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Research Article

Weighted Inequalities for Potential Operators on Differential Forms

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We develop the weak-type and strong-type inequalities for potential operators under two-weight conditions to the versions of differential forms. We also obtain some estimates for potential operators applied to the solutions of the nonhomogeneous A-harmonic equation.

1. Introduction

In recent years, differential forms as the extensions of functions have been rapidly developed. Many important results have been obtained and been widely used in PDEs, potential theory, nonlinear elasticity theory, and so forth; see [1–3]. In many cases, the process to solve a partial differential equation involves various norm estimates for operators. In this paper, we are devoted to develop some two-weight norm inequalities for potential operator P to the versions of differential forms.

We first introduce some notations. Throughout this paper we always use E to denote an open subset of $\mathbb{R}^n$, $n \geq 2$. Assume that $B \subset \mathbb{R}^n$ is a ball and σB is the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. Let $\wedge^k = \wedge^k(\mathbb{R}^n)$, $k = 0, 1, \ldots, n$, be the linear space of all k-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1, i_2, \ldots, i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ with summation over all ordered k-tuples $I = (i_1, i_2, \ldots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. The Grassman algebra $\wedge = \oplus_{k=0}^n \wedge^k$ is a graded algebra with respect to the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$. Moreover, if the coefficient $\omega_I(x)$ of k-form $\omega(x)$ is differential on E, then we call $\omega(x)$ a differential k-form on E and use $D'(E, \wedge^k)$ to denote the space of all differential k-forms on E. In fact, a differential k-form $\omega(x)$ is a Schwarz distribution on E with value in $\wedge^k(\mathbb{R}^n)$. For any $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is defined by $(\alpha, \beta) = \sum \alpha^I \beta^I$ with summation over all k-tuples I and all $k = 0, 1, \ldots, n$. As usual, we still use $\star$ to denote the Hodge star operator. Moreover, the norm of $\omega \in \wedge$ is given by $|\omega|^2 = (\omega, \omega) = \star(\omega \wedge \star\omega) \in \wedge^0 = \mathbb{R}$. Also, we use $d : D'(E, \wedge^k) \to D'(E, \wedge^{k+1})$ to denote the differential operator and use $d^\star : D'(E, \wedge^{k+1}) \to D'(E, \wedge^k)$ to denote the Hodge codifferential operator defined by $d^\star = (-1)^{nk+1} \star d\star$ on $D'(E, \wedge^{k+1})$, $k = 0, 1, \ldots, n-1$.

A weight $w(x)$ is a nonnegative locally integrable function on $\mathbb{R}^n$. The Lebesgue measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$. $L^p(E, \wedge^k)$ is a Banach space with norm

$$\|\omega\|_{p,E} = \left(\int_E |\omega(x)|^p dx\right)^{1/p} = \left(\int_E \left(\sum_I |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}. \tag{1.1}$$

Similarly, for a weight $w(x)$, we use $L^p(E, \wedge^k, w)$ to denote the weighted L^p space with norm $\|\omega\|_{p,E,w} = (\int_E |\omega(x)|^p w\, dx)^{1/p}$.

From [1], if ω is a differential form defined in a bounded, convex domain M, then there is a decomposition

$$\omega = d(T\omega) + T(d\omega), \tag{1.2}$$

where T is called a homotopy operator. Furthermore, we can define the k-form $\omega_M \in D'(M, \wedge^k)$ by

$$\omega_M = |M|^{-1} \int_M \omega(y) dy, \quad k = 0, \qquad \omega_M = d(T\omega), \quad k = 1, 2, \ldots, n \tag{1.3}$$

for all $\omega \in L^p(M, \wedge^k)$, $1 \leq p < \infty$.

