$

Using (3.5) and the convexity of $\|\cdot\|^2$, for each $q \in F \subset C_n$, we obtain

$$\begin{aligned}
\phi(q, z_n) &= \phi\left(q, J^{-1}(\alpha_nJx_{n-1} + \beta_nJT_nx_n + \gamma_nJS_nw_n)\right) \\
&= \left\|q\right\|^2 - 2\alpha_n\langle q, Jx_{n-1}\rangle - 2\beta_n\langle q, JT_nx_n\rangle - 2\gamma_n\langle q, JS_nw_n\rangle \\
&\quad + \left\|\alpha_nJx_{n-1} + \beta_nJT_nx_n + \gamma_nJS_nw_n\right\|^2 \\
&\le \left\|q\right\|^2 - 2\alpha_n\langle q, Jx_{n-1}\rangle - 2\beta_n\langle q, JT_nx_n\rangle - 2\gamma_n\langle q, JS_nw_n\rangle \\
&\quad + \alpha_n\|Jx_{n-1}\|^2 + \beta_n\|JT_nx_n\|^2 + \gamma_n\|JS_nw_n\|^2 \\
&= \alpha_n\phi(q, x_{n-1}) + \beta_n\phi(q, T_nx_n) + \gamma_n\phi(q, S_nw_n) \\
&\le \alpha_n\phi(q, x_{n-1}) + \beta_n\phi(q, x_n) + \gamma_n\phi(q, w_n) \\
&\le \alpha_n\phi(q, x_{n-1}) + \beta_n\phi(q, x_n) + \gamma_n\phi(q, x_n) \\
&= \alpha_n\phi(q, x_{n-1}) + (1 - \alpha_n)\phi(q, x_n).
\end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$\begin{aligned}\phi(q,y_n) &= \phi\Big(q, J^{-1}(\delta_n Jx_1 + (1-\delta_n)Jz_n)\Big)\\ &= \|q\|^2 - 2\delta_n\langle q, Jx_1\rangle - 2(1-\delta_n)\langle q, Jz_n\rangle + \|\delta_n Jx_1 + (1-\delta_n)Jz_n\|^2\\ &\le \|q\|^2 - 2\delta_n\langle q, Jx_1\rangle - 2(1-\delta_n)\langle q, Jz_n\rangle + \delta_n\|x_1\|^2 + (1-\delta_n)\|z_n\|^2\\ &= \delta_n\phi(q,x_1) + (1-\delta_n)\phi(q,z_n)\\ &\le \delta_n\phi(q,x_1) + (1-\delta_n)\big[\alpha_n\phi(q,x_{n-1}) + (1-\alpha_n)\phi(q,x_n)\big].\end{aligned} \tag{3.7}$$

So, $q \in C_{n+1}$. Then by induction, $F \subset C_n$ for all $n \ge 1$ and hence the sequence $\{x_n\}$ generated by (3.1) is well defined. Next, we show that $\{x_n\}$ is a convergent sequence in C. From $x_n = \Pi_{C_n}x_1$, we have

$$\langle x_n - u, Jx_1 - Jx_n\rangle \ge 0, \quad \forall u \in C_n. \tag{3.8}$$

It follows from $F \subset C_n$ for all $n \ge 1$ that

$$\langle x_n - z, Jx_1 - Jx_n\rangle \ge 0, \quad \forall z \in F. \tag{3.9}$$

From Lemma 2.4, we have

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}x_1, x_1) \le \phi(u, x_1) - \phi(u, x_n) \le \phi(u, x_1), \tag{3.10}$$

for each $u \in F \subset C_n$ and for all $n \ge 1$. Therefore, the sequence $\{\phi(x_n, x_1)\}$ is bounded. Furthermore, since $x_n = \Pi_{C_n}x_1$ and $x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_1) \le \phi(x_{n+1}, x_1), \quad \forall n \ge 1. \tag{3.11}$$

This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing and hence $\lim_{n\to\infty}\phi(x_n, x_1)$ exists. Similarly, by Lemma 2.4, we have, for any positive integer m, that

$$\begin{aligned}\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n}x_1) \le \phi(x_{n+m}, x_1) - \phi(\Pi_{C_n}x_1, x_1)\\ &= \phi(x_{n+m}, x_1) - \phi(x_n, x_1), \quad \forall n \ge 1.\end{aligned} \tag{3.12}$$

The existence of $\lim_{n\to\infty}\phi(x_n, x_1)$ implies that $\phi(x_{n+m}, x_n) \to 0$ as $n \to \infty$. From Lemma 2.2, we have

$$\|x_{n+m} - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.13}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Now, we will show that $p \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \cap VI(A, C)$.

