Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 604217, 29 pages
doi:10.1155/2010/604217

*Research Article*

# Wiman and Arima Theorems for Quasiregular Mappings

**O. Martio,[1] V. M. Miklyukov,[2] and M. Vuorinen[3]**

[1] *Department of Mathematics and Statistics, University of Helsinki, 00014 Helsinki, Finland*
[2] *Department of Mathematics, Volgograd State University, 2 Prodolnaya 30, Volgograd 400062, Russia*
[3] *Department of Mathematics, University of Turku, 20014 Turku, Finland*

Correspondence should be addressed to M. Vuorinen, vuorinen@utu.fi

Received 28 December 2009; Accepted 11 February 2010

Academic Editor: Shusen Ding



Wiman's theorem says that an entire holomorphic function of order less than 1/2 has a minimum modulus converging to $\infty$ along a sequence. Arima's theorem is a refinement of Wiman's theorem. Here we generalize both results to quasiregular mappings in the manifold setup. The so called fundamental frequency has an important role in this study.

## 1. Main Results

It follows from the Ahlfors theorem that an entire holomorphic function $f$ of order $\rho$ has no more than $[2\rho]$ distinct asymptotic curves where $[r]$ stands for the largest integer $\leq r$. This theorem does not give any information if $\rho < 1/2$, This case is covered by two theorems: *if an entire holomorphic function $f$ has order $\rho < 1/2$ then* $\limsup_{r\to\infty}\min_{|z|=r}|f(z)| = \infty$ (Wiman [1]) and *if $f$ is an entire holomorphic function of order $\rho > 0$ and $l$ is a number satisfying the conditions $0 < l \leq 2\pi, l < \pi/\rho$, then there exists a sequence of circular arcs $\{|z| = r_k,\ \theta_k \leq \arg z \leq \theta_k + l\}, r_k \to \infty, 0 \leq \theta_k < 2\pi$, along which $|f(z)|$ tends to $\infty$ uniformly with respect to* $\arg z$ (Arima [2]).

Below we prove generalizations of these theorems for quasiregular mappings for $n \geq 2$. The next two theorems are generalizations of the theorems of Wiman and of Arima for quasiregular mappings on manifolds.

**Theorem 1.1.** *Let $\mathcal{M}, \mathcal{N}$ be $n$-dimensional noncompact Riemannian manifolds without boundary. Assume that $h : \mathcal{M} \to (0,\infty)$ is a special exhaustion function of the manifold $\mathcal{M}$ and $u$ is a nonnegative growth function on the manifold $\mathcal{N}$, which is a subsolution of* (3.4) *with the structure conditions* (3.2), (3.3) *and the structure constants $p = n$, $\nu_1$, $\nu_2$.*

*Let $f : \mathcal{M} \to \mathcal{N}$ be a nonconstant quasiregular mapping. Suppose that the manifold $\mathcal{M}$ is such that*

$$\int^{\infty} \lambda_n(\Sigma_h(t);1)dt = \infty. \tag{1.1}$$

*If now*

$$\liminf_{\tau\to\infty} \max_{h(m)=\tau} u(f(m)) \exp\left\{-C\int^{\tau} \lambda_n(\Sigma_h(t);1)dt\right\} = 0, \tag{1.2}$$

*then*

$$\limsup_{\tau\to\infty} \min_{h(m)=\tau} u(f(m)) = \infty. \tag{1.3}$$

*Here*

$$C = \left(n-1+n\left(\left(\frac{\nu_2}{\nu_1}\right)^2 K^2(f)-1\right)^{1/2}\right)^{-1} \tag{1.4}$$

*is a constant, $K(f)$ is the maximal dilatation of $f$, $\Sigma_h(t)$ is an $h$-sphere in the manifold $\mathcal{M}$, $\lambda_n(U)$ is a fundamental frequency of an open subset $U \subset \Sigma_h(t)$, and $\lambda_n(\Sigma_h(t);1) = \inf \lambda_n(U)$, where the infimum is taken over all open sets $U \subset \Sigma_h(t)$ with $U \neq \Sigma_h(t)$. (See Sections 4 and 6.)*

**Theorem 1.2.** *Let $\mathcal{M}, \mathcal{N}$ be $n$-dimensional noncompact Riemannian manifolds without boundary. Assume that $h : \mathcal{M} \to (0,\infty)$ is a special exhaustion function of the manifold $\mathcal{M}$ and $u$ is a nonnegative growth function on the manifold $\mathcal{N}$, which is a subsolution of* (3.4) *with the structure conditions* (3.2), (3.3) *and the structure constants $p = n$, $\nu_1$, $\nu_2$.*

*Let $f : \mathcal{M} \to \mathcal{N}$ be a quasiregular mapping and $M(\tau) = \max_{\Sigma_h(\tau)} u(f(m))$. If for some $\gamma > 0$ the mapping $f$ satisfies the condition*

$$\liminf_{\tau\to\infty} M(\tau+1) \exp\left\{-\gamma\int^{\tau} \lambda_n(\Sigma_h(t);1)dt\right\} = 0, \tag{1.5}$$

*then for each $k = 1,2,\ldots$ there exists an $h$-sphere $\Sigma_h(t_k)$ and an open set $U \subset \Sigma_h(t_k)$, for which*

$$u(f)|_U \geq k, \qquad \lambda_n(U) < \frac{n\gamma}{C}\lambda_n(\Sigma_h(t_k);1). \tag{1.6}$$

The proofs of these results are based upon Phragmén-Lindelöf's and Ahlfors' theorems for differential forms of $\mathcal{WT}$-classes obtained in [3].

For $n$-harmonic functions on abstract cones, similar theorems were obtained in [4].

Our notation is as in [3, 5]. We assume that the results of [3] are known to the reader and we only recall some results on qr-mappings.

## 2. Quasiregular Mappings

Let $\mathcal{M}$ and $\mathcal{N}$ be Riemannian manifolds of dimension $n$. A continuous mapping $F : \mathcal{M} \to \mathcal{N}$ of the class $W^1_{n,\mathrm{loc}}(\mathcal{M})$ is called a quasiregular mapping if $F$ satisfies

$$\left|F'(m)\right|^n \le K J_F(m) \tag{2.1}$$

almost everywhere on $\mathcal{M}$. Here $F'(m) : T_m(\mathcal{M}) \to T_{F(m)}(\mathcal{N})$ is the formal derivative of $F(m)$, further, $|F'(m)| = \max_{|h|=1}|F'(m)h|$. We denote by $J_F(m)$ the Jacobian of $F$ at the point $m \in \mathcal{M}$, that is, the determinant of $F'(m)$.