For any differential k-form $\omega(x)$, we define the potential operator P by

$$P\omega(x) = \sum_I \int_E K(x,y) \omega_I(y) dy\, dx_I, \tag{1.4}$$

where the kernel $K(x,y)$ is a nonnegative measurable function defined for $x \neq y$ and the summation is over all ordered k-tuples I. It is easy to find that the case $k = 0$ reduces to the usual potential operator. That is,

$$Pf(\mathrm{x}) = \int_E K(x,y) f(y) dy, \tag{1.5}$$

where $f(x)$ is a function defined on $E \subset \mathbb{R}^n$. Associated with P, the functional φ is defined as

$$\varphi(B) = \sup_{x,y \in B,\ |x-y| \geq Cr} K(x,y), \tag{1.6}$$

where C is some sufficiently small constant and $B \subset E$ is a ball with radius r. Throughout this paper, we always suppose that φ satisfies the following conditions: there exists C_φ such that

$$\varphi(2B) \leq C_\varphi \varphi(B) \quad \text{for all balls } B \subset E, \tag{1.7}$$

and there exists $\varepsilon > 0$ such that

$$\varphi(B_1)\mu(B_1) \leq C_\varphi \left(\frac{r(B_1)}{r(B_2)}\right)^\varepsilon \varphi(B_2)\mu(B_2) \quad \text{for all balls } B_1 \subset B_2. \tag{1.8}$$

On the potential operator P and the functional φ, see [4] for details.

For any locally L^p-integrable form ω, the Hardy-Littlewood maximal operator $\mathbb{M}_p$ is defined by

$$\mathbb{M}_p(\omega) = \sup_{r>0}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |\omega(y)|^p dy\right)^{1/p}, \tag{1.9}$$

where $B(x,r)$ is the ball of radius r, centered at x, $1 \leq p < \infty$.

Consider the nonhomogeneous A-harmonic equation for differential forms as follows:

$$d^\star A(x, d\omega) = B(x, d\omega), \tag{1.10}$$

where $A : E \times \wedge^k(\mathbb{R}^n) \to \wedge^k(\mathbb{R}^n)$ and $B : E \times \wedge^k(\mathbb{R}^n) \to \wedge^{k-1}(\mathbb{R}^n)$ are two operators satisfying the conditions

$$|A(x,\xi)| \leq a|\xi|^{p-1}, \qquad A(x,\xi)\cdot\xi \geq |\xi|^p, \qquad |B(x,\xi)| \leq b|\xi|^{p-1} \tag{1.11}$$

for almost every $x \in E$ and all $\xi \in \wedge^k(\mathbb{R}^n)$. Here $a, b > 0$ are some constants and $1 < p < \infty$ is a fixed exponent associated with (1.10). A solution to (1.10) is an element of the Sobolev space $W^{1,p}_{\text{loc}}(E, \wedge^{k-1})$ such that

$$\int_E A(x, d\omega)\cdot d\varphi + B(x, d\omega)\cdot\varphi = 0 \tag{1.12}$$

for all $\varphi \in W^{1,p}_{\text{loc}}(E, \wedge^{k-1})$ with compact support. Here $W^{1,p}(E, \wedge^k)$ are those differential k-forms on E whose coefficients are in $W^{1,p}(E, \mathbb{R}^n)$. The notation $W^{1,p}_{\text{loc}}(E, \wedge^k)$ is self-explanatory.

2. Weak Type (p, p) Inequalities for Potential Operators

In this section, we establish the weighted weaks type (p, p) inequalities for potential operators applied to differential forms. To state our results, we need the following definitions and lemmas.

We first need the following generalized Hölder inequality.

Lemma 2.1. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$, and $1/s = 1/\alpha + 1/\beta$. If f and g are two measurable functions on $\mathbb{R}^n$, then*

$$\|fg\|_{s,E} \leq \|f\|_{\alpha,E} \cdot \|g\|_{\beta,E} \tag{2.1}$$

for any $E \subset \mathbb{R}^n$.