(I) We first show that $p \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i)$. Indeed, taking $m = 1$ in (3.12), we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.14}$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.15}$$

This implies that

$$\lim_{n\to\infty} \|x_{n+l} - x_n\| = 0, \quad \forall l \in \{1, 2, \ldots, N\}. \tag{3.16}$$

The property of the function ϕ implies that

$$\lim_{n\to\infty} \phi(x_{n+l}, x_n) = 0, \quad \forall l \in \{1, 2, \ldots, N\}. \tag{3.17}$$

Since $x_{n+1} \in C_{n+1}$, we obtain

$$\phi(x_{n+1}, y_n) \le \delta_n \phi(x_{n+1}, x_n) + (1 - \delta_n)\left[\alpha_n \phi(x_{n+1}, x_{n-1}) + (1 - \alpha_n)\phi(x_{n+1}, x_n)\right]. \tag{3.18}$$

It follows from the condition (3.14) and (3.17) that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{3.19}$$

From Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.20}$$

Combining (3.15) and (3.20), we have

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.21}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.22}$$

On the other hand, noticing

$$\|Jy_n - Jz_n\| = \delta_n \|Jx_1 - Jz_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.23}$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{3.24}$$

Using (3.15), (3.20), and (3.24) that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.25}$$

Taking the constant $r = \sup_{n\geq 1}\{\|x_{n+1}\|, \|T_n x_n\|, \|S_n w_n\|\}$, we have, from Lemma 2.6, that there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ satisfying the inequality (2.7) and $g(0) = 0$.

Case 1. Assume that (a) holds. Applying (2.7) and (3.5), we can calculate

$$\begin{aligned}
\phi(u, z_n) &= \phi\Big(u, J^{-1}(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n)\Big) \\
&= \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle \\
&\quad + \big\|\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n\big\|^2 \\
&\leq \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle \\
&\quad + \alpha_n\|Jx_{n-1}\|^2 + \beta_n\|JT_n x_n\|^2 + \gamma_n\|JS_n w_n\|^2 - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|) \\
&\leq \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, T_n x_n) + \gamma_n\phi(u, S_n w_n) - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|) \\
&\leq \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, x_n) + \gamma_n\phi(u, w_n) - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|) \\
&\leq \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, x_n) + \gamma_n\phi(u, x_n) \\
&\quad + 2r_n\gamma_n\left(\frac{2}{c^2}r_n - \alpha\right)\|Ax_n - Au\|^2 - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|) \\
&\leq \alpha_n\phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n) + 2r_n\gamma_n\left(\frac{2}{c^2}r_n - \alpha\right)\|Ax_n - Au\|^2 \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|).
\end{aligned} \tag{3.26}$$

This implies that

$$\alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|) \leq \alpha_n\left[\phi(u, x_{n-1}) - \phi(u, x_n)\right] + \phi(u, x_n) - \phi(u, z_n). \tag{3.27}$$

We observe that

$$\begin{aligned}
&\alpha_n\left[\phi(u,x_{n-1})-\phi(u,x_n)\right]+\phi(u,x_n)-\phi(u,z_n)\\
&\quad\le \alpha_n\left[\|x_{n-1}\|^2-\|x_n\|^2-2\langle u,Jx_{n-1}-Jx_n\rangle\right]\\
&\qquad+\|x_n\|^2-\|z_n\|^2-2\langle u,Jx_n-Jz_n\rangle\\
&\quad\le \alpha_n\left[\|x_{n-1}-x_n\|(\|x_{n-1}\|+\|x_n\|)+2\|u\|\|Jx_{n-1}-Jx_n\|\right]\\
&\qquad+\|x_n-z_n\|(\|x_n\|+\|z_n\|)+2\|u\|\|Jx_n-Jz_n\|.
\end{aligned} \tag{3.28}$$

It follows from (3.15), (3.22), (3.23) and (3.25) that

$$\lim_{n\to\infty}\alpha_n\left[\phi(u,x_{n-1})-\phi(u,x_n)\right]+\phi(u,x_n)-\phi(u,z_n)=0. \tag{3.29}$$

From $\liminf_{n\to\infty}\alpha_n\beta_n>0$ and (3.27), we get

$$\lim_{n\to\infty}g(\|Jx_{n-1}-JT_nx_n\|)=0. \tag{3.30}$$

By the property of function g, we obtain that

$$\lim_{n\to\infty}\|Jx_{n-1}-JT_nx_n\|=0. \tag{3.31}$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|x_{n-1}-T_nx_n\|=\lim_{n\to\infty}\left\|J^{-1}(Jx_{n-1})-J^{-1}(JT_nx_n)\right\|=0. \tag{3.32}$$