The best constant $K \ge 1$ in the inequality (2.1) is called the outer dilatation of $F$ and denoted by $K_O(F)$. If $F$ is quasiregular, then the least constant $K \ge 1$ for which we have

$$J_F(m) \le K l\left(F'(m)\right)^n \tag{2.2}$$

almost everywhere on $\mathcal{M}$ is called the inner dilatation and denoted by $K_I(F)$. Here

$$l\left(F'(m)\right) = \min_{|h|=1}\left|F'(m)h\right|. \tag{2.3}$$

The quantity

$$K(F) = \max\{K_O(F), K_I(F)\} \tag{2.4}$$

is called the maximal dilatation of $F$ and if $K(F) \le K$, then the mapping $F$ is called $K$-quasiregular.

If $F : \mathcal{M} \to \mathcal{N}$ is a quasiregular homeomorphism, then the mapping $F$ is called quasiconformal. In this case, the inverse mapping $F^{-1}$ is also quasiconformal in the domain $F(\mathcal{M}) \subset \mathcal{N}$ and $K(F^{-1}) = K(F)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be Riemannian manifolds of dimensions $\dim \mathcal{A} = k$ and $\dim \mathcal{B} = n - k$, $1 \le k < n$, and with scalar products $\langle , \rangle_A$, $\langle , \rangle_B$, respectively. The Cartesian product $\mathcal{N} = \mathcal{A} \times \mathcal{B}$ has the natural structure of a Riemannian manifold with the scalar product

$$\langle , \rangle = \langle , \rangle_{\mathcal{A}} + \langle , \rangle_{\mathcal{B}}. \tag{2.5}$$

We denote by $\pi : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ and $\eta : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ the natural projections of the manifold $\mathcal{N}$ onto submanifolds.

If $w_{\mathcal{A}}$ and $w_{\mathcal{B}}$ are volume forms on $\mathcal{A}$ and $\mathcal{B}$, respectively, then the differential form $w_{\mathcal{N}} = \pi^* w_{\mathcal{A}} \wedge \eta^* w_{\mathcal{B}}$ is a volume form on $\mathcal{N}$.

**Theorem 2.1** (see [5]). *Let $F : \mathcal{M} \to \mathcal{N}$ be a quasiregular mapping and let $f = \pi \circ F : \mathcal{M} \to \mathcal{A}$. Then the differential form $f^* w_{\mathcal{A}}$ is of the class $\mathcal{WT}_2$ on $\mathcal{M}$ with the structure constants $p = n/k$, $\nu_1 = \nu_1(n, k, K_O)$, and $\nu_2 = \nu_2(n, k, K_O)$.*

*Remark 2.2.* The structure constants can be chosen to be

$$\nu_1^{-1} = \left(k + \frac{n-k}{\overline{c}^2}\right)^{-n/2} n^{n/2}\ K_O, \qquad \nu_2^{-1} = \underline{c}^{n-k}, \tag{2.6}$$

where $\overline{c} = \overline{c}(k, n, K_O)$ and $\underline{c} = \underline{c}(k, n, K_O)$ are, respectively, the greatest and smallest positive roots of the equation

$$\left(k\xi^2 + (n-k)\right)^{n/2} - n^{n/2} K_O \xi^k = 0. \tag{2.7}$$

## 3. Domains of Growth

Let $D \subset \mathbf{C}$ be an unbounded domain and let $w = f(z)$ be a holomorphic function continuous on the closure $\overline{D}$. The Phragmén-Lindelöf principle [6] traditionally refers to the alternatives of the following type:

($\alpha$) if $\operatorname{Re} f(z) \leq 1$ everywhere on $\partial D$, then either $\operatorname{Re} f(z)$ grows with a certain rate as $z \to \infty$ or $\operatorname{Re} f(z) \leq 1$ for all $z \in D$;

($\beta$) if $|f(z)| \leq 1$ on $\partial D$, then either $|f(z)|$ grows with a certain rate as $|z| \to \infty$ or $|f(z)| \leq 1$ for all $z \in D$.

Here the rate of growth of the quantities $\operatorname{Re} f(z)$ and $|f(z)|$ depends on the "width" of the domain $D$ near infinity.

It is not difficult to prove that these conditions are equivalent with the following conditions:

($\alpha_1$) if $\operatorname{Re} f(z) = 1$ on $\partial D$ and $\operatorname{Re} f(z) \geq 1$ in $D$, then either $\operatorname{Re} f(z)$ grows with a certain rate as $z \to \infty$ or $f \equiv \text{const}$;

($\beta_1$) if $|f(z)| = 1$ on $\partial D$ and $|f(z)| \geq 1$ in $D$, then either $|f(z)|$ grows with a certain rate as $z \to \infty$ or $f \equiv \text{const}$.

Let $D$ be an unbounded domain in $\mathbf{R}^n$ and let $f = (f_1, f_2, \ldots, f_n) : D \to \mathbf{R}^n$ be a quasiregular mapping. We assume that $f \in C^0(\overline{D})$. It is natural to consider the Phragmén-Lindelöf alternative under the following assumptions:

(a) $f_1(x)|_{\partial D} = 1$ and $f_1(x) \geq 1$ everywhere in $D$;

(b) $\sum_{i=1}^p f_i^2(x)|_{\partial D} = 1$ and $\sum_{i=1}^p f_i^2(x) \geq 1$ on $D$, $1 < p < n$;

(c) $|f(x)| = 1$ on $\partial D$ and $|f(x)| \geq 1$ on $D$.

Several formulations of the Phragmén-Lindelöf theorem under various assumptions can be found in [7–11]. However, these results are mainly of qualitative character. Here we give a new approach to Phragmén-Lindelöf type theorems for quasiregular mappings, based on isoperimetry, that leads to almost sharp results. Our approach can be used to prove Phragmén-Lindelöf type results for quasiregular mappings of Riemannian manifolds.

Let $\mathcal{N}$ be an $n$-dimensional noncompact Riemannian $C^2$-manifold with piecewise smooth boundary $\partial\mathcal{N}$ (possibly empty). A function $u \in C^0(\overline{\mathcal{N}}) \cap W^1_{n,\mathrm{loc}}(\mathcal{N})$ is called a *growth function* with $\mathcal{N}$ as a *domain of growth* if (i) $u \geq 1$, (ii) $u \mid \partial\mathcal{N} = 1$ if $\partial\mathcal{N} \neq \emptyset$, and $\sup_{y \in \mathcal{N}} u(y) = +\infty$.

We consider a quasiregular mapping $f : \mathcal{M} \to \mathcal{N}$, $f \in C^0(\mathcal{M} \cup \partial M)$, where $\mathcal{M}$ is a noncompact Riemannian $C^2$-manifold, $\dim \mathcal{M} = n$, and $\partial \mathcal{M} \neq \emptyset$. We assume that $f(\partial \mathcal{M}) \subset \partial \mathcal{N}$. In what follows, we mean by the Phragmén-Lindelöf principle an alternative of the form: either the function $u(f(m))$ has a certain rate of growth in $\mathcal{M}$ or $f(m) \equiv \text{const}$.