Definition 2.2. *A pair of weights $(w_1(x), w_2(x))$ satisfies the $A_{r,\lambda}(E)$-condition in a set $E \subset \mathbb{R}^n$; write $(w_1(x), w_2(x)) \in A_{r,\lambda}(E)$ for some $\lambda \geq 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$ if*

$$\sup_{B \subset E} \left(\frac{1}{|B|}\int_B w_1^{\lambda} dx\right)^{1/\lambda r} \left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{1/\lambda r'} < \infty. \tag{2.2}$$

Proposition 2.3. *If $(w_1(x), w_2(x)) \in A_{r,\lambda}(E)$ for some $\lambda \geq 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$, then $(w_1(x), w_2(x))$ satisfies the following condition:*

$$\sup_{B \subset E} \left(\frac{1}{|B|}\int_B w_1^{\lambda} dx\right)^{1/\lambda r} \left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{1/(r-1)} dx\right)^{(r-1)/r} < \infty. \tag{2.3}$$

Proof. Choose $r - 1 = (r-1)/\lambda + 1/s$ and $1/r + 1/r' = 1$. From the Hölder inequality, we have the estimate

$$\begin{aligned} &\left(\frac{1}{|B|}\int_B w_1^{\lambda} dx\right)^{1/\lambda r} \left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{1/(r-1)} dx\right)^{(r-1)/r} \\ &\quad\leq |B|^{-1/\lambda r-(r-1)/r+1/rs} \left(\int_B w_1^{\lambda} dx\right)^{1/\lambda r} \left(\int_B \left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{1/\lambda r'} \\ &\quad= \left(\frac{1}{|B|}\int_B w_1^{\lambda} dx\right)^{1/\lambda r} \left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{1/\lambda r'}. \end{aligned} \tag{2.4}$$

Since

$$\sup_{B \subset E} \left(\frac{1}{|B|}\int_B w_1^{\lambda} dx\right)^{1/\lambda r} \left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{1/\lambda r'} < \infty, \tag{2.5}$$

we obtain that $(w_1(x), w_2(x))$ satisfies (2.3) as required. □

In [4], Martell proved the following two-weight weak type norm inequality applied to functions.

Lemma 2.4. *Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Assume that P is the potential operator defined in* (1.5) *and that φ is a functional satisfying* (1.7) *and* (1.8). *Let $(w_1(x), w_2(x))$ be a pair of weights for which there exists $r > 1$ such that*

$$\sup_{B\subset E} \varphi(B)|B| \left(\frac{1}{|B|}\int_B w_1^r dx\right)^{1/rp} \left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{p'-1} dx\right)^{1/p'} < \infty. \tag{2.6}$$

Then the potential operator P verifies the following weak type (p, p) inequality:

$$\sup_{\lambda>0} \lambda\mu(\{x \in E : |Pf(x)| > \lambda\})^{1/p} \leq C\left(\int_E |f(x)|^p d\nu\right)^{1/p}, \tag{2.7}$$

where $\mu(D) = \int_D w_1 dx$ for any set $D \subset \mathbb{R}^n$ and $d\nu = w_2 dx$.

The following definition is introduced in [5].

Definition 2.5. *A kernel K on $\mathbb{R}^n \times \mathbb{R}^n$ satisfies the standard estimates if there exist δ, $0 < \delta \leq 1$, and constant C such that for all distinct points x and y in $\mathbb{R}^n$, and all z with $|x - z| < (1/2)|x - y|$, the kernel K satisfies* (1) $K(x, y) \leq C|x - y|^{-n}$; (2) $|K(x, y) - K(z, y)| \leq C|x - z|^{\delta}|x - y|^{-n-\delta}$; (3) $|K(y, x) - K(y, z)| \leq C|x - z|^{\delta}|x - y|^{-n-\delta}$.

Theorem 2.6. *Let P be the potential operator defined in* (1.4) *with the kernel $K(x, y)$ satisfying the condition* (1) *of the standard estimates and let $\omega \in D'(E, \wedge^k)$, $k = 0, 1, \ldots, n$ be a differential form in a domain E. Assume that $(w_1(x), w_2(x))$ satisfies* (2.3) *for some $r > 1$ and $1 < p < \infty$. Then, there exists a constant C, independent of ω, such that the potential operator P satisfies the following weak type (p, p) inequality:*

$$\sup_{\lambda>0} \lambda\mu(\{x \in E : |P\omega(x)| > \lambda\})^{1/p} \leq C\left(\int_E |\omega(x)|^p d\nu\right)^{1/p}, \tag{2.8}$$

where $\mu(D) = \int_D w_1 dx$ for any set $D \subset \mathbb{R}^n$ and $d\nu = w_2 dx$.

