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Research Article

Asymptotic Behavior for a Class of Modified α-Potentials in a Half Space

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A class of α-potentials represented as the sum of modified Green potential and modified Poisson integral are proved to have the growth estimates $R_{\alpha,l,l}(x) = o(x_n^{\beta}|x|^{l-2\beta+2}h(|x|)^{-1})$ at infinity in the upper-half space of the n-dimensional Euclidean space, where the function $h(|x|)$ is a positive non-decreasing function on the interval $(0,\infty)$ satisfying certain conditions. This result generalizes the growth properties of analytic functions, harmonic functions, and superharmonic functions.

1. Introduction and Main Results

Let $\mathbf{R}^n$ $(n \geq 2)$ denote the n-dimensional Euclidean space with points $x = (x_1, x_2, \ldots, x_{n-1}, x_n) = (x', x_n)$, where $x' \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open Ω of $\mathbf{R}^n$ are denoted by $\partial\Omega$ and $\overline{\Omega}$, respectively. The upper half-space is the set $H = \{x = (x', x_n) \in \mathbf{R}^n;\ x_n > 0\}$, whose boundary is ∂H. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x, y \in \mathbf{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1})$, $y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbf{R}^{n-1}$ and putting $x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n$, $|x| = \sqrt{x \cdot x}$, $|x'| = \sqrt{x' \cdot x'}$.

For $x \in \mathbf{R}^n$ and $r > 0$, let $B_n(x, r)$ denote the open ball with center at x and radius r in $\mathbf{R}^n$.

It is well known that (see, e.g., [1, Chapter 6]) the positive powers of the Laplace operator Δ can be defined by

$$(-\Delta)^{\alpha/2} f(x) = \mathcal{F}^{-1}\left(|\xi|^{\alpha} \widehat{f}(\xi)\right), \tag{1.1}$$

where $\alpha > 0$, f is a Schwarz function and

$$\mathcal{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-ix\xi} dx. \tag{1.2}$$

It follows that we can extend definition (1.1) to certain negative powers of $-\Delta$, $(-\Delta)^{-\alpha/2}$ for $0 < \alpha < n$ and define an operator I_α by

$$I_\alpha f = (-\Delta)^{-\alpha/2} f = \mathcal{F}^{-1}\left(|\xi|^{-\alpha} \widehat{f}\right), \tag{1.3}$$

where $0 < \alpha < n$ and f is a function in the Schwartz class.

If I_α is defined as the inverse Fourier transform of $|\xi|^{-\alpha}$ (in the sense of distributions), one can show that

$$I_\alpha(x) = \gamma_\alpha |x|^{\alpha - n}, \tag{1.4}$$

where γ_α is a certain constant (see, e.g., [1, page 414] for the exact value of γ_α).

The function I_α is known as the Riesz kernel. It follows immediately from the rules for manipulating Fourier transforms that any Schwartz function f can be written as a Riesz potential,

$$f(x) = I_\alpha g(x) = (I_\alpha * g)(x) = \gamma_\alpha \int_{\mathbf{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}} dy, \tag{1.5}$$

where $0 < \alpha < n$ and $g = (-\Delta)^{\alpha/2} f$.

This Riesz kernel I_α in $\mathbf{R}^n$ inspired us to introduce the modified Riesz kernel for H. To do this, we first set

$$E_\alpha(x) = \begin{cases} -\log|x| & \text{if } \alpha = n = 2, \\ |x|^{\alpha-n} & \text{if } 0 < \alpha < n. \end{cases} \tag{1.6}$$

Let $G_\alpha(x,y)$ be the modified Riesz kernel for H, that is,

$$G_\alpha(x,y) = E_\alpha(x-y) - E_\alpha(x-y^*), \quad x, y \in \overline{H},\ x \neq y,\ 0 < \alpha \leq n, \tag{1.7}$$

where $*$ denotes reflection in the boundary plane ∂H just as $y^* = (y_1, y_2, \ldots, y_{n-1}, -y_n)$.

We define the kernel function $P_\alpha(x, y')$ when $x \in H$ and $y' \in \partial H$ by

$$P_\alpha(x,y') = \left.\frac{\partial G_\alpha(x,y)}{\partial y_n}\right|_{y_n=0} = C_\alpha \frac{x_n}{|x-y'|^{n-\alpha+2}}, \tag{1.8}$$

where $C_\alpha = 2(n-\alpha)$ if $0 < \alpha < n$ and $= 2$ if $\alpha = n = 2$.

We remark that $G_2(x,y)$ and $P_2(x,y')$ are the classical Green function and classical Poisson kernel for H respectively (see, e.g., [2, page 127]).