By choosing the domain of growth $\mathcal{N}$ and the growth function $u$ in a special way, we can obtain several formulations of Phragmén-Lindelöf theorems for quasiregular mappings. In view of the examples in [7], the best results are obtained if an $n$-harmonic function is chosen as a growth function. In the case (a), the domain of growth is $\mathcal{N} = \{y = (y_1, \ldots, y_n) \in \mathbf{R}^n : y_1 \geq 0\}$ and as the function of growth, it is natural to choose $u(y) = y_1 + 1$; in the case (b), the domain $\mathcal{N}$ is the set $\{y = (y_1, \ldots, y_n) \in \mathbf{R}^n : \sum_{i=1}^p y_i^2 \geq 1\}$, $1 < p < n$, and $u(y) = (\sum_{i=1}^p y_i^2)^{(n-p)/2(n-1)}$; in the case (c), the domain of growth is $\mathcal{N} = \{y \in \mathbf{R}^n : |y| > 1\}$ and $u(y) = \log|y| + 1$.

In the general case, we shall consider growth functions which are $A$-solutions of elliptic equations [12]. Namely, let $\mathcal{M}$ be a Riemannian manifold and let

$$A : T(\mathcal{M}) \longrightarrow T(\mathcal{M}) \tag{3.1}$$

be a mapping defined a.e. on the tangent bundle $T(\mathcal{M})$. Suppose that for a.e. $m \in \mathcal{M}$ the mapping $A$ is continuous on the fiber $T_m$, that is, for a.e. $m \in \mathcal{M}$, the function $A(m, \cdot) : T_m \to T_m$ is defined and continuous; the mapping $m \mapsto A_m(X)$ is measurable for all measurable vector fields $X$ (see [12]).

Suppose that for a.e. $m \in \mathcal{M}$ and for all $\xi \in T_m$, the inequalities

$$\nu_1 |\xi|^p \leq \langle \xi, A(m, \xi) \rangle, \tag{3.2}$$

$$|A(m, \xi)| \leq \nu_2 |\xi|^{p-1} \tag{3.3}$$

hold with $p > 1$ and for some constants $\nu_1, \nu_2 > 0$. It is clear that we have $\nu_1 \leq \nu_2$.

We consider the equation

$$\operatorname{div} A(m, \nabla f) = 0. \tag{3.4}$$

Solutions to (3.4) are understood in the weak sense, that is, $A$-solutions are $W^1_{p,\text{loc}}$-functions satisfying the integral identity

$$\int_{\mathcal{M}} \langle \nabla \theta, A(m, \nabla f) \rangle * \mathbb{1}_{\mathcal{M}} = 0 \tag{3.5}$$

for all $\theta \in W^1_p(\mathcal{M})$ with compact support in $\mathcal{M}$.

A function $f$ in $W^1_{p,\text{loc}}(\mathcal{M})$ is an *$A$-subsolution* of (3.4) in $\mathcal{M}$ if

$$\operatorname{div} A(m, \nabla f) \geq 0 \tag{3.6}$$

weakly in $\mathcal{M}$, that is,

$$\int_{\mathcal{M}} \langle \nabla\theta, A(m, \nabla f)\rangle * \mathbb{1}_{\mathcal{M}} \leq 0, \tag{3.7}$$

whenever $\theta \in W_p^1(\mathcal{M})$, is nonnegative with compact support in $\mathcal{M}$.

A basic example of such an equation is the $p$-Laplace equation

$$\operatorname{div}\left(|\nabla f|^{p-2}\nabla f\right) = 0. \tag{3.8}$$

## 4. Exhaustion Functions

Below we introduce exhaustion and special exhaustion functions on Riemannian manifolds and give illustrating examples.

### *4.1. Exhaustion Functions of Boundary Sets*

Let $h : \mathcal{M} \to (0, h_0)$, $0 < h_0 \leq \infty$, be a locally Lipschitz function such that

$$\operatorname{ess}\inf_Q |\nabla h| > 0 \quad \forall Q \subset\subset \mathcal{M}. \tag{4.1}$$

For arbitrary $t \in (0, h_0)$, we denote by

$$B_h(t) = \{m \in \mathcal{M} : h(m) < t\}, \qquad \Sigma_h(t) = \{m \in \mathcal{M} : h(m) = t\} \tag{4.2}$$

the $h$-balls and $h$-spheres, respectively.

Let $h : \mathcal{M} \to \mathbf{R}$ be a locally Lipschitz function such that there exists a compact $K \subset \mathcal{M}$ with $|\nabla h(x)| > 0$ for a.e. $m \in \mathcal{M} \setminus K$. We say that the function $h$ is an exhaustion function for a boundary set $\Xi$ of $\mathcal{M}$ if for an arbitrary sequence of points $m_k \in \mathcal{M}$, $k = 1, 2, \ldots$, the function $h(m_k) \to h_0$ if and only if $m_k \to \xi$.

It is easy to see that this requirement is satisfied if and only if for an arbitrary increasing sequence $t_1 < t_2 < \cdots < h_0$, the sequence of the open sets $V_k = \{m \in \mathcal{M} : h(m) > t_k\}$ is a chain, defining a boundary set $\xi$. Thus the function $h$ exhausts the boundary set $\xi$ in the traditional sense of the word.

The function $h : \mathcal{M} \to (0, h_0)$ is called the exhaustion function of the manifold $\mathcal{M}$ if the following two conditions are satisfied:

(i) for all $t \in (0, h_0)$, the $h$-ball $\overline{B_h(t)}$ is compact;

(ii) for every sequence $t_1 < t_2 < \cdots < h_0$ with $\lim_{k\to\infty} t_k = h_0$, the sequence of $h$-balls $\{B_h(t_k)\}$ generates an exhaustion of $\mathcal{M}$, that is,

$$B_h(t_1) \subset B_h(t_2) \subset \cdots \subset B_h(t_k) \subset \cdots, \qquad \bigcup_k B_h(t_k) = \mathcal{M}. \tag{4.3}$$

*Example 4.1.* Let $\mathcal{M}$ be a Riemannian manifold. We set $h(m) = \operatorname{dist}(m, m_0)$ where $m_0 \in \mathcal{M}$ is a fixed point. Because $|\nabla h(m)| = 1$ almost everywhere on $\mathcal{M}$, the function $h$ defines an exhaustion function of the manifold $\mathcal{M}$.