Proof. Since $K(x, y)$ satisfies condition (1) of the standard estimates, for any ball $B \subset E$ of radius r, we have

$$|B|\varphi(B) = |B| \sup_{x,y\in B,\ |x-y|\geq C_1 r} K(x, y) \leq |B| \sup_{x,y\in B,\ |x-y|\geq C_1 r} C_2|x - y|^{-n} \leq \frac{C_3|B|}{r^n} \leq C_4. \tag{2.9}$$

Here C_1 and C_2 are two constants independent of B. Therefore, C_3 and C_4 are some constants independent of B. Thus, from $(w_1(x), w_2(x))$ satisfying (2.3) for some $r > 1$ and $1 < p < \infty$, it follows that

$$\sup_{B\subset E} \varphi(B)|B|\left(\frac{1}{|B|}\int_B w_1^r dx\right)^{1/rp}\left(\frac{1}{|B|}\int_B w_2^{1/(1-p)} dx\right)^{(p-1)/p} < \infty. \tag{2.10}$$

Set $D = \{x \in E : |P\omega(x)| > \lambda\}$ and $D_I = \{x \in D : |P\omega_I(x)| > \lambda/\sqrt{m}\}$, where I corresponds to all ordered k-tuples and $m = C_n^k$. It is easy to find that there must exist some J such that $|P\omega_J(x)| > \lambda/\sqrt{m}$ whenever $x \in D$. Since the reverse is obvious, we immediately get $D = \bigcup_I D_I$. Thus, using Lemma 2.4 and the elementary inequality $|a+b|^s \le 2^s(|a|^s + |b|^s)$, where $s > 0$ is any constant, we have

$$\begin{aligned}
\mu(\{x \in E : |P\omega(x)| > \lambda\})^{1/p} &= \left(\int_{\bigcup_I D_I} w_1(x)dx\right)^{1/p} \\
&\le \left(\sum_I \int_{D_I} w_1(x)dx\right)^{1/p} \\
&\le C_5 \sum_I \left(\int_{D_I} w_1(x)dx\right)^{1/p}.
\end{aligned} \tag{2.11}$$

Combining the above inequality (2.11), the elementary inequality and Lemma 2.4 yield

$$\begin{aligned}
\lambda^p \mu(\{x \in E : |P\omega(x)| > \lambda\}) &\le C_6 \sum_I \lambda^p \left(\int_{D_I} w_1(x)dx\right) \\
&\le C_7 \sum_I \left(\frac{\lambda}{\sqrt{m}}\right)^p \mu\left(\left\{x \in E : |P\omega_I(x)| > \frac{\lambda}{\sqrt{m}}\right\}\right) \\
&\le C_7 \sum_I \left(\int_E |\omega_I(x)|^p d\nu\right) \\
&\le C_7 \int_E \left(\sum_I |\omega_I(x)|^2\right)^{p/2} d\nu \\
&= C_7 \int_E |\omega(x)|^p d\nu.
\end{aligned} \tag{2.12}$$

We complete the proof of Theorem 2.6. □

3. The Strong Type (p, p) Inequalities for Potential Operators

In this section, we give the strong type (p, p) inequalities for potential operators applied to differential forms. The result in last section shows that $A_{r,\lambda}$-weights are stronger than those of condition (2.3), which is sufficient for the weak (p, p) inequalities, while the following conclusions show that $A_{r,\lambda}$-condition is sufficient for strong (p, p) inequalities.

The following weak reverse Hölder inequality appears in [6].

Lemma 3.1. *Let $\omega \in D'(E, \wedge^k)$, $k = 0, 1, \ldots, n$ be a solution of the nonhomogeneous A-harmonic equation in E, $\rho > 1$ and $0 < s, t < \infty$. Then there exists a constant C, independent of ω, such that*

$$\|\omega\|_{s,B} \leq C|B|^{(t-s)/st}\|\omega\|_{t,\rho B} \tag{3.1}$$

for all balls B with $\rho B \subset E$.

The following two-weight inequality appears in [7].