Next we use the following modified kernel function $P_{\alpha,m}(x,y')$ defined by

$$P_{\alpha,m}(x,y') = \begin{cases} P_\alpha(x,y') & \text{if } |y'| < 1, \\ P_\alpha(x,y') - \displaystyle\sum_{k=0}^{m-1} \frac{C_\alpha x_n |x|^k}{|y|^{n-\alpha+2+k}} C_k^{(n-\alpha+2)/2}\left(\frac{x \cdot y'}{|x||y'|}\right) & \text{if } |y'| \geq 1, \end{cases} \tag{1.9}$$

where m is a nonnegative integer; $C_k^\omega(t)$ $\omega = (n-\alpha)/2$ is the ultraspherical (or Gegenbauer) polynomials (see [3]). The Gegenbauer polynomials come from the generating function

$$\left(1 - 2\,\mathrm{tr} + r^2\right)^{-\omega} = \sum_{k=0}^{\infty} C_k^\omega(t) r^k, \tag{1.10}$$

where $|r| < 1$, $|t| \leq 1$, and $\omega > 0$. The coefficients $C_k^\omega(t)$ are called the ultraspherical (or Gegenbauer) polynomials of degree k associated with ω, each function $C_k^\omega(t)$ is a polynomial of degree k in t. Here note that $P_{2,m}(x,y')$ is the modified Poisson kernel in H, which has been used by several authors (see, e.g., [4–8]).

Motivated by this modified kernel function $P_{\alpha,m}(x,y')$, it is natural to ask if the function $G_\alpha(x,y)$ can also be modified? In this paper, we give an affirmative answer to this question.

First we consider the modified kernel function in case $\alpha = n = 2$, which is defined by

$$E_{n,l}(x-y) = \begin{cases} E_n(x-y) & \text{if } |y| < 1, \\ E_n(x-y) + \Re\left(\log y - \displaystyle\sum_{k=1}^{l-1}\left(\frac{x^k}{k y^k}\right)\right) & \text{if } |y| \geq 1. \end{cases} \tag{1.11}$$

In case $0 < \alpha < n$, we define

$$E_{\alpha,l}(x-y) = \begin{cases} E_\alpha(x-y) & \text{if } |y| < 1, \\ E_\alpha(x-y) - \displaystyle\sum_{k=0}^{l-1} \frac{|x|^k}{|y|^{n-\alpha+k}} C_k^{n-\alpha/2}\left(\frac{x \cdot y}{|x||y|}\right) & \text{if } |y| \geq 1, \end{cases} \tag{1.12}$$

where l is a nonnegative integer, $x, y \in \overline{H}$, and $x \neq y$.

Then we define the modified kernel function $G_{\alpha,l}(x,y)$ by

$$G_{\alpha,l}(x,y) = \begin{cases} E_{n,l+1}(x-y) - E_{n,l+1}(x-y^*) & \text{if } \alpha = n = 2, \\ E_{\alpha,l+1}(x-y) - E_{\alpha,l+1}(x-y^*) & \text{if } 0 < \alpha < n. \end{cases} \tag{1.13}$$

Write

$$G_{\alpha,l}(x,\mu) = \int_H G_{\alpha,l}(x,y)\,d\mu(y),$$
$$U_{\alpha,m}(x,\nu) = \int_{\partial H} P_{\alpha,m}(x,y')\,d\nu(y'), \tag{1.14}$$

where μ(resp., ν) is a nonnegative measure on H(resp., ∂H). Here note that $G_{\alpha,0}(x,\mu)$ is nothing but the Green potential of general order (see [9–11]).

Following Fuglede (see [6]), we set

$$k(y,\mu) = \int_E k(y,x)\,d\mu(x), \qquad k(\mu,x) = \int_E k(y,x)\,d\mu(y), \tag{1.15}$$

for a nonnegative Borel measurable function k on $\mathbf{R}^n \times \mathbf{R}^n$ and a nonnegative measure μ on a Borel set $E \subset \mathbf{R}^n$. We define a capacity C_k by

$$C_k(E) = \sup\, \mu(\mathbf{R}^n), \quad E \subset H, \tag{1.16}$$

where the supremum is taken over all nonnegative measures μ such that S_μ (the support of μ) is contained in E and $k(y,\mu) \le 1$ for every $y \in H$.

For $\beta \le 1$ and $\delta \le 1$, we consider the function $k_{\alpha,\beta,\delta}$ defined by

$$k_{\alpha,\beta,\delta}(y,x) = x_n^{-\beta} y_n^{-\delta} G_\alpha(x,y) \quad \text{for } x,y \in H. \tag{1.17}$$

If $\beta = \delta = 1$, then $k_\alpha = k_{\alpha,1,1}$ is extended to be continuous on $\overline{H} \times \overline{H}$ in the extended sense, where $\overline{H} = H \cup \partial H$.

Now we will discuss the behavior at infinity of the modified Green potential and modified Poisson integral in the upper-half space, respectively. For related results, we refer the readers to the papers by Mizuta (see [9]), Siegel and Talvila (see [8]), and Mizuta and Shimomura (see [7]).