From (3.15) and (3.32), we have

$$\lim_{n\to\infty}\|x_n-T_nx_n\|=0. \tag{3.33}$$

Noticing that

$$\|x_n-T_{n+l}x_n\|\le\|x_n-x_{n+l}\|+\|x_{n+l}-T_{n+l}x_{n+l}\|+\|T_{n+l}x_{n+l}-T_{n+l}x_n\|, \tag{3.34}$$

for all $l\in\{1,2,\dots,N\}$. By the uniformly continuity of T_l, (3.16) and (3.33), we obtain

$$\lim_{n\to\infty}\|x_n-T_{n+l}x_n\|=0,\quad\forall l\in\{1,2,\dots,N\}. \tag{3.35}$$

Thus

$$\lim_{n\to\infty}\|x_n-T_lx_n\|=0,\quad\forall l\in\{1,2,\dots,N\}. \tag{3.36}$$

From the closeness of T_i, we get $p = T_i p$. Therefore $p \in \bigcap_{i=1}^{N} F(T_i)$. In the same manner, we can apply the condition $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$ to conclude that

$$\lim_{n\to\infty} \|x_n - S_n w_n\| = 0. \tag{3.37}$$

Again, by (C2) and (3.26), we have

$$\begin{aligned} 2\gamma_n\left(\alpha - \frac{2}{c^2}b\right)\|Ax_n - Au\|^2 &\le \frac{1}{a}\left[\alpha_n\phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n) - \phi(u, z_n)\right] \\ &= \frac{1}{a}\left[\alpha_n\left(\phi(u, x_{n-1}) - \phi(u, x_n)\right) + \phi(u, x_n) - \phi(u, z_n)\right]. \end{aligned} \tag{3.38}$$

It follows from (3.29) and $\liminf_{n\to\infty} \gamma_n \ge \liminf_{n\to\infty} \beta_n \gamma_n > 0$ that

$$\liminf_{n\to\infty} \|Ax_n - Au\| \le 0. \tag{3.39}$$

Since $\liminf_{n\to\infty} \|Ax_n - Au\| \ge 0$, we have

$$\lim_{n\to\infty} \|Ax_n - Au\| = 0. \tag{3.40}$$

From Lemmas 2.4, 2.5, and (3.4), we have

$$\begin{aligned} \phi(x_n, w_n) &= \phi\left(x_n, \Pi_C J^{-1}(Jx_n - r_n Ax_n)\right) \le \phi\left(x_n, J^{-1}(Jx_n - r_n Ax_n)\right) = V(x_n, Jx_n - r_n Ax_n) \\ &\le V(x_n, (Jx_n - r_n Ax_n) + r_n Ax_n) - 2\left\langle J^{-1}(Jx_n - r_n Ax_n) - x_n, r_n Ax_n \right\rangle \\ &= \phi(x_n, x_n) + 2\left\langle J^{-1}(Jx_n - r_n Ax_n) - x_n, -r_n Ax_n \right\rangle \\ &= 2\left\langle J^{-1}(Jx_n - r_n Ax_n) - x_n, -r_n Ax_n \right\rangle \le \frac{4}{c^2} b^2 \|Ax_n - Au\|^2. \end{aligned} \tag{3.41}$$

It follows from (3.40) that

$$\lim_{n\to\infty} \phi(x_n, w_n) = 0. \tag{3.42}$$

Lemma 2.2 implies that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.43}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jw_n\| = 0. \tag{3.44}$$

Combining (3.37) and (3.43), we also obtain

$$\lim_{n\to\infty}\|w_n - S_n w_n\| = 0. \tag{3.45}$$

Moreover

$$\|w_n - w_{n+1}\| \le \|w_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - w_{n+1}\|. \tag{3.46}$$

By (3.43), (3.15), we have

$$\lim_{n\to\infty}\|w_n - w_{n+1}\| = 0. \tag{3.47}$$

This implies that

$$\lim_{n\to\infty}\|w_n - w_{n+l}\| = 0, \quad \forall l \in \{1,2,\ldots,N\}. \tag{3.48}$$

Noticing that

$$\|w_n - S_{n+l}w_n\| \le \|w_n - w_{n+l}\| + \|w_{n+l} - S_{n+l}w_{n+l}\| + \|S_{n+l}w_{n+l} - S_{n+l}w_n\|, \tag{3.49}$$

for all $l \in \{1,2,\ldots,N\}$. Since S_l is uniformly continuous, we can show that $\lim_{n\to\infty}\|w_n - S_l w_n\| = 0$. From the closeness of S_l, we get $p = S_l p$. Therefore $p \in \bigcap_{i=1}^N F(S_i)$. Hence $p \in \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$.