Lemma 3.2. *Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Assume that P is the potential operator defined in* (1.5) *and φ is a functional satisfying* (1.7) *and* (1.8). *Let (w_1, w_2) be a pair of weights for which there exists $r > 1$ such that*

$$\sup_{B\subset E} \varphi(B)|B|\left(\frac{1}{|B|}\int_B w_1^r dx\right)^{1/rp}\left(\frac{1}{|B|}\int_B w_2^{(1-p')r} dx\right)^{1/rp'} < \infty. \tag{3.2}$$

Then, there exists a constant C, independent of f, such that

$$\|Pf(x)\|_{p,E,w_1} \leq \|f(x)\|_{p,E,w_2}. \tag{3.3}$$

Lemma 3.3. *Let $\omega \in L^p(E, \wedge^k)$, $k = 0, 1, \ldots, n$, $1 < p < \infty$, be a differential form defined in a domain E and P be the potential operator defined in* (1.4) *with the kernel $k(x, y)$ satisfying condition* (1) *of standard estimates. Assume that $(w_1, w_2) \in A_{r,\lambda}(E)$ for some $\lambda \geq 1$ and $1 < r < \infty$. Then, there exists a constant C, independent of ω, such that*

$$\|P(\omega)\|_{p,E,w_1} \leq C\|\omega\|_{p,E,w_2}. \tag{3.4}$$

Proof. By the proof of Theorem 2.6, note that (3.2) still holds whenever (w_1, w_2) satisfies the $A_{r,\lambda}(E)$-condition. Therefore, using Lemma 3.2, we have

$$\begin{aligned}\|P(\omega)\|_{p,E,w_1}^p &= \int_E |P(\omega)|^p w_1 dx \\ &= \int_E \left(\sum_I |P\omega_I(x)|^2\right)^{p/2} w_1 dx\end{aligned}$$

$$\le C_1 \int_E \sum_I |P\omega_I(x)|^p w_1 dx$$
$$= C_1 \sum_I \|P\omega_I(x)\|^p_{p,E,w_1}. \tag{3.5}$$

Also, Lemma 3.2 yields that

$$\|P\omega_I(x)\|^p_{p,E,w_1} \le C_I \|\omega_I(x)\|^p_{p,E,w_2} \tag{3.6}$$

for all ordered k-tuples I. From (3.5) and (3.6), it follows that

$$\begin{aligned} \|P(\omega)\|^p_{p,E,w_1} &\le C_2 \sum_I \|\omega_I(x)\|^p_{p,E,w_2} \\ &= C_2 \int_E \sum_I |\omega_I(x)|^p w_2 dx \\ &\le C_3 \int_E \left(\sum_I |\omega_I(x)|^2\right)^{p/2} w_2 dx \\ &= C_3 \|\omega\|^p_{p,E,w_2}. \end{aligned} \tag{3.7}$$

We complete the proof of Lemma 3.3. □

Lemma 3.3 shows that the two-weight strong (p,p) inequality still holds for differential forms. Next, we develop the inequality to the parametric version.

Theorem 3.4. *Let $\omega \in L^p(E, \wedge^k)$, $k = 0, 1, \ldots, n$, $1 < p < \infty$, be the solution of the nonhomogeneous A-harmonic equation in a domain E and let P be the potential operator defined in* (1.4) *with the kernel $k(x,y)$ satisfying condition* (1) *of standard estimates. Assume that $(w_1, w_2) \in A_{r,\lambda}(E)$ for some $\lambda \ge 1$ and $1 < r < \infty$. Then, there exists a constant C, independent of ω, such that*

$$\|P(\omega)\|_{p,B,w_1^\alpha} \le \|\omega\|_{p,\sigma B,w_2^\alpha} \tag{3.8}$$

for all balls $B \subset E$ with $\sigma B \subset E$. Here $\sigma > 1$ and α are constants with $0 < \alpha < \lambda$.