Theorem 1.1. *Let $h(r)$ be a positive nondecreasing function on the interval $(0,\infty)$ such that*

(a) $r^{\beta-1}h(r)$ *is nondecreasing on* $(0,\infty)$,

(b) $r^{\beta-2}h(r)$ *is nonincreasing on* $(0,\infty)$ *and* $\lim_{r\to\infty} r^{\beta-2}h(r) = 0$,

(c) *there exists a positive constant M such that $h(2r) \le Mh(r)$ for any $r > 0$.*

Let μ be a nonnegative measure on H satisfying

$$\int_H \frac{y_n^\delta h(|y|)}{(1+|y|)^{n+l-\alpha-\beta+\delta+2}}\,d\mu(y) < \infty. \tag{1.18}$$

Then there exists a Borel set $E' \subset H$ with properties

(i) $\lim_{|x|\to\infty, x\in H-E'} x_n^{-\beta}|x|^{-l+2\beta-2}h(|x|)G_{\alpha,l}(x,\mu)=0$;

(ii) $\sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+\delta)} C_{k_{\alpha,\beta,\delta}}(E_i')<\infty$,

where $E_i'=\{x\in E' : 2^i\le |x|<2^{i+1}\}$.

Corollary 1.2. *Let μ be a nonnegative measure on H satisfying*

$$\int_H \frac{y_n}{(1+|y|)^{n+l-\alpha+2}}d\mu(y)<\infty. \tag{1.19}$$

Then there exists a Borel set $E\subset H$ with properties

(i) $\lim_{|x|\to\infty, x\in H-E} x_n^{-\beta}|x|^{\beta-l-1}G_{\alpha,l}(x,\mu)=0$;

(ii) $\sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+1)} C_{k_{\alpha,\beta,1}}(E_i)<\infty$,

where $E_i=\{x\in E : 2^i\le |x|<2^{i+1}\}$.

Theorem 1.3. *Let h be defined as in Theorem 1.1 and ν a nonnegative measure on ∂H satisfying*

$$\int_{\partial H} \frac{h(|y'|)}{(1+|y'|)^{n+m-\alpha-\beta+3}}d\nu(y')<\infty. \tag{1.20}$$

Then there exists a Borel set $F\subset H$ with properties

(i) $\lim_{|x|\to\infty, x\in H-F} x_n^{-\beta}|x|^{-m+2\beta-2}h(|x|)U_{\alpha,m}(x,\nu)=0$;

(i) $\sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+1)} C_{k_{\alpha,\beta,1}}(F_i)<\infty$,

where $F_i=\{x\in F : 2^i\le |x|<2^{i+1}\}$.

Remark 1.4. In the case $m=l$, $F=E$.

Corollary 1.5. *Let ν be a nonnegative measure on ∂H satisfying*

$$\int_{\partial H} \frac{1}{(1+|y'|)^{n+l-\alpha+2}}d\nu(y')<\infty. \tag{1.21}$$

Then there exists a Borel set $E\subset H$ satisfying Corollary 1.2 *(ii) such that*

$$\lim_{|x|\to\infty, x\in H-E} x_n^{-\beta}|x|^{\beta-l-1}U_{\alpha,l}(x,\nu)=0. \tag{1.22}$$

We define the modified α-potentials on H by

$$R_{\alpha,l,m}(x)=G_{\alpha,l}(x,\mu)+U_{\alpha,m}(x,\nu), \tag{1.23}$$

where $0<\alpha\le n$ and μ (resp., ν) is a nonnegative measure on H (resp., ∂H) satisfying (1.18) *($\delta=1$) (resp.,* (1.20)*). Clearly, $R_{2,0,m}(x)$ is a superharmonic function on H.*

The following theorem follows readily from Theorems 1.1 and 1.3.

Theorem 1.6. *Let h be defined as in Theorem 1.1 and $R_{\alpha,l,l}(x)$ defined by* (1.23)*. Then there exists a Borel set $E \subset H$ satisfying* Corollary 1.2 *(ii) such that*

$$\lim_{|x|\to\infty, x\in H-E} x_n^{-\beta}|x|^{-l+2\beta-2}h(|x|)R_{\alpha,l,l}(x) = 0. \tag{1.24}$$

Remark 1.7. In the case $h(|x|) = |x|^{1-\beta}(0 \le \beta \le 1)$, by using Lemma 2.5 below, we can easily show that Corollary 1.2 (ii) with $\alpha = 2$ means that E is β-rarefied at infinity in the sense of [12]. In particular, This condition with $\alpha = 2$, $\beta = 1$, and $h(|x|) \equiv 1$(resp., $\alpha = 2$, $\beta = 0$, and $h(|x|) = |x|$) means that E is minimally thin at infinity (resp., rarefied at infinity) in the sense of [13].