Case 2. Assume that (b) holds. Using the inequalities (2.7) and (3.5), we obtain

$$\begin{aligned}
\phi(u,z_n) &= \phi\Big(u, J^{-1}(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n)\Big) \\
&= \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle \\
&\quad + \left\|\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n\right\|^2 \\
&\le \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle \\
&\quad + \alpha_n\|Jx_{n-1}\|^2 + \beta_n\|JT_n x_n\|^2 + \gamma_n\|JS_n w_n\|^2 - \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|) \\
&\le \alpha_n\phi(u,x_{n-1}) + \beta_n\phi(u,T_n x_n) + \gamma_n\phi(u,S_n w_n) - \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|) \\
&\le \alpha_n\phi(u,x_{n-1}) + \beta_n\phi(u,x_n) + \gamma_n\phi(u,w_n) - \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|) \\
&\le \alpha_n\phi(u,x_{n-1}) + \beta_n\phi(u,x_n) + \gamma_n\phi(u,x_n) \\
&\quad + 2r_n\gamma_n\left(\frac{2}{c^2}r_n - \alpha\right)\|Ax_n - Au\|^2 - \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|) \\
&\le \alpha_n\phi(u,x_{n-1}) + (1-\alpha_n)\phi(u,x_n) + 2r_n\gamma_n\left(\frac{2}{c^2}r_n - \alpha\right)\|Ax_n - Au\|^2 \\
&\quad - \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|).
\end{aligned} \tag{3.50}$$

This implies that

$$\begin{aligned}
\beta_n\gamma_n g(\|JT_nx_n - JS_nw_n\|) &\le \alpha_n\left[\phi(u,x_{n-1}) - \phi(u,x_n)\right] + \phi(u,x_n) - \phi(u,z_n)\\
&\le \alpha_n\left[\|x_{n-1}\|^2 - \|x_n\|^2 - 2\langle u, Jx_{n-1} - Jx_n\rangle\right]\\
&\quad + \|x_n\|^2 - \|z_n\|^2 - 2\langle u, Jx_n - Jz_n\rangle\\
&\le \alpha_n[\|x_{n-1} - x_n\|(\|x_{n-1}\| + \|x_n\|) + 2\|u\|\|Jx_{n-1} - Jx_n\|]\\
&\quad + \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|u\|\|Jx_n - Jz_n\|.
\end{aligned} \tag{3.51}$$

It follows from (3.21), (3.24) and the condition $\liminf_{n\to\infty}\beta_n\gamma_n > 0$ that

$$\lim_{n\to\infty} g(\|JT_nx_n - JS_nw_n\|) = 0. \tag{3.52}$$

By the property of function g, we obtain that

$$\lim_{n\to\infty}\|JT_nx_n - JS_nw_n\| = 0. \tag{3.53}$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|T_nx_n - S_nw_n\| = \lim_{n\to\infty}\left\|J^{-1}(JT_nx_n) - J^{-1}(JS_nw_n)\right\| = 0. \tag{3.54}$$

On the other hand, we can calculate

$$\begin{aligned}
\phi(T_nx_n, z_n) &= \phi\Big(T_nx_n, J^{-1}(\alpha_nJx_{n-1} + \beta_nJT_nx_n + \gamma_nJS_nw_n)\Big)\\
&= \|T_nx_n\|^2 - 2\langle T_nx_n, \alpha_nJx_{n-1} + \beta_nJT_nx_n + \gamma_nJS_nw_n\rangle\\
&\quad + \left\|\alpha_nJx_{n-1} + \beta_nJT_nx_n + \gamma_nJS_nw_n\right\|^2\\
&\le \|T_nx_n\|^2 - 2\alpha_n\langle T_nx_n, Jx_n\rangle - 2\beta_n\langle T_nx_n, JT_nx_n\rangle - 2\gamma_n\langle T_nx_n, JS_nw_n\rangle\\
&\quad + \alpha_n\|x_n\|^2 + \beta_n\|T_nx_n\|^2 + \gamma_n\|S_nw_n\|^2\\
&\le \alpha_n\phi(T_nx_n, x_n) + \gamma_n\phi(T_nx_n, S_nw_n).
\end{aligned} \tag{3.55}$$

Observe that

$$\begin{aligned}
\phi(T_nx_n, S_nw_n) &= \|T_nx_n\|^2 - 2\langle T_nx_n, JS_nw_n\rangle + \|S_nw_n\|^2\\
&= \|T_nx_n\|^2 - 2\langle T_nx_n, JT_nx_n\rangle + 2\langle T_nx_n, JT_nx_n - JS_nw_n\rangle + \|S_nw_n\|^2\\
&\le \|S_nw_n\|^2 - \|T_nx_n\|^2 + 2\|T_nx_n\|\|JT_nx_n - JS_nw_n\|\\
&\le \|S_nw_n - T_nx_n\|(\|S_nw_n\| + \|T_nx_n\|) + 2\|T_nx_n\|\|JT_nx_n - JS_nw_n\|.
\end{aligned} \tag{3.56}$$