Proof. Take $t = p\lambda/\alpha$. By $1/p = 1/t + 1/k$, where $k = pt/(p-t)$ and the Hölder inequality, we have

$$\|P(\omega)\|_{p,B,w_1^\alpha} = \left(\int_B \left(|P(\omega)| w_1^{\alpha/p}\right)^p dx\right)^{1/p} \le \left(\int_B |P(\omega)|^k dx\right)^{1/k} \left(\int_B w_1^\lambda dx\right)^{\alpha/p\lambda} \tag{3.9}$$

for all balls B with $B \subset E$. Choosing E to be a ball and $w_1(x) = w_2(x) = 1$ in Lemma 3.3, then there exists a constant C_1, independent of ω, such that

$$\|P(\omega)\|_{k,B} \leq C_1 \|\omega\|_{k,B}. \tag{3.10}$$

Choosing $s = \lambda p/(\lambda + \alpha(r-1))$ and using Lemma 3.1, we obtain

$$\|\omega\|_{k,B} \leq C_2 |B|^{(s-k)/sk} \|\omega\|_{s,\sigma B}, \tag{3.11}$$

where $\sigma > 1$. Combining (3.9), (3.10), and (3.11), it follows that

$$\|P(\omega)\|_{p,B,w_1^\alpha} \leq C_3 |B|^{(s-k)/sk} \|\omega\|_{s,\sigma B} \left(\int_B w_1^\lambda dx \right)^{\alpha/p\lambda}. \tag{3.12}$$

Since $s < p$, using the Hölder inequality with $1/s = 1/p + (p-s)/sp$, we obtain

$$\|\omega\|_{s,\sigma B} = \left(\int_{\sigma B} \left(|\omega| w_2^{\alpha/p} w_2^{-\alpha/p} \right)^s dx \right)^{1/s} \leq \left(\int_{\sigma B} |\omega|^p w_2^\alpha dx \right)^{1/p} \left(\int_{\sigma B} w_2^{s\alpha/(s-p)} dx \right)^{(p-s)/sp}. \tag{3.13}$$

From the condition $(w_1(x), w_2(x)) \in A_{r,\lambda}(E)$, we have

$$\begin{aligned}
&\left(\int_B w_1^\lambda dx \right)^{\alpha/p\lambda} \left(\int_{\sigma B} w_2^{s\alpha/(s-p)} dx \right)^{(p-s)/sp} \\
&\quad \leq \left(\int_{\sigma B} w_1^\lambda dx \right)^{\alpha/p\lambda} \left(\int_{\sigma B} \left(\frac{1}{w_2} \right)^{\lambda r'/r} dx \right)^{\alpha(r-1)/\lambda p} \\
&\quad \leq C_4 |B|^{1/t+1/s-1/p} \left(\left(\frac{1}{|\sigma B|} \int_{\sigma B} w_1^\lambda dx \right)^{1/\lambda r} \left(\frac{1}{|\sigma B|} \int_{\sigma B} \left(\frac{1}{w_2} \right)^{\lambda r'/r} dx \right)^{1/\lambda r'} \right)^{\alpha r/p} \\
&\quad \leq C_5 |B|^{1/t+1/s-1/p}.
\end{aligned} \tag{3.14}$$

Combining (3.12), (3.13), and (3.14) yields

$$\|P(\omega)\|_{p,B,w_1^\alpha} \leq C_6 \|\omega\|_{p,\sigma B,w_2^\alpha} \tag{3.15}$$

for all balls B with $\sigma B \subset E$. Thus, we complete the proof of Theorem 3.4. □

Next, we extend the weighted inequality to the global version, which needs the following lemma about Whitney cover that appears in [6].

Lemma 3.5. *Each open set $E \subset \mathbb{R}^n$ has a modified Whitney cover of cubes $\vartheta = \{Q_i\}$ such that*

$$\bigcup_i Q_i = E, \tag{3.16}$$

$$\sum_{Q\in\vartheta} \chi_{\sqrt{5/4}Q}(x) \le N\chi_E(x), \tag{3.17}$$

for all $x \in \mathbb{R}^n$ and some $N > 1$, where χ_D is the characteristic function for a set D.