Theorem 1.6 is the best possibility as to the size of the exceptional set. In fact we have the following result. The proof of it is essentially due to Mizuta (see [9, Theorem 2]), so we omit the proof here.

Theorem 1.8. *Let $E \subset H$ be a Borel set satisfying* Corollary 1.2 *(ii), h defined as in* Theorem 1.1*, and $R_{\alpha,l,l}(x)$ defined by* (1.23)*. Then we can find a nonnegative measure λ defined on $\overline{H}$ satisfying*

$$\int_{\overline{H}} \frac{h(|y|)}{(1+|y|)^{n+l-\alpha-\beta+3}} d\lambda(y) < \infty, \tag{1.25}$$

such that

$$\limsup_{|x|\to\infty, x\in E} x_n^{-\beta}|x|^{-l+2\beta-2}h(|x|)R_{\alpha,l,l}(x) = \infty, \tag{1.26}$$

where $d\lambda(y) = y_n d\mu(y)(y \in H)$ and $d\lambda(y') = d\nu(y')(y' \in \partial H)$.

2. Some Lemmas

Throughout this paper, let M denote various constants independent of the variables in questions, which may be different from line to line.

Lemma 2.1. *There exists a positive constant M such that $G_\alpha(x,y) \le M(x_n y_n/|x-y|^{n-\alpha+2})$, where $0 < \alpha \le n$, $x = (x_1, x_2, \ldots, x_n)$, and $y = (y_1, y_2, \ldots, y_n)$ in H.*

This can be proved by simple calculation.

Lemma 2.2. *Gegenbauer polynomials have the following properties:*

(i) $|C_k^\omega(t)| \le C_k^\omega(1) = \Gamma(2\omega+k)/\Gamma(2\omega)\Gamma(k+1)$, $|t| \le 1$;

(ii) $(d/dt)C_k^\omega(t) = 2\omega C_{k-1}^{\omega+1}(t)$, $k \ge 1$;

(iii) $\sum_{k=0}^\infty C_k^\omega(1)r^k = (1-r)^{-2\omega}$;

(iv) $|C_k^{(n-\alpha)/2}(t) - C_k^{(n-\alpha)/2}(t^*)| \le (n-\alpha)C_{k-1}^{(n-\alpha+2)/2}(1)|t-t^*|$, $|t| \le 1$, $|t^*| \le 1$.

Proof. (i) and (ii) can be derived from [3]. (iii) follows by taking $t = 1$ in (1.10); (iv) follows by (i), (ii) and the Mean Value Theorem for Derivatives. □

Lemma 2.3. *Let l be a nonnegative integer and $x, y \in \mathbf{R}^n (\alpha = n = 2)$, then one has the following properties:*

(i) $|\Im \sum_{k=0}^{l}(x^k/y^{k+1})| \le \sum_{k=0}^{l-1}(2^k x_n|x|^k/|y|^{k+2})$;

(ii) $|\Im \sum_{k=0}^{\infty}(x^{k+l+1}/y^k)| \le 2^{l+1}x_n|x|^l$;

(iii) $|G_{n,l}(x,y) - G_n(x,y)| \le M\sum_{k=1}^{l}(kx_ny_n|x|^{k-1}/|y|^{k+1})$;

(iv) $|G_{n,l}(x,y)| \le M\sum_{k=l+1}^{\infty}(kx_ny_n|x|^{k-1}/|y|^{k+1})$.

Lemma 2.4 (see [14]). *Let m be a nonnegative integer and $M > 0$.*

(i) *If $1 \le |y'| \le |x|/2$, then $|P_{\alpha,m}(x,y')| \le M(x_n|x|^{m-1}/|y'|^{n+m-\alpha+1})$.*

(ii) *If $|y'| \ge 2|x|$ and $|y'| \ge 1$, then $|P_{\alpha,m}(x,y')| \le M(x_n|x|^m/|y'|^{n+m-\alpha+2})$.*

The following lemma can be proved by using Fuglede ([6, Théorèm 7.8]).

Lemma 2.5. *For any Borel set E in H, we have $C_{k_{\alpha,\beta,1}}(E) = \widehat{C}_{k_{\alpha,\beta,1}}(E)$ and*

$$\widehat{C}_{k_{\alpha,\beta,\delta}}(E) = \inf\lambda(H)\ \left(\textit{resp.}\ \inf\lambda\left(\overline{H}\right)\right)\quad \textit{if } \delta < 1\ (\textit{resp.},\ \delta = 1), \tag{2.1}$$

where the infimum is taken over all nonnegative measures λ on H (resp., $\overline{H}$) such that $k_{\alpha,\beta,\delta}(\lambda,x) \ge 1$ for every $x \in E$.