It follows from (3.53) and (3.54) that

$$\lim_{n\to\infty} \phi(T_n x_n, S_n w_n) = 0. \tag{3.57}$$

Applying $\lim_{n\to\infty} \alpha_n = 0$ and (3.57) and the fact that $\{\phi(T_n x_n, x_n)\}$ is bounded to (3.55), we obtain

$$\lim_{n\to\infty} \phi(T_n x_n, z_n) = 0. \tag{3.58}$$

From Lemma 2.2, one obtains

$$\lim_{n\to\infty} \|T_n x_n - z_n\| = 0. \tag{3.59}$$

We observe that

$$\|T_n x_n - x_n\| \le \|T_n x_n - z_n\| + \|z_n - x_n\|. \tag{3.60}$$

This together with (3.25) and (3.59), we obtain

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{3.61}$$

Noticing that

$$\|x_n - T_{n+l} x_n\| \le \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| + \|T_{n+l} x_{n+l} - T_{n+l} x_n\|, \tag{3.62}$$

for all $l \in \{1, 2, \ldots, N\}$. By the uniformly continuity of T_l, (3.16) and (3.61), we obtain

$$\lim_{n\to\infty} \|x_n - T_{n+l} x_n\| = 0, \quad \forall l \in \{1, 2, \ldots, N\}. \tag{3.63}$$

Thus

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l \in \{1, 2, \ldots, N\}. \tag{3.64}$$

From the closeness of T_i, we get $p = T_i p$. Therefore $p \in \bigcap_{i=1}^{N} F(T_i)$. By the same proof as in Case 1, we obtain that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.65}$$

Hence $w_n \to p$ as $n \to \infty$ for each $i \in I$ and

$$\lim_{n\to\infty} \|J x_n - J w_n\| = 0. \tag{3.66}$$

Combining (3.54), (3.61), and (3.65), we also have

$$\lim_{n\to\infty}\|S_n w_n - w_n\| = 0. \tag{3.67}$$

Moreover

$$\|w_n - w_{n+1}\| \le \|w_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - w_{n+1}\|. \tag{3.68}$$

By (3.43), (3.15), we have

$$\lim_{n\to\infty}\|w_n - w_{n+1}\| = 0. \tag{3.69}$$

This implies that

$$\lim_{n\to\infty}\|w_n - w_{n+l}\| = 0, \quad \forall l \in \{1,2,\ldots,N\}. \tag{3.70}$$

Noticing that

$$\|w_n - S_{n+l}w_n\| \le \|w_n - w_{n+l}\| + \|w_{n+l} - S_{n+l}w_{n+l}\| + \|S_{n+l}w_{n+l} - S_{n+l}w_n\|, \tag{3.71}$$

for all $l \in \{1,2,\ldots,N\}$. Since S_l is uniformly continuous, we can show that $\lim_{n\to\infty}\|w_n - S_l w_n\| = 0$. From the closeness of S_i, we get $p = S_i p$. Therefore $p \in \bigcap_{i=1}^N F(S_i)$. Hence $p \in \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$.

(II) We next show that $p \in VI(C,A)$.
Let $T \subset E \times E^*$ be an operator defined by:

$$Tv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{3.72}$$

By Lemma 2.7, T is maximal monotone and $T^{-1}0 = VI(A,C)$. Let $(v,w) \in G(T)$, since $w \in Tv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. From $x_n = \Pi_{C_n}x \in C_n \subset C$, we get

$$\langle v - x_n, w - Av\rangle \ge 0. \tag{3.73}$$

Since A is α-inverse-strong monotone, we have

$$\langle v - x_n, w\rangle \ge \langle v - x_n, Av\rangle = \langle v - x_n, Av - Ax_n\rangle + \langle v - x_n, Ax_n\rangle \ge \langle v - x_n, Ax_n\rangle. \tag{3.74}$$

On other hand, from $w_n = \Pi_C J^{-1}(Jx_n - r_n Ax_n)$ and Lemma 2.3, we have $\langle v - w_n, Jw_n - (Jx_n - r_n Ax_n)\rangle \ge 0$, and hence

$$\left\langle v - w_n, \frac{Jx_n - Jw_n}{r_n} - Ax_n \right\rangle \le 0. \tag{3.75}$$