### *4.2. Special Exhaustion Functions*

Let $\mathcal{M}$ be a noncompact Riemannian manifold with the boundary $\partial\mathcal{M}$ (possibly empty). Let $A$ satisfy (3.2) and (3.3) and let $h : \mathcal{M} \to (0, h_0)$ be an exhaustion function, satisfying the following additional conditions:

$(a_1)$ there is $h' > 0$ such that $h^{-1}((0, h'))$ is compact and $h$ is a solution of (3.4) in the open set $K = h^{-1}((h', h_0))$;

$(a_2)$ for a.e. $t_1, t_2 \in (h', h_0)$, $t_1 < t_2$,

$$\int_{\Sigma_h(t_2)} \left\langle \frac{\nabla h}{|\nabla h|}, A(x, \nabla h) \right\rangle d\mathcal{H}^{n-1} = \int_{\Sigma_h(t_1)} \left\langle \frac{\nabla h}{|\nabla h|}, A(x, \nabla h) \right\rangle d\mathcal{H}^{n-1}. \tag{4.4}$$

Here $d\mathcal{H}^{n-1}$ is the element of the $(n-1)$-dimensional Hausdorff measure on $\Sigma_h$. Exhaustion functions with these properties will be called *the special exhaustion functions of $\mathcal{M}$ with respect to $A$*. In most cases, the mapping $A$ will be the $p$-Laplace operator (3.8) and, unless otherwise stated, $A$ is the $p$-Laplace operator.

Since the unit vector $\nu = \nabla h / |\nabla h|$ is orthogonal to the $h$-sphere $\Sigma_h$, the condition $(a_2)$ means that the flux of the vector field $A(m, \nabla h)$ through $h$-spheres $\Sigma_h(t)$ is constant.

In the following, we consider domains $D$ in $\mathbf{R}^n$ as manifolds $\mathcal{M}$. However, the boundaries $\partial D$ of $D$ are allowed to be rather irregular. To handle this situation, we introduce $(A, h)$-transversality property for $\mathcal{M}$.

Let $h : \mathcal{M} \to (0, h_0)$ be a $C^2$-exhaustion function. We say that $\mathcal{M}$ satisfies the $(A, h)$-transversality property if for a.e. $t_1,\ t_2, h < t_1 < t_2 < h_0$, and for every $\varepsilon > 0$, there exists an open set

$$G = G_\varepsilon(t_1, t_2) \subset B_h(t_2) \setminus \overline{B}_h(t_1) \tag{4.5}$$

with piecewise regular boundary such that

$$\mathcal{H}^{n-1}(\Sigma_h(t_1) \cap \Sigma_h(t_2) \setminus \partial G) < \varepsilon, \tag{4.6}$$

$$\mathcal{H}^{n}\left(\left(B_h(t_2) / \overline{B}_h(t_1)\right) \setminus G\right) < \varepsilon, \tag{4.7}$$

$$\langle A(m, \nabla h(m), v) \rangle = 0, \tag{4.8}$$

where $v$ is the unit inner normal to $\partial G$.

We say that $\mathcal{M}$ satisfies the $h$-transversality condition if $\mathcal{M}$ satisfies the $(A, h)$-transversality condition for the $p$-Laplace operator $A(m, \xi) = |\xi|^{p-2}\xi$. In this case, (4.8) reduces to

$$\langle \nabla h(m), v \rangle = 0. \tag{4.9}$$

*Example 4.2.* Let $D$ be a bounded domain in $\mathbf{R}^2$ and let

$$\mathcal{M} = \left\{(x_1, x_2, x_3) \in \mathbf{R}^3 : (x_1, x_2) \in D, x_3 > 0\right\} \tag{4.10}$$

be a cylinder with base $D$. The function $h : (0,\infty) \to \mathbf{R}$, $h(x) = x_3$, is an exhaustion function for $\mathcal{M}$. Since every domain $D$ in $\mathbf{R}^2$ can be approximated by smooth domains $D'$ from inside, it is easy to see that for $0 < t_1 < t_2 < \infty$ the domain $G = D' \times (t_1, t_2)$ can be used as an approximating domain $G_\varepsilon(t_1, t_2)$. Note that the transversality condition (4.8) is automatically satisfied for the $p$-Laplace operator $A(m, \xi) = |\xi|^{p-2}\xi$.

**Lemma 4.3.** *Suppose that an exhaustion function $h \in C^2(\mathcal{M} \setminus K)$ satisfies* (3.4) *in $\mathcal{M} \setminus K$ and that the function $A(m,\xi)$ is continuously differentiable. If $\mathcal{M}$ satisfies the $(A,h)$-transversality condition, then $h$ is a special exhaustion function on the manifold $\mathcal{M}$.*

*Proof.* It suffices to show $(a_2)$. Let $h' < t_1 < t_2 < h_0$ and $\varepsilon > 0$. Choose an open set $G$ as in the definition of the $(A,h)$-transversality condition. $|A(m, \nabla h(m))| \leq M < \infty$ for every $m \in \mathcal{M}$, and (4.6)–(4.8) together with the Gauss formula imply for a.e. $t_1, t_2$

$$\begin{aligned}
&\left| \int_{\Sigma_h(t_2)} \left\langle \frac{\nabla h}{|\nabla h|}, A(m,\nabla h) \right\rangle d\mathcal{H}^{n-1} - \int_{\Sigma_h(t_1)} \left\langle \frac{\nabla h}{|\nabla h|}, A(m,\nabla h) \right\rangle d\mathcal{H}^{n-1} \right| \\
&\quad \leq \left| \int_{\partial G \cup \Sigma_h(t_2)} \left\langle \frac{\nabla h}{|\nabla h|}, A(m,\nabla h) \right\rangle d\mathcal{H}^{n-1} - \int_{\partial G \cup \Sigma_h(t_1)} \left\langle \frac{\nabla h}{|\nabla h|}, A(m,\nabla h) \right\rangle d\mathcal{H}^{n-1} \right| + \varepsilon M \\
&\quad = \left| \int_{\partial G} \left\langle \frac{\nabla h}{|\nabla h|}, A(m,\nabla h) \right\rangle d\mathcal{H}^{n-1} \right| + \varepsilon M = \left| \int_{\partial G} \langle v, A(m,\nabla h) \rangle d\mathcal{H}^{n-1} \right| + \varepsilon M \\
&\quad = \left| \int_G \operatorname{div} A(m,\nabla h) d\mathcal{H}^{n} \right| + \varepsilon M = \varepsilon M.
\end{aligned} \tag{4.11}$$

Since $\varepsilon > 0$ is arbitrary, $(a_2)$ follows. □

*Example 4.4.* Fix $1 \leq n \leq p$. Let $x_1, x_2, \ldots, x_n$ be an orthonormal system of coordinates in $\mathbf{R}^n, 1 \leq n < p$. Let $D \subset \mathbf{R}^n$ be an unbounded domain with piecewise smooth boundary and let $\mathcal{B}$ be a $(p-n)$-dimensional compact Riemannian manifold with or without boundary. We consider the manifold $\mathcal{M} = D \times \mathcal{B}$.

We denote by $x \in D$, $b \in \mathcal{B}$, and $(x,b) \in \mathcal{M}$ the points of the corresponding manifolds. Let $\pi : D \times \mathcal{B} \to D$ and $\eta : D \times \mathcal{B} \to \mathcal{B}$ be the natural projections of the manifold $\mathcal{M}$.

Assume now that the function $h$ is a function on the domain $D$ satisfying the conditions $(b_1)$, $(b_2)$, and (3.8). We consider the function $h^* = h \circ \pi : \mathcal{M} \to (0,\infty)$.