3. Proof of Theorem 1.1

For any $\epsilon_1 > 0$, there exists $R_{\epsilon_1} > 2$ such that

$$\int_{\{y\in H, |y|\ge R_{\epsilon_1}\}} \frac{y_n^{\delta}h(|y|)}{(1+|y|)^{n+l-\alpha-\beta+\delta+2}}d\mu(y) < \epsilon_2. \tag{3.1}$$

For fixed $x \in H$ and $|x| \ge 2R_{\epsilon_1}$, we write

$$\begin{aligned} G_{\alpha,l}(x,\mu) &= \int_{H_1} G_\alpha(x,y)d\mu(y) + \int_{H_2} G_\alpha(x,y)d\mu(y) + \int_{H_3}[G_{\alpha,l}(x,y) - G_\alpha(x,y)]d\mu(y) \\ &\quad + \int_{H_4} G_{\alpha,l}(x,y)d\mu(y) + \int_{H_5} G_\alpha(x,y)d\mu(y) + \int_{H_6}[G_{\alpha,l}(x,y) - G_\alpha(x,y)]d\mu(y) \\ &\quad + \int_{H_7} G_{\alpha,l}(x,y)d\mu(y) \\ &= V_1(x) + V_2(x) + V_3(x) + V_4(x) + V_5(x) + V_6(x) + V_7(x), \end{aligned} \tag{3.2}$$

where

$$H_1 = \left\{ y \in H : |y| \geq R_{\epsilon_1}, |x-y| \leq \frac{|x|}{2} \right\}, \qquad H_2 = \left\{ y \in H : |y| \geq R_{\epsilon_1}, \frac{|x|}{2} < |x-y| \leq 3|x| \right\},$$
$$H_3 = \{ y \in H : |y| \geq R_{\epsilon_1}, |x-y| \leq 3|x| \}, \qquad H_4 = \{ y \in H : |y| \geq R_{\epsilon_1}, |x-y| > 3|x| \},$$
$$H_5 = H_6 = \{ y \in H : 1 \leq |y| < R_{\epsilon_1} \}, \qquad H_7 = \{ y \in H : |y| < 1 \}. \tag{3.3}$$

We distinguish the following two cases.

Case 1 ($0 < \alpha < n$). Note that $V_1(x) = x_n^{\beta} \int_{H_1} k_{\alpha,\beta,\delta}(y,x) y_n^{\delta} d\mu(y)$. In view of (1.18), we can find a sequence $\{a_i\}$ of positive numbers such that $\lim_{i\to\infty} a_i = \infty$ and $\sum_{i=1}^{\infty} a_i b_i < \infty$, where

$$b_i = \int_{\{y\in H: 2^{i-1}<|y|<2^{i+2}\}} \frac{y_n^{\delta} h(|y|)}{|y|^{n+l-\alpha-\beta+\delta+2}} d\mu(y). \tag{3.4}$$

Consider the sets

$$E_i' = \left\{ x \in H : 2^i \leq |x| < 2^{i+1}, x_n^{-\beta} V_1(x) \geq a_i^{-1} 2^{i(l-2\beta+2)} h\left(2^{i+1}\right)^{-1} \right\}, \tag{3.5}$$

for $i = 1, 2, \ldots$. If ω is a nonnegative measure on H such that $S_\omega \subset E_i'$ and $k_{\alpha,\beta,\delta}(y,\omega) \leq 1$ for $y \in H$, then we have

$$\begin{aligned}
&\int_H d\omega \\
&\leq a_i 2^{-i(l-2\beta+2)} h\left(2^{i+1}\right) \int x_n^{-\beta} V_1(x) d\omega(x) \\
&= a_i 2^{-i(l-2\beta+2)} h\left(2^{i+1}\right) \int_{\{y\in H: 2^{i-1}<|y|<2^{i+2}\}} k_{\alpha,\beta,\delta}(y,\omega) y_n^{\delta} d\mu(y) \\
&\leq M a_i 2^{-i(l-2\beta+2)} h\left(2^{i+1}\right) \int_{\{y\in H: 2^{i-1}<|y|<2^{i+2}\}} y_n^{\delta} d\mu(y) \\
&= M a_i 2^{-i(l-2\beta+2)} h\left(2^{i+1}\right) \int_{\{y\in H: 2^{i-1}<|y|<2^{i+2}\}} \frac{|y|^{2-\beta}}{h(|y|)} \frac{|y|^{n+l-\alpha+2}}{(1+|y|)^{2-\delta}} \frac{y_n^{\delta} h(|y|)}{(1+|y|)^{n+l-\alpha-\beta+\delta+2}} d\mu(y) \\
&\leq M 2^{-i(l-2\beta+2)} 2^{(i+2)(2-\beta)} 2^{(i+2)(n+l-\alpha+2)} 2^{-i(2-\delta)} \int_{\{y\in H: 2^{i-1}<|y|<2^{i+2}\}} \frac{h(|y'|)}{|y'|^{n+m-\alpha-\beta+3}} d\mu(y) \\
&\leq M 2^{i(n-\alpha+\beta+\delta)} a_i b_i.
\end{aligned} \tag{3.6}$$