Because A is $1/\alpha$ constricted, it holds from (3.74) and (3.75) that

$$\begin{aligned}\langle v - x_n, w\rangle &\geq \langle v - x_n, Ax_n\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{r_n} - Ax_n\right\rangle \\ &= \langle v - w_n, Ax_n\rangle + \langle w_n - x_n, Ax_n\rangle - \langle v - w_n, Ax_n\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{r_n}\right\rangle \\ &= \langle w_n - x_n, Ax_n\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{r_n}\right\rangle \\ &\geq -\|w_n - x_n\| \cdot \|Ax_n\| - \|v - w_n\| \cdot \frac{\|Jx_n - Jw_n\|}{a},\end{aligned} \tag{3.76}$$

for all $n \in \mathbb{N} \cup \{0\}$. By taking the limit as $n \to \infty$ in (3.76) and from (3.43) and (3.44), we have $\langle v - p, w\rangle \geq 0$ as $n \to \infty$. By the maximality of T we obtain $p \in T^{-1}0$ and hence $p \in VI(A, C)$. Hence we conclude that

$$p \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \cap VI(A, C). \tag{3.77}$$

Finally, we show that $p \in \Pi_F x_1$. Indeed, taking the limit as $n \to \infty$ in (3.9), we obtain

$$\langle p - z, Jx_1 - Jp\rangle \geq 0, \quad \forall z \in F \tag{3.78}$$

and hence $p = \Pi_F\, x_1$ by Lemma 2.3. This complete the proof. □

Remark 3.2. Theorem 3.1 improves and extends main results of Iiduka and Takahashi [15], Xu and Ori [19], Qin et al. [21], and Cai and Hu [22] because it can be applied to solving the problem of finding the common element of the set of common fixed points of two families of relatively weak quasi-nonexpansive mappings and the set of solutions of the variational inequality for an inverse-strongly monotone operator.

Strong convergence theorem for approximating a common fixed point of two finite families of closed relatively weak quasi-nonexpansive mappings in Banach spaces may not require that E is 2-uniformly convex. In fact, we have the following theorem.

Corollary 3.3. *Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $\{T_1, T_2, \dots, T_N\}$ and $\{S_1, S_2, \dots, S_N\}$ be two finite families of closed relatively weak quasi-nonexpansive mappings from C into itself with $F \neq \emptyset$, where $F := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i)$. Assume that T_i and S_i are uniformly continuous for all $i \in \{1, 2, \dots, N\}$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
\begin{aligned}
&x_0 = x \in C, \quad \textit{chosen arbitrary}, \\
&C_1 = C, \quad x_1 = \Pi_{C_1} x_0, \\
&z_n = J^{-1}\left(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n x_n\right), \\
&y_n = J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n), \\
&C_{n+1} = \left\{u \in C_n : \phi(u, y_n) \leq \delta_n \phi(u, x_1) + (1-\delta_n)\left[\alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n)\right]\right\}, \\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{aligned}
\tag{3.79}
$$

where $T_n = T_{n(\bmod N)}, S_n = S_{n(\bmod N)}$, and J is the normalized duality mapping on E. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0,1]$ satisfying the following restrictions:

(C1) $\lim_{n\to\infty} \delta_n = 0$;

(C2) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

(a) $\liminf_{n\to\infty} \alpha_n \beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$ *and*

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n \gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_1$, where Π_F is the generalized projection from C onto F.

Proof. Put $A \equiv 40$ in Theorem 3.1. Then, we get that $w_n = x_n$. Thus, the method of the proof of Theorem 3.1 gives the required assertion without the requirement that E is 2-uniformly convex. □

Remark 3.4. Corollary 3.3 improves Theorem 3.1 of Cai and Hu [22] from a finite family of of relatively weak quasi-nonexpansive mappings to two finite families of relatively weak quasi-nonexpansive mappings.

If $E = H$, a Hilbert space, then E is 2-uniformly convex (we can choose $c = 1$) and uniformly smooth real Banach space and closed relatively weak quasi-nonexpansive map reduces to closed weak quasi-nonexpansive map. Furthermore, $J = I$, identity operator on H and $\Pi_C = P_C$, projection mapping from H into C. Thus, the following corollaries hold.