We have

$$\begin{aligned}
\nabla h^* &= \nabla(h \circ \pi) = (\nabla_x h) \circ \pi, \\
\operatorname{div}\left(|\nabla h^*|^{p-2} \nabla h^*\right) &= \operatorname{div}\left(|\nabla (h\circ\pi)|^{p-2} \nabla (h \circ \pi)\right) \\
&= \operatorname{div}\left(|\nabla_x h|^{p-2} \circ \pi (\nabla_x h) \circ \pi\right) = \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla_x h|^{p-2} \frac{\partial h}{\partial x_i}\right)\right) \circ \pi.
\end{aligned} \tag{4.12}$$

Because $h$ is a special exhaustion function of $D$, we have

$$\operatorname{div}\left(|\nabla h^*|^{p-2}\nabla h^*\right) = 0. \tag{4.13}$$

Let $(x,b) \in \partial\mathcal{M}$ be an arbitrary point where the boundary $\partial\mathcal{M}$ has a tangent hyperplane and let $\nu$ be a unit normal vector to $\partial\mathcal{M}$.

If $x \in \partial D$, then $\nu = \nu_1 + \nu_2$ where the vector $\nu_1 \in \mathbf{R}^k$ is orthogonal to $\partial D$ and $\nu_2$ is a vector from $T_b(\mathcal{B})$. Thus

$$\langle \nabla h^*, \nu\rangle = \langle (\nabla_x h) \circ \pi, \nu_1\rangle = 0, \tag{4.14}$$

because $h$ is a special exhaustion function on $D$ and satisfies the property $(b_2)$ on $\partial D$. If $b \in \partial\mathcal{B}$, then the vector $\nu$ is orthogonal to $\partial\mathcal{B} \times \mathbf{R}^n$ and

$$\langle \nabla h^*, \nu\rangle = \langle (\nabla_x h) \circ \pi, \nu\rangle = 0, \tag{4.15}$$

because the vector $(\nabla_x h) \circ \pi$ is parallel to $\mathbf{R}^n$.

The other requirements for a special exhaustion function for the manifold $\mathcal{M}$ are easy to verify.

Therefore, *the function*

$$h^* = h^*(x,b) = h \circ \pi : \mathcal{M} \longrightarrow (0,\infty) \tag{4.16}$$

*is a special exhaustion function on the manifold* $\mathcal{M} = D \times \mathcal{B}$.

*Example 4.5.* We fix an integer $k$, $1 \le k \le n$, and set

$$d_k(x) = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}. \tag{4.17}$$

It is easy to see that $|\nabla d_k(x)| = 1$ everywhere in $\mathbf{R}^n \setminus \Sigma_0$, where $\Sigma_0 = \{x \in \mathbf{R}^n : d_k(x) = 0\}$. We shall call the set

$$B_k(t) = \{x \in \mathbf{R}^n : d_k(x) < t\} \tag{4.18}$$

a $k$-ball and the set

$$\Sigma_k(t) = \{x \in \mathbf{R}^n : d_k(x) = t\} \tag{4.19}$$

a $k$-sphere in $\mathbf{R}^n$.

We shall say that an unbounded domain $D \subset \mathbf{R}^n$ is $k$-admissible if for each $t > \inf_{x\in D} d_k(x)$, the set $D \cap B_k(t)$ has compact closure.

It is clear that every unbounded domain $D \subset \mathbf{R}^n$ is $n$-admissible. In the general case, the domain $D$ is $k$-admissible if and only if the function $d_k(x)$ is an exhaustion function of $D$.

It is not difficult to see that if a domain $D \subset \mathbf{R}^n$ is $k$-admissible, then it is $l$-admissible for all $k < l < n$.

Fix $1 \le k < n$. Let $\Delta$ be a bounded domain in the $(n-k)$-plane $x_1 = \cdots = x_k = 0$ and let

$$D = \{x = (x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in \mathbf{R}^n : (x_{k+1}, \dots, x_n) \in \Delta\}. \tag{4.20}$$

The domain $D$ is $k$-admissible. The $k$-spheres $\Sigma_k(t)$ are orthogonal to the boundary $\partial D$ and therefore $\langle \nabla d_k, \nu \rangle = 0$ everywhere on the boundary. The function

$$h(x) = \begin{cases} \log d_k(x), & p = k, \\ d_k^{(p-k)/(p-1)}(x), & p \neq k, \end{cases} \tag{4.21}$$

satisfies (3.4). By Lemma 4.3, the function $h$ is a special exhaustion function of the domain $D$. Therefore, the domain $D$ has $p$-parabolic type for $p \ge k$ and $p$-hyperbolic type for $p < k$.

*Example 4.6.* Fix $1 \le k < n$. Let $\Delta$ be a bounded domain in the plane $x_1 = \cdots = x_k = 0$ with a (piecewise) smooth boundary and let

$$D = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : (x_{k+1}, \dots, x_n) \in \Delta\} = \mathbf{R}^{n-k} \times \Delta \tag{4.22}$$

be the cylinder domain with base $\Delta$.

The domain $D$ is $k$-admissible. The $k$-spheres $\Sigma_k(t)$ are orthogonal to the boundary $\partial D$ and therefore $\langle \nabla d_k, \nu \rangle = 0$ everywhere on the boundary, where $d_k$ is as in Example 4.5.

Let $h = \phi(d_k)$ where $\phi$ is a $C^2$-function with $\phi' \ge 0$. We have $\nabla h = \phi' \nabla d_k$ and since $|\nabla d_k| = 1$, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left( |\nabla h|^{n-2} \frac{\partial h}{\partial x_i} \right) &= \sum_{i=1}^{k} \frac{\partial}{\partial x_i}\left( (\phi')^{n-1} \frac{\partial d_k}{\partial x_i} \right) \\ &= (n-1)(\phi')^{n-2}\phi'' + \frac{k-1}{d_k}(\phi')^{n-1}. \end{aligned} \tag{4.23}$$

From the equation

$$(n-1)\phi'' + \frac{k-1}{d_k}\phi' = 0, \tag{4.24}$$

we conclude that *the function*

$$h(x) = (d_k(x))^{(n-k)/(n-1)} \tag{4.25}$$

*satisfies* (3.8) *in $D \setminus K$ and thus it is a special exhaustion function of the domain $D$.*

*Example 4.7.* Let $(r,\theta)$, where $r \geq 0$, $\theta \in S^{n-1}(1)$, be the spherical coordinates in $\mathbf{R}^n$. Let $U \subset S^{n-1}(1)$, $\partial U \neq \emptyset$, be an arbitrary domain with a piecewise smooth boundary on the unit sphere $S^{n-1}(1)$. We fix $0 \leq r_1 < \infty$ and consider the domain

$$D = \{(r,\theta) \in \mathbf{R}^n : r_1 < r < \infty,\ \theta \in U\}. \tag{4.26}$$

As above it is easy to verify that the given domain is $n$-admissible and *the function*

$$h(|x|) = \log \frac{|x|}{r_1} \tag{4.27}$$

*is a special exhaustion function of the domain* $D$ for $p = n$.