So that

$$C_{k_{\alpha,\beta,\delta}}(E_i') \leq M2^{i(n-\alpha+\beta+\delta)}a_i b_i, \tag{3.7}$$

which yields

$$\sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+\delta)} C_{k_{\alpha,\beta,\delta}}(E_i') < \infty. \tag{3.8}$$

Setting $E' = \bigcup_{i=1}^{\infty} E_i'$, we see that Theorem 1.1 (ii) is satisfied and

$$\limsup_{|x|\to\infty, x\in H-E'} x_n^{-\beta}|x|^{-l+2\beta-2}h(|x|)V_1(x) \leq \limsup_{i\to\infty} a_i^{-1} = 0. \tag{3.9}$$

Moreover by Lemma 2.1,

$$\begin{aligned}|V_2(x)| &\leq Mx_n \int_{H_2} \frac{y_n}{|x-y|^{n-\alpha+2}} d\mu(y) \\ &\leq Mx_n|x|^{\alpha-n-2} \int_{H_2} \frac{|y|^{2-\beta}}{h(|y|)} |y|^{n+l-\alpha+1} \frac{y_n^{\delta}h(|y|)}{(1+|y|)^{n+l-\alpha-\beta+\delta+2}} d\mu(y) \\ &\leq M\epsilon_2 x_n|x|^{l-\beta+1}h(4|x|)^{-1}.\end{aligned} \tag{3.10}$$

Note that $C_0^{\omega}(t) \equiv 1$. By (iii) and (iv) in Lemma 2.2, we take $t = (x\cdot y)/|x||y|$, $t^* = (x\cdot y^*)/|x||y^*|$ in Lemma 2.2 (iv) and obtain

$$\begin{aligned}|V_3(x)| &\leq \int_{H_3} \sum_{k=1}^{l} \frac{|x|^k}{|y|^{n-\alpha+k}} 2(n-\alpha) C_{k-1}^{(n-\alpha+2)/2}(1) \frac{x_n y_n}{|x||y|} d\mu(y) \\ &\leq Mx_n|x|^{l-1} \sum_{k=1}^{l-1} \frac{1}{4^{k-1}} C_{k-1}^{(n-\alpha+2)/2}(1) \int_{H_3} \frac{|y|^{2-\beta}}{h(|y|)} \frac{y_n^{\delta}h(|y|)}{(1+|y|)^{n+l-\alpha-\beta+\delta+2}} d\mu(y) \\ &\leq M\epsilon_1 x_n|x|^{l-\beta+1}h(4|x|)^{-1}.\end{aligned} \tag{3.11}$$

Similarly, we have by (iii) and (iv) in Lemma 2.2

$$\begin{aligned}|V_4(x)| &\leq \int_{H_4} \sum_{k=l+1}^{\infty} \frac{|x|^k}{|y|^{n-\alpha+k}} 2(n-\alpha) C_{k-1}^{(n-\alpha+2)/2}(1) \frac{x_n y_n}{|x||y|} d\mu(y) \\ &\leq Mx_n|x|^{l} \sum_{k=l+1}^{\infty} \frac{1}{2^{k-1}} C_{k-1}^{(n-\alpha+2)/2}(1) \int_{H_4} \frac{|y|^{1-\beta}}{h(|y|)} \frac{y_n^{\delta}h(|y|)}{(1+|y|)^{n+l-\alpha-\beta+\delta+2}} d\mu(y) \\ &\leq M\epsilon_1 x_n|x|^{l-\beta+1}h(2|x|)^{-1}.\end{aligned} \tag{3.12}$$

By Lemma 2.1, we have

$$\begin{aligned}|V_5(x)| &\le Mx_n \int_{H_5} \frac{y_n}{|x-y|^{n-\alpha+2}} d\mu(y) \\ &\le Mx_n|x|^{\alpha-n-2} \int_{H_5} \frac{|y|^{2-\beta}}{h(|y|)} |y|^{n+l-\alpha+1} \frac{y_n^{\delta} h(|y|)}{(1+|y|)^{n+l-\alpha-\beta+\delta+2}} d\mu(y) \\ &\le Mx_n|x|^{l-1} R_{\epsilon_1}^{2-\beta} h(R_{\epsilon_2})^{-1}.\end{aligned} \tag{3.13}$$