Corollary 3.5. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $\{T_1, T_2, \ldots, T_N\}$ and $\{S_1, S_2, \ldots, S_N\}$ be two finite families of closed weak quasi-nonexpansive mappings from C into itself with $F \neq \emptyset$, where $F := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap VI(A, C)$ with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in F$. Assume that T_i and S_i are uniformly continuous for all $i \in \{1, 2, \ldots, N\}$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
\begin{aligned}
&x_0 = x \in C, \quad \textit{chosen arbitrary}, \\
&C_1 = C, \quad x_1 = P_{C_1} x_0, \\
&w_n = P_C(x_n - r_n A x_n), \\
&z_n = \left(\alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n S_n w_n\right), \\
&y_n = (\delta_n x_1 + (1-\delta_n) z_n), \\
&C_{n+1} = \left\{u \in C_n : \|u - y_n\|^2 \leq \delta_n \|u - x_1\|^2 + (1-\delta_n)\left[\alpha_n \|u - x_{n-1}\|^2 + (1-\alpha_n)\|u - x_n\|^2\right]\right\}, \\
&x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{aligned}
\tag{3.80}
$$

where $T_n = T_{n(\bmod N)}$, $S_n = S_{n(\bmod N)}$, and J is the normalized duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, and $\{r_n\}$ are the sequences in $[0,1]$ satisfying the restrictions:

(C1) $\lim_{n\to\infty}\delta_n = 0$;

(C2) $r_n \subset [a,b]$ *for some* a,b *with* $0 < a < b < c^2\alpha/2$, *where* $1/c$ *is the 2-uniformly convexity constant of* E;

(C3) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

(a) $\liminf_{n\to\infty}\alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty}\alpha_n\gamma_n > 0$ *and*

(b) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\liminf_{n\to\infty}\beta_n\gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $P_F x_1$, where P_F is the metric projection from C onto F.

Let X be a nonempty closed convex cone in E, and let A be an operator from X into E^*. We define its *polar* in E^* to be the set

$$X^* = \{y^* \in E^* : \langle x, y^*\rangle \geq 0 \,\forall x \in X\}. \tag{3.81}$$

Then an element x in X is called a *solution of the complementarity problem* if

$$Ax \in X^*, \qquad \langle x, Ax\rangle = 0. \tag{3.82}$$

The set of all solutions of the complementarity problem is denoted by $CP(A,X)$. Several problem arising in different fields, such as mathematical programming, game theory, mechanics, and geometry, are to find solutions of the complementarity problems.

Theorem 3.6. *Let X be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E, let A be an α-inverse-strongly monotone mapping of X into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in X$ and $q \in F$. Let $\{T_1, T_2, \ldots, T_N\}$ and $\{S_1, S_2, \ldots, S_N\}$ be two finite families of closed relatively weak quasi-nonexpansive mappings from X into itself with $F \neq \emptyset$, where $F := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap CP(A,X)$. Assume that T_i and S_i are uniformly continuous for all $i \in \{1,2,\ldots,N\}$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned}
&x_0 = x \in X, \quad \text{chosen arbitrary},\\
&C_1 = X, \quad x_1 = \Pi_{C_1}x_0,\\
&w_n = \Pi_C J^{-1}(Jx_n - r_n Ax_n),\\
&z_n = J^{-1}\left(\alpha_n Jx_{n-1} + \beta_n JT_n x_n + \gamma_n JS_n w_n\right),\\
&y_n = J^{-1}(\delta_n Jx_1 + (1-\delta_n)Jz_n),\\
&C_{n+1} = \left\{u \in C_n : \phi(u,y_n) \leq \delta_n\phi(u,x_1) + (1-\delta_n)\left[\alpha_n\phi(u,x_{n-1}) + (1-\alpha_n)\phi(u,x_n)\right]\right\},\\
&x_{n+1} = \Pi_{C_{n+1}}x_1, \quad \forall n \geq 1,
\end{aligned} \tag{3.83}$$

where $T_n = T_{n(\text{mod } N)}, S_n = S_{n(\text{mod } N)}$, *and* J *is the normalized duality mapping on* E. *Assume that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ *and* $\{r_n\}$ *are the sequences in* $[0,1]$ *satisfying the restrictions:*

(C1) $\lim_{n\to\infty} \delta_n = 0$;

(C2) $r_n \subset [a,b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$, *where* $1/c$ *is the 2-uniformly convexity constant of* E;

(C3) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

(a) $\liminf_{n\to\infty}\alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty}\alpha_n\gamma_n > 0$ *and*

(b) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\liminf_{n\to\infty}\beta_n\gamma_n > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_F x_1$, *where* Π_F *is the generalized projection from* X *onto* F.

Proof. From [25, Lemma 7.1.1], we have $VI(A,X) = CP(A,X)$. From Theorem 3.1, we can obtain the desired conclusion easily. □

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References

[1] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, vol. 88 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1980.

[2] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[3] W. Takahashi, *Convex Analysis and Approximation Fixed Points*, vol. 2 of *Mathematical Analysis Series*, Yokohama Publishers, Yokohama, Japan, 2000.

[4] M. M. Vainberg, *Variational Methods and Method of Monotone Operators*, John Wiley & Sons, New York, NY, USA, 1973.