*Example 4.8.* Let $\mathcal{A}$ be a compact Riemannian manifold, $\dim \mathcal{A} = k$, with piecewise smooth boundary or without boundary. We consider the Cartesian product $\mathcal{M} = \mathcal{A} \times \mathbf{R}^n$, $n \geq 1$. We denote by $a \in \mathcal{A}$, $x \in \mathbf{R}^n$, and $(a,x) \in \mathcal{M}$ the points of the corresponding spaces. It is easy to see that the function

$$h(a,x) = \begin{cases} \log|x|, & p = n, \\ |x|^{(p-n)/(p-1)}, & p \neq n, \end{cases} \tag{4.28}$$

is a special exhaustion function for the manifold $\mathcal{M}$. Therefore, for $p \geq n$, the given manifold has $p$-parabolic type and for $p < n$, $p$-hyperbolic type.

*Example 4.9.* Let $(r,\theta)$, where $r \geq 0$, $\theta \in S^{n-1}(1)$, be the spherical coordinates in $\mathbf{R}^n$. Let $U \subset S^{n-1}(1)$ be an arbitrary domain on the unit sphere $S^{n-1}(1)$. We fix $0 \leq r_1 < r_2 < \infty$ and consider the domain

$$D = \{(r,\theta) \in \mathbf{R}^n : r_1 < r < r_2,\ \theta \in U\} \tag{4.29}$$

with the metric

$$ds^2_{\mathcal{M}} = \alpha^2(r)dr^2 + \beta^2(r)dl^2_\theta, \tag{4.30}$$

where $\alpha(r), \beta(r) > 0$ are $C^0$-functions on $[r_1, r_2)$ and $dl_\theta$ is an element of length on $S^{n-1}(1)$.

The manifold $\mathcal{M} = (D, ds^2_{\mathcal{M}})$ is a warped Riemannian product. In the cases, $\alpha(r) \equiv 1$, $\beta(r) = 1$, and $U = S^{n-1}$ the manifold $\mathcal{M}$ is isometric to a cylinder in $\mathbf{R}^{n+1}$. In the cases, $\alpha(r) \equiv 1$, $\beta(r) = r$, $U = S^{n-1}$ the manifold $\mathcal{M}$ is a spherical annulus in $\mathbf{R}^n$.

The volume element in the metric (4.30) is given by the expression

$$d\sigma_{\mathcal{M}} = \alpha(r)\beta^{n-1}(r)dr\, dS^{n-1}(1). \tag{4.31}$$

If $\phi(r,\theta) \in C^1(D)$, then the length of the gradient $\nabla\phi$ in $\mathcal{M}$ takes the form

$$|\nabla\phi|^2 = \frac{1}{\alpha^2}(\phi'_r)^2 + \frac{1}{\beta^2}|\nabla_\theta\phi|^2, \tag{4.32}$$

where $\nabla_\theta\phi$ is the gradient in the metric of the unit sphere $S^{n-1}(1)$.

For the special exhaustion function $h(r,\theta) \equiv h(r)$, (3.8) reduces to the following form:

$$\frac{d}{dr}\left(\left(\frac{1}{\alpha(r)}\right)^{p-1}(h'_r(r))^{p-1}\beta^{n-1}(r)\right) = 0. \tag{4.33}$$

Solutions of this equation are the functions

$$h(r) = C_1\int_{r_1}^{r}\frac{\alpha(t)}{\beta^{(n-1)/(p-1)}(t)}dt + C_2, \tag{4.34}$$

where $C_1$ and $C_2$ are constants.

Because the function $h$ satisfies obviously the boundary condition $(a_2)$ as well as the other conditions of special exhaustion functions listed in (4.2), we see that under the assumption

$$\int^{r_2}\frac{\alpha(t)}{\beta^{(n-1)/(p-1)}(t)}dt = \infty, \tag{4.35}$$

the function

$$h(r) = \int_{r_1}^{r}\frac{\alpha(t)}{\beta^{(n-1)/(p-1)}(t)}dt \tag{4.36}$$

is a special exhaustion function on the manifold $\mathcal{M}$.

**Theorem 4.10.** *Let $h : \mathcal{M} \to (0,h_0)$ be a special exhaustion function of a boundary set $\xi$ of the manifold $\mathcal{M}$. Then*

(i) *if $h_0 = \infty$, the set $\xi$ has $p$-parabolic type,*

(ii) *if $h_0 < \infty$, the set $\xi$ has $p$-hyperbolic type.*

*Proof.* Choose $0 < t_1 < t_2 < h_0$ such that $K \subset B_h(t_1)$. We need to estimate the $p$-capacity of the condenser $(B_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M})$. We have

$$\operatorname{cap}_p\left(\overline{B}_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M}\right) = \frac{J}{(t_2 - t_1)^{p-1}}, \tag{4.37}$$

where

$$J = \int_{\Sigma_h(t)}|\nabla h|^{p-1}d\mathcal{H}^{n-1}_{\mathcal{M}} \tag{4.38}$$

is a quantity independent of $t > h(K) = \sup\{h(m) : m \in K\}$. Indeed, for the variational problem [3, (2.9)], we choose the function $\varphi_0$, $\varphi_0(m) = 0$ for $m \in B_h(t_1)$,

$$\varphi_0(m) = \frac{h(m) - t_1}{t_2 - t_1}, \quad m \in B_h(t_2) \setminus B_h(t_1), \tag{4.39}$$

and $\varphi_0(m) = 1$ for $m \in \mathcal{M} \setminus B_h(t_2)$. Using the Kronrod-Federer formula [13, Theorem 3.2.22], we get

$$\begin{aligned}
\operatorname{cap}_p(B_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M}) &\leq \int_{\mathcal{M}} |\nabla \varphi_0|^p * \mathbb{1}_{\mathcal{M}} \\
&\leq \frac{1}{(t_2 - t_1)^p} \int_{t_1 < h(m) < t_2} |\nabla h(m)|^p * \mathbb{1}_{\mathcal{M}} \\
&= \int_{t_1}^{t_2} dt \int_{\Sigma_h(t)} |\nabla h(m)|^{p-1} d\mathcal{H}^{n-1}_{\mathcal{M}}.
\end{aligned} \tag{4.40}$$