Similarly as $V_3(x)$, we obtain

$$\begin{aligned}|V_6(x)| &\le \int_{H_6} \sum_{k=1}^{l} \frac{|x|^k}{|y|^{n-\alpha+k}} 2(n-\alpha) C_{k-1}^{n-\alpha+2/2}(1) \frac{x_n y_n}{|x||y|} d\mu(y) \\ &\le Mx_n \sum_{k=1}^{l} C_{k-1}^{(n-\alpha+2)/2}(1)|x|^{k-1} R_{\epsilon_2}^{l-k+1} \int_{H_6} \frac{|y|^{1-\beta}}{h(|y|)} \frac{y_n^{\delta} h(|y|)}{(1+|y|)^{n+l-\alpha-\beta+\delta+2}} d\mu(y) \\ &\le MR_{\epsilon_1}^{l} x_n|x|^{l-1} h(1)^{-1}.\end{aligned} \tag{3.14}$$

Finally, by Lemma 2.1, we have

$$|V_7(x)| \le Mx_n|x|^{l-1} h(1)^{-1}. \tag{3.15}$$

Combining (3.9)–(3.15), we prove Case 1.

Case 2 ($\alpha = n = 2$). In this case, the growth estimates of $V_1(x)$, $V_2(x)$, $V_5(x)$ and $V_7(x)$ can be proved similarly as in Case 1. Inequations (3.9), (3.10), (3.13) and (3.15) still hold.

Moreover we have by Lemma 2.3 (iii)

$$\begin{aligned}|V_3(x)| &\le M \int_{H_3} \sum_{k=1}^{l} \frac{k x_n y_n |x|^{k-1}}{|y|^{k+1}} |y|^{l+1} \frac{|y|^{2-\beta}}{h(|y|)} \frac{y_n^{\delta} h(|y|)}{(1+|y|)^{l-\beta+\delta+2}} d\mu(y) \\ &\le Mx_n|x|^{l-1} \sum_{k=1}^{l} \frac{k}{4^{k-1}} \int_{H_3} \frac{|y|^{2-\beta}}{h(|y|)} \frac{y_n^{\delta} h(|y|)}{(1+|y|)^{l-\beta+\delta+2}} d\mu(y) \\ &\le M\epsilon_1 x_n|x|^{l-\beta+1} h(4|x|)^{-1}.\end{aligned} \tag{3.16}$$

By Lemma 2.3 (iv), we have

$$\begin{aligned}|V_4(x)| &\le M\int_{H_4}\sum_{k=l+1}^{\infty}\frac{kx_ny_n|x|^{k-1}}{|y|^{k+1}}|y|^{l+2}\frac{|y|^{1-\beta}}{h(|y|)}\frac{y_n^{\delta}h(|y|)}{(1+|y|)^{l-\beta+\delta+2}}d\mu(y)\\ &\le Mx_n|x|^{l}\sum_{k=l+1}^{\infty}\frac{k}{2^{k-1}}\int_{H_4}\frac{|y|^{1-\beta}}{h(|y|)}\frac{y_n^{\delta}h(|y|)}{(1+|y|)^{l-\beta+\delta+2}}d\mu(y)\\ &\le M\epsilon_1x_n|x|^{l-\beta+1}h(2|x|)^{-1}.\end{aligned} \tag{3.17}$$

Similarly as $V_3(x)$, we have

$$|V_6(x)| \le MR_{\epsilon_1}^{l}x_n|x|^{l-1}h(1)^{-1}. \tag{3.18}$$

Combining (3.9), (3.10), (3.13), (3.15), and (3.16)–(3.18), we prove Case 2.

Hence we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.3

For any $\epsilon_2 > 0$, there exists $R_{\epsilon_2} > 2$ such that

$$\int_{\{y'\in\partial H,|y'|\ge R_{\epsilon_2}\}}\frac{h(|y'|)}{(1+|y'|)^{n+m-\alpha-\beta+3}}d\nu(y') < \epsilon_2. \tag{4.1}$$

For fixed $x \in H$ and $|x| \ge 2R_{\epsilon_2}$, we write

$$\begin{aligned}U_{\alpha,m}(x,\nu) &= \int_{G_1}P_{\alpha,m}(x,y')d\nu(y') + \int_{G_2}P_{\alpha,m}(x,y')d\nu(y')\\ &\quad+\int_{G_3}\left[P_{\alpha,m}(x,y') - P_{\alpha}(x,y')\right]d\nu(y') + \int_{G_4}P_{\alpha}(x,y')d\nu(y')\\ &\quad+\int_{G_5}P_{\alpha,m}(x,y')d\nu(y')\\ &= U_1(x)+U_2(x)+U_3(x)+U_4(x)+U_5(x),\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}&G_1 = \{y'\in\partial H : |y'|<1\}, \qquad G_2 = \left\{y'\in\partial H : 1\le|y'|<\frac{|x|}{2}\right\},\\ &G_3 = G_4 = \left\{y'\in\partial H : \frac{|x|}{2}\le|y'|<2|x|\right\}, \qquad G_5 = \{y'\in\partial H : |y'|\ge 2|x|\}.\end{aligned} \tag{4.3}$$