[5] Ya. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, A. G. Kartsatos, Ed., vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 15–50, Marcel Dekker, New York, NY, USA, 1996.

[6] Ya. I. Alber and S. Reich, "An iterative method for solving a class of nonlinear operator equations in Banach spaces," *PanAmerican Mathematical Journal*, vol. 4, no. 2, pp. 39–54, 1994.

[7] D. Butnariu, S. Reich, and A. J. Zaslavski, "Weak convergence of orbIts of nonlinear operators in reflexive Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 24, no. 5-6, pp. 489–508, 2003.

[8] S. Reich, "A weak convergence theorem for the alternating method with Bregman distances," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, A. G. Kartsatos, Ed., vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 313–318, Marcel Dekker, New York, NY, USA, 1996.

[9] D. Butnariu, S. Reich, and A. J. Zaslavski, "Asymptotic behavior of relatively nonexpansive operators in Banach spaces," *Journal of Applied Analysis*, vol. 7, no. 2, pp. 151–174, 2001.

[10] Y. Censor and S. Reich, "Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization," *Optimization*, vol. 37, no. 4, pp. 323–339, 1996.

[11] F. Kohsaka and W. Takahashi, "Strong convergence of an iterative sequence for maximal monotone operators in a Banach space," *Abstract and Applied Analysis*, vol. 2004, no. 3, pp. 239–249, 2004.

[12] S. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.
[13] S. Plubtieng and K. Ungchittrakool, "Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 149, no. 2, pp. 103–115, 2007.
[14] X. Qin and Y. Su, "Strong convergence theorems for relatively nonexpansive mappings in a Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 6, pp. 1958–1965, 2007.
[15] H. Iiduka and W. Takahashi, "Weak convergence of a projection algorithm for variational inequalities in a Banach space," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 668–679, 2008.
[16] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.
[17] H. Iiduka, W. Takahashi, and M. Toyoda, "Approximation of solutions of variational inequalities for monotone mappings," *Panamerican Mathematical Journal*, vol. 14, no. 2, pp. 49–61, 2004.
[18] F. Liu and M. Z. Nashed, "Regularization of nonlinear ill-posed variational inequalities and convergence rates," *Set-Valued Analysis*, vol. 6, no. 4, pp. 313–344, 1998.
[19] H.-K. Xu and R. G. Ori, "An implicit iteration process for nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 22, no. 5-6, pp. 767–773, 2001.
[20] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.
[21] X. Qin, Y. J. Cho, S. M. Kang, and H. Zhou, "Convergence of a modified Halpern-type iteration algorithm for quasi-ϕ-nonexpansive mappings," *Applied Mathematics Letters*, vol. 22, no. 7, pp. 1051–1055, 2009.
[22] G. Cai and C. S. Hu, "On the strong convergence of the implicit iterative processes for a finite family of relatively weak quasi-nonexpansive mappings," *Applied Mathematics Letters*, vol. 23, no. 1, pp. 73–78, 2010.
[23] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.
[24] S. Reich, "Geometry of Banach spaces, duality mappings and nonlinear problems," *Bulletin of the American Mathematical Society*, vol. 26, no. 2, pp. 367–370, 1992.
[25] W. Takahashi, *Nonlinear Functional Analysis*, Fixed Point Theory and Its Applications, Yokohama Publishers, Yokohama, Japan, 2000.
[26] K. Ball, E. A. Carlen, and E. H. Lieb, "Sharp uniform convexity and smoothness inequalities for trace norms," *Inventiones Mathematicae*, vol. 115, no. 3, pp. 463–482, 1994.
[27] Y. Takahashi, K. Hashimoto, and M. Kato, "On sharp uniform convexity, smoothness, and strong type, cotype inequalities," *Journal of Nonlinear and Convex Analysis*, vol. 3, no. 2, pp. 267–281, 2002.
[28] B. Beauzamy, *Introduction to Banach Spaces and Their Geometry*, vol. 68 of *North-Holland Mathematics Studies*, North-Holland, Amsterdam, The Netherlands, 2nd edition, 1985.
[29] R. E. Bruck and S. Reich, "Nonexpansive projections and resolvents of accretive operators in Banach spaces," *Houston Journal of Mathematics*, vol. 3, no. 4, pp. 459–470, 1977.
[30] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.
[31] Y. J. Cho, H. Zhou, and G. Guo, "Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings," *Computers & Mathematics with Applications*, vol. 47, no. 4-5, pp. 707–717, 2004.
[32] R. T. Rockafellar, "On the maximality of sums of nonlinear monotone operators," *Transactions of the American Mathematical Society*, vol. 149, pp. 75–88, 1970.