Theorem 3.6. *Let $\omega \in L^p(E, \wedge^k)$, $k = 0, 1, \ldots, n$, $1 < p < \infty$, be the solution of the nonhomogeneous A-harmonic equation in a domain E and let P be the potential operator defined in* (1.4) *with the kernel $k(x, y)$ satisfying condition* (1) *of standard estimates. Assume that $(w_1, w_2) \in A_{r,\lambda}(E)$ for some $\lambda \ge 1$ and $1 < r < \infty$. Then, there exists a constant C, independent of ω, such that*

$$\|P(\omega)\|_{p,E,w_1^\alpha} \le C\|\omega\|_{p,E,w_2^\alpha}, \tag{3.18}$$

where α is some constant with $0 < \alpha < \lambda$.

Proof. From Lemma 3.5, we note that E has a modified Whitney cover $\vartheta = \{Q_i\}$. Hence, by Theorem 3.4, we have that

$$\begin{aligned}
\|P(\omega)\|_{p,E,w_1^\alpha} &\le \sum_{Q_i\in\vartheta} \|P(\omega)\|_{p,Q_i,w_1^\alpha} \\
&\le \sum_{Q_i\in\vartheta} \left(C_i\|\omega\|_{p,\sigma_i Q_i,w_2^\alpha}\right) \\
&\le \sum_{Q_i\in\vartheta} \left(C_i\|\omega\|_{p,\sigma_i Q_i,w_2^\alpha}\right)\chi_{\sqrt{5/4}Q_i}(x) \\
&\le C_1\|\omega\|_{p,E,w_2^\alpha} \sum_{Q_i\in\vartheta} \chi_{\sqrt{5/4}Q_i}(x) \\
&\le C_2\|\omega\|_{p,E,w_2^\alpha}.
\end{aligned} \tag{3.19}$$

This completes the proof of Theorem 3.6. □

Remark 3.7. Note that if we choose the kernel $k(x, y) = \phi(x - y)$ to satisfy the standard estimates, then the potential operators P reduce to the Calderón-Zygmund singular integral operators. Hence, Theorems 3.4 and 3.6 as well as Theorem 2.6 in last section still hold for the Calderón-Zygmund singular integral operators applied to differential forms.

4. Applications

In this section, we apply our results to some special operators. We first give the estimate for composite operators. The following lemma appears in [8].

Lemma 4.1. *Let $\mathbb{M}_s$ be the Hardy-Littlewood maximal operator defined in* (1.9) *and let $\omega \in L^t(E, \wedge^k)$, $k = 1, 2, \ldots, n$, $1 \le s < t < \infty$, be a differential form in a domain E. Then, $\mathbb{M}_s(\omega) \in L^t(E)$ and*

$$\|\mathbb{M}_s(\omega)\|_{t,E} \le C\|\omega\|_{t,E} \tag{4.1}$$

for some constant C independent of ω.

Observing Lemmas 4.1 and 3.3, we immediately have the following estimate for the composition of the Hardy-Littlewood maximal operator $\mathbb{M}_s$ and the potential operator P.

Theorem 4.2. *Let $\omega \in L^p(E, \wedge^k)$, $k = 1, 2, \ldots, n$, $1 < p < \infty$, be a differential form defined in a domain E, $\mathbb{M}_s$ be the Hardy-Littlewood maximal operator defined in* (1.9), *$1 \le s < p < \infty$, and let P be the potential operator with the kernel $k(x, y)$ satisfying condition* (1) *of standard estimates. Then, there exists a constant C, independent of ω, such that*

$$\|\mathbb{M}_s(P(\omega))\|_{p,E} \le C\|\omega\|_{p,E}. \tag{4.2}$$

Next, applying our results to some special kernels, we have the following estimates.
Consider that the function $\varphi(x)$ is defined by

$$\varphi(x) = \frac{1}{c}\exp\left\{\frac{1}{|x|^2 - 1}\right\} \quad \text{if } |x| < 1, \qquad \varphi(x) = 0 \quad \text{if } |x| \ge 1, \tag{4.3}$$

where $c = \int_{B(0,1)} e^{1/(|x|^2-1)}dx$. For any $\varepsilon > 0$, we write $\varphi_\varepsilon(x) = (1/\varepsilon^n)\varphi(x/\varepsilon)$. It is easy to see that $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}\varphi_\varepsilon(x)dx = 1$. Such functions are called mollifiers. Choosing the kernel $k(x, y) = \varphi_\varepsilon(x - y)$ and setting each coefficient of $\omega \in D'(E, \wedge^k)$ satisfing $\sup p\omega_I \subset E$, we have the following estimate.