First note that

$$\begin{aligned}|U_1(x)| &\le Mx_n\left(\frac{|x|}{2}\right)^{\alpha-n-2}\int_{G_1}d\nu(y')\\ &\le Mx_n|x|^{\alpha-n-2}\int_{G_1}\frac{|y'|^{2-\beta}}{h(|y'|)}|y'|^{n+m-\alpha+1}\frac{h(|y'|)}{(1+|y'|)^{n+m-\alpha-\beta+3}}\nu(y')\\ &\le Mx_n|x|^{m-1}h(1)^{-1}.\end{aligned} \tag{4.4}$$

Write

$$U_2(x)=U_{21}(x)+U_{22}(x), \tag{4.5}$$

where

$$\begin{aligned}U_{21}(x)&=\int_{G_2\bigcap B_{n-1}(0,R_{\epsilon_2})}P_{\alpha,m}(x,y')d\nu(y'),\\ U_{22}(x)&=\int_{G_2-B_{n-1}(0,R_{\epsilon_2})}P_{\alpha,m}(x,y')d\nu(y').\end{aligned} \tag{4.6}$$

We obtain by Lemma 2.4 (i)

$$\begin{aligned}|U_2(x)| &\le Mx_n|x|^{m-1}\int_{G_2}\frac{1}{|y'|^{n+m-\alpha+1}}d\nu(y')\\ &\le Mx_n|x|^{m-1}\int_{G_2}\frac{|y'|^{2-\beta}}{h(|y'|)}\frac{h(|y'|)}{(1+|y'|)^{n+m-\alpha-\beta+3}}\nu(y').\end{aligned} \tag{4.7}$$

For $|x|>2R_{\epsilon_2}$, by (4.7) we have

$$|U_{21}(x)|\le Mx_n|x|^{m-1}R_{\epsilon_2}^{2-\beta}h(R_{\epsilon_2})^{-1}. \tag{4.8}$$

On the other hand, (4.7) yields that

$$|U_{22}(x)|\le M\epsilon_2x_n|x|^{m-\beta+1}h\left(\frac{|x|}{2}\right)^{-1}. \tag{4.9}$$

Combining (4.8) and (4.9), we have

$$\lim_{|x|\to\infty,x\in H}x_n^{-\beta}|x|^{-m+2\beta-2}h(|x|)U_2(x)=0. \tag{4.10}$$

We have by Lemma 2.2 (iii)

$$\begin{aligned}|U_3(x)| &\leq M\int_{G_3}\sum_{k=0}^{m-1}\frac{x_n|x|^k}{|y'|^{n-\alpha+2+k}}C_k^{n-\alpha+2/2}(1)d\nu(y') \\ &\leq Mx_n|x|^m\sum_{k=0}^{m-1}\frac{1}{2^k}C_k^{(n-\alpha+2)/2}(1)\int_{G_3}\frac{|y'|^{1-\beta}}{h(|y'|)}\frac{h(|y'|)}{|y'|^{n+m-\alpha-\beta+3}}d\nu(y') \\ &\leq M\varepsilon x_n|x|^{m-\beta+1}h\left(\frac{|x|}{2}\right)^{-1}.\end{aligned} \tag{4.11}$$

By Lemma 2.4 (ii), we obtain

$$\begin{aligned}|U_5(x)| &\leq Mx_n|x|^m\int_{G_5}\frac{1}{|y'|^{n+m-\alpha+2}}d\nu(y') \\ &\leq Mx_n|x|^m\int_{G_5}\frac{|y'|^{1-\beta}}{h(|y'|)}\frac{h(|y'|)}{(1+|y'|)^{n+m-\alpha-\beta+3}}\nu(y') \\ &\leq M\epsilon_2 x_n|x|^{m-\beta+1}h(2|x|)^{-1}.\end{aligned} \tag{4.12}$$

Note that $U_4(x) = x_n^{\beta}\int_{G_4}k_{\alpha,\beta,1}(y',x)d\nu(y')$. By the lower semicontinuity of $k_{\alpha,\beta,1}(y',x)$, we can prove the following fact in the same way as $V_1(x)$ in the proof of Theorem 1.1:

$$\limsup_{|x|\to\infty, x\in H-F} x_n^{-\beta}|x|^{-m+2\beta-2}h(|x|)U_4(x) = 0, \tag{4.13}$$

where $F = \bigcup_{i=1}^{\infty}F_i, F_i = \{x \in F : 2^i \leq |x| < 2^{i+1}\}$, and $\sum_{i=1}^{\infty}2^{-i(n-\alpha+\beta+1)}C_{k_{\alpha,\beta,1}}(F_i) < \infty$.

Combining (4.4) and (4.10)–(4.13), we complete the proof of Theorem 1.1.

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