Because the special exhaustion function satisfies (3.8) and the boundary condition $(a_2)$, one obtains for arbitrary $\tau_1, \tau_2$, $h(K) < \tau_1 < \tau_2 < h_0$

$$\begin{aligned}
&\int_{\Sigma_h(t_2)} |\nabla h|^{p-1} d\mathcal{H}^{n-1}_{\mathcal{M}} - \int_{\Sigma_h(t_1)} |\nabla h|^{p-1} d\mathcal{H}^{n-1}_{\mathcal{M}} \\
&\quad = \int_{\Sigma_h(t_2)} |\nabla h|^{p-2} \langle \nabla h, \nu \rangle d\mathcal{H}^{n-1}_{\mathcal{M}} - \int_{\Sigma_h(t_1)} |\nabla h|^{p-2} \langle \nabla h, \nu \rangle d\mathcal{H}^{n-1}_{\mathcal{M}} \\
&\quad = \int_{t_1 < h(m) < t_2} \operatorname{div}_{\mathcal{M}} \left( |\nabla h|^{p-2} \nabla h \right) * \mathbb{1}_{\mathcal{M}} = 0.
\end{aligned} \tag{4.41}$$

Thus we have established the inequality

$$\operatorname{cap}_p(B_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M}) \leq \frac{J}{(t_2 - t_1)^{p-1}}. \tag{4.42}$$

By the conditions, imposed on the special exhaustion function, the function $\varphi_0$ is an extremal in the variational problem [3, (2.9)]. Such an extremal is unique and therefore the preceding inequality holds as an equality. This conclusion proves (4.37).

If $h_0 = \infty$, then letting $t_2 \to \infty$ in (4.37) we conclude the parabolicity of the type of $\xi$. Let $h_0 < \infty$. Consider an exhaustion $\{\mathcal{U}_k\}$ and choose $t_0 > 0$ such that the $h$-ball $B_h(t_0)$ contains the compact set $K$.

Set $t_k = \sup_{m \in \partial \mathcal{U}_k} h(m)$. Then for $t_k > t_0$, we have

$$\operatorname{cap}_p\left(\overline{U}_{k_0}, \mathcal{U}_k; \mathcal{M}\right) \geq \operatorname{cap}_p(B_h(t_0), B_h(t_k); \mathcal{M}) = \frac{J}{(t_k - t_0)^{p-1}}, \tag{4.43}$$

and hence

$$\liminf_{k\to\infty} \operatorname{cap}_p\left(\overline{U}_{k_0}, \mathcal{U}_k; \mathcal{M}\right) \geq \frac{J}{(h_0 - t_0)^{p-1}} > 0, \tag{4.44}$$

and the boundary set $\xi$ has $p$-hyperbolic type. □

## 5. Wiman Theorem

Now we will prove Theorem 1.1.

### 5.1. Fundamental Frequency

Let $U \subset \Sigma_h(\tau)$ be an open set. We need further the following quantity:

$$\lambda_p(U) = \inf \frac{\left(\int_U |\nabla h|^{-1} \left|\nabla_2 \varphi\right|^p d\mathcal{H}^{n-1}_{\mathcal{M}}\right)^{1/p}}{\left(\int_U |\nabla h|^{p-1} \left|\varphi\right|^p d\mathcal{H}^{n-1}_{\mathcal{M}}\right)^{1/p}}, \tag{5.1}$$

where the infimum is taken over all functions $\varphi \in W^1_p(U)$ with $\operatorname{supp} \varphi \subset U$ (By the definition, $\varphi$ is a $W^1_p$-function on an open set $U$, if $\varphi$ belongs to this class on every component of $U$.). Here $\nabla_2 \varphi$ is the gradient of $\varphi$ on the surface $\Sigma_h(\tau)$.

In the case $|\nabla h| \equiv 1$, this quantity is well-known and can be interpreted, in particular, as the best constant in the Poincaré inequality. Following [14], we shall call this quantity the fundamental frequency of the rigidly supported membrane $U$.

Observe a useful property of the fundamental frequency.

**Lemma 5.1.** *Let $U \subset \Sigma_h(\tau)$ be an open set and let $U_i$ be the components of $U$, $i = 1, 2, \ldots$. Then*

$$\lambda_p(U) = \inf_i \lambda_p(U_i). \tag{5.2}$$

*Proof.* To prove this property, we fix arbitrary functions $\varphi_i$ with $\operatorname{supp} \varphi_i \subset U_i$. Set $\varphi(m) = \varphi_i(m)$ for $m \in U_i$ and $\varphi = 0$ for $U \setminus (\bigcup_i U_i)$. Hence

$$\lambda^p_p(U_i) \int_{U_i} |\nabla h|^{p-1} \left|\varphi_i\right|^p d\mathcal{H}^{n-1} \leq \int_{U_i} |\nabla h|^{-1} \left|\nabla_2 \varphi_i\right|^p d\mathcal{H}^{n-1}. \tag{5.3}$$

Summation yields

$$\left(\inf_i \lambda^p_p(U_i)\right) \sum_i \int_{U_i} |\nabla h|^{p-1} \left|\varphi_i\right|^p d\mathcal{H}^{n-1} \leq \sum_i \int_{U_i} |\nabla h|^{-1} \left|\nabla_2 \varphi_i\right|^p d\mathcal{H}^{n-1} \tag{5.4}$$

and we obtain

$$\left(\inf_i \lambda_p^p(U_i)\right) \int_U |\nabla h|^{p-1} |\varphi|^p d\mathcal{H}^{n-1} \le \int_U |\nabla h|^{-1} |\nabla_2 \varphi|^p d\mathcal{H}^{n-1}. \tag{5.5}$$

This gives

$$\inf_i \lambda_p(U_i) \le \lambda_p(U). \tag{5.6}$$

The reverse inequality is evident. Indeed, if $U_i$ is a component of $U$, then evidently

$$\lambda_p(U) \le \lambda_p(U_i) \tag{5.7}$$

and hence

$$\lambda_p(U) \le \inf_i \lambda_p(U_i). \tag{5.8}$$

□

We also need the following statement.

**Lemma 5.2.** *Under the above assumptions for a.e.* $\tau \in (0, h_0)$, *we have*

$$\varepsilon(\tau; \mathcal{F}_B) \ge \frac{\lambda_p(\Sigma_h(\tau))}{c}, \tag{5.9}$$

*where* $\lambda_p$ *is the fundamental frequency of the membrane* $\Sigma_h(\tau)$ *defined by formula* (5.1) *and*

$$c = c(\nu_1, \nu_2, p) = \begin{cases} c_1 & \text{for } p \le 2, \\ c_2 & \text{for } p \ge 2, \end{cases} \tag{5.10}$$

*where*

$$\begin{gathered} c_1 = \sqrt{\nu_2^2 - \nu_1^2} + 2^{(2-p)/2}\ \nu_1 p^{-1} (p-1)^{(p-1)/p}, \\ c_2 = \sqrt{\nu_2^2 - \nu_1^2} + \nu_1 \frac{p-1}{p}. \end{gathered} \tag{5.11}$$

For the proof, see Lemma 4.3 in [10].

We now use these estimates for proving Phragmén-Lindelöf type theorems for the solutions of quasilinear equations on manifolds.