Theorem 4.3. *Let $\omega \in D'(E, \wedge^k)$, $k = 0, 1, \ldots, n-1$, be a differential form defined in a bounded, convex domain E, and let ω_I be coefficient of ω with $\operatorname{supp}\omega_I \subset E$ for all ordered k-tuples I. Assume that $1 < p < \infty$ and P is the potential operator with $k(x, y) = \varphi_\varepsilon(x - y)$ for any $\varepsilon > 0$. Then, there exists a constant C, independent of ω, such that*

$$\|P(\omega) - (P(\omega))_E\|_{p,E} \le C|E|\operatorname{diam}(E)\|\omega\|_{p,E}. \tag{4.4}$$

Proof. By the decomposition for differential forms, we have

$$P(\omega) - (P(\omega))_E = T(d(P(\omega))), \tag{4.5}$$

where T is the homotopy operator. Also, from [1], we have

$$\|T(\omega)\|_{p,E} \leq C_1|E| \operatorname{diam}(E)\|\omega\|_{p,E} \tag{4.6}$$

for any differential form ω defined in E. Therefore,

$$\|P(\omega) - (P(\omega))_E\|_{p,E} = \|T(d(P(\omega)))\|_{p,E} \leq C_1|E| \operatorname{diam}(E)\|d(P(\omega))\|_{p,E}. \tag{4.7}$$

Note that

$$\begin{aligned} dP(\omega) &= d\left(\sum_I \int_E k(x-y)\omega_I(y)dy\,dx_I\right) \\ &= d\left(\sum_I \varphi_\varepsilon * \omega_I(x)dx_I\right) \\ &= \sum_I \sum_{i=1}^n \left(\frac{\partial \varphi_\varepsilon}{\partial x_i} * \omega_I\right)(x)dx_i \wedge dx_I, \end{aligned} \tag{4.8}$$

where the notation $*$ denotes convolution. Hence, we have

$$\begin{aligned} \|d(P(\omega))\|_{p,E}^p &= \int_E \left|\sum_I \sum_{i=1}^n \left(\frac{\partial \varphi_\varepsilon}{\partial x_i} * \omega_I\right)(x)dx_i \wedge dx_I\right|^p dx \\ &\leq C_2 \sum_I \sum_{i=1}^n \int_E \left|\frac{\partial \varphi_\varepsilon}{\partial x_i} * \omega_I\right|^p dx \\ &= C_2 \sum_I \sum_{i=1}^n \left\|\frac{\partial \varphi_\varepsilon}{\partial x_i} * \omega_I\right\|_{p,E}^p \\ &\leq C_3 \sum_I \sum_{i=1}^n \left\|\frac{\partial \varphi_\varepsilon}{\partial x_i}\right\|_{1,E}^p \|\omega_I\|_{p,E}^p \\ &= C_3 \left(\sum_{i=1}^n \left\|\frac{\partial \varphi_\varepsilon}{\partial x_i}\right\|_{1,E}^p\right)\left(\sum_I \|\omega_I\|_{p,E}^p\right). \end{aligned} \tag{4.9}$$

Since $\varphi_\varepsilon(x) \in C_0^\infty(\mathbb{R}^n)$, it is easy to find that $\sum_{i=1}^n \|\partial\varphi_\varepsilon/\partial x_i\|_{1,E}^p < \infty$. Therefore, we have

$$\|dP(\omega)\|_{p,E}^p \leq C_4 \sum_I \|\omega_I\|_{p,E}^p = C_4 \sum_I \int_E |\omega_I|^p dx \leq C_5 \int_E \left(\sum_I |\omega_I|^2\right)^{p/2} dx = C_5\|\omega\|_{p,E}^p. \tag{4.10}$$

From (4.7) and (4.10), we obtain

$$\|P(\omega) - (P(\omega))_E\|_{p,E} \leq C|E| \operatorname{diam}(E)\|\omega\|_{p,E}. \tag{4.11}$$

This ends the proof of Theorem 4.3. □

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