**Theorem 5.3.** *Let* $h : \mathcal{M} \to (0, \infty)$ *be an exhaustion function. Suppose that the manifold* $\mathcal{M}$ *satisfies the condition*

$$\int^\infty \lambda_p(\Sigma_h(t)) dt = \infty. \tag{5.12}$$

*Let $f$ be a continuous solution of* (3.4) *with* (3.2) *and* (3.3) *on $\mathcal{M}$ such that*

$$\limsup_{m \to m_0} f(m) \le 0, \quad \forall m_0 \in \partial \mathcal{M}. \tag{5.13}$$

*Then either $f(m) \le 0$ everywhere on $\mathcal{M}$ or*

$$\liminf_{\tau \to \infty} \int_{\tau < h(m) < \tau+1} |\nabla h| |f(m)| |\nabla f(m)|^{p-1} * \mathbb{1} \exp\left\{ -c_3 \int^{\tau} \lambda_p(\Sigma_h(t))dt \right\} > 0, \tag{5.14}$$

$$\liminf_{\tau \to \infty} \int_{\tau < h(m) < \tau+1} |\nabla h(m)|^p |f(m)|^p * \mathbb{1} \exp\left\{ -c_3 \int^{\tau} \lambda_p(\Sigma_h(t))dt \right\} > 0. \tag{5.15}$$

*In particular, if h is a special exhaustion function on $\mathcal{M}$, then*

$$\liminf_{\tau \longrightarrow \infty} M(\tau+1) \exp\left\{ -\frac{c_3}{p} \int^{\tau} \lambda_p(\Sigma_h(t))dt \right\} > 0. \tag{5.16}$$

*Here*

$$M(t) = \sup_{m \in \Sigma_h(t)} |f(m)| \tag{5.17}$$

*and $c_3 = \nu_1 c^{-1}$ where $c$ is the constant of Lemma 5.2.*

*Proof.* We assume that at some point $m_1 \in \operatorname{int} \mathcal{M}$ we have $f(m_1) > 0$. We consider the set

$$\mathcal{O} = \{ m \in \mathcal{M} : f(m) > f(m_1) \}. \tag{5.18}$$

By, [3, Corollary 4.57] the set $\mathcal{O}$ is noncompact.

The function $h$ is an exhaustion function on $\mathcal{O}$. Using the relation [3, 6.74] for the function $f(m) - f(m_1)$ on $\mathcal{O}$, we have

$$\liminf_{\tau \to \infty} \int_{\mathcal{O}(\tau)} |\nabla h| |f(m) - f(m_1)| |A(m, \nabla f)| * \mathbb{1} \exp\left\{ -\nu_1 \int_{\tau_0}^{\tau} \varepsilon(t; \mathcal{F}_{\mathcal{O}})dt \right\} > 0, \tag{5.19}$$

where $\mathcal{O}(\tau) = \{ m \in \mathcal{O} : \tau < h(m) < \tau + 1 \}$.

By Lemma 5.2, the following inequality holds

$$\varepsilon(t; \mathcal{F}_{\mathcal{O}}) \ge \frac{\lambda_p(\Sigma_h(t) \cap \mathcal{O})}{c}. \tag{5.20}$$

Because $\Sigma_h(t) \cap \mathcal{O} \subset \Sigma_h(t)$, it follows that $\lambda_p(\Sigma_h(t) \cap \mathcal{O}) \ge \lambda_p(\Sigma_h(t))$ and hence

$$\varepsilon(t; \mathcal{F}_{\mathcal{O}}) \ge \frac{\lambda_p(\Sigma_h(t))}{c}. \tag{5.21}$$

Thus using the requirement (3.3) for (3.4), we arrive at the estimate

$$\liminf_{\tau\to\infty}\int_{\mathcal{O}(\tau)}|\nabla h(m)|\left|f(m)-f(m_1)\right|\left|\nabla f(m)\right|^{p-1}*\mathbb{1}\exp\left\{-c_3\int^{\tau}\lambda_p(\Sigma_h(t))dt\right\}>0. \tag{5.22}$$

Further we observe that from the condition $f(m)>f(m_1)>0$ on $\mathcal{O}$, it follows that

$$\begin{aligned}\int_{\mathcal{O}(\tau)}|\nabla h|\left|f(m)-f(m_1)\right|\left|\nabla f(m)\right|^{p-1}*\mathbb{1}&=\int_{\mathcal{O}(\tau)}f(m)|\nabla h|\left|\nabla f(m)\right|^{p-1}*\mathbb{1}\\&\quad-f(m_1)\int_{\mathcal{O}(\tau)}|\nabla h|\ \left|\nabla f(m)\right|^{p-1}*\mathbb{1}\\&\le\int_{\tau<h(m)<\tau+1}|\nabla h|\left|f(m)\right|\left|\nabla f(m)\right|^{p-1}*\mathbb{1}.\end{aligned} \tag{5.23}$$

From this relation, we arrive at (5.14).

The proof of (5.15) is carried out exactly in the same way by means of the inequality [3, 5.75].

In order to convince ourselves of the validity of (5.16), we observe that by the maximum principle we have

$$\int_{\tau<h(m)<\tau+1}|\nabla h(m)|^p\left|f(m)\right|^p*\mathbb{1}\le M^p(\tau+1)\int_{\tau<h(m)<\tau+1}|\nabla h(m)|^p*\mathbb{1}. \tag{5.24}$$

But $h$ is a special exhaustion function and therefore by (4.37) we can write

$$\int_{\tau<h(m)<\tau+1}|\nabla h(m)|^p*\mathbb{1}=J, \tag{5.25}$$

where $J$ is a number independent of $\tau$.

The relation (5.15) implies then that (5.16) holds. □

*Example 5.4.* Let $\mathcal{A}$ be a compact Riemannian manifold with nonempty piecewise smooth boundary, $\dim\mathcal{A}=k\ge 1$, and let $\mathcal{M}=\mathcal{A}\times\mathbf{R}^n$, $n\ge 1$. Choosing as a special exhaustion function of $\mathcal{M}$ the function $h(a,x)$, defined in Example 4.8, we have

$$\Sigma_h(t)=\mathcal{A}\times S^{n-1}(t). \tag{5.26}$$

Then using the fact that $h(a,x)|_{\Sigma_h(t)}=t$, we find

$$|\nabla h(a,x)|_{\Sigma_h(t)}=h'(t)=\begin{cases}e^{-t} & \text{for } p=n\\ \dfrac{p-n}{p-1}t^{(1-n)/(p-n)} & \text{for } p\neq n.\end{cases} \tag{5.27}$$

Therefore, on the basis of (5.1) we get

$$\lambda_p(\Sigma_h(t)) = \frac{1}{h'(t)} \inf \frac{\left(\int_{\mathcal{A}\times S^{n-1}(t)} |\nabla_2\phi|^p d\mathcal{H}^{n-1}_{\mathcal{M}}\right)^{1/p}}{\left(\int_{\mathcal{A}\times \mathbf{R}^n} |\phi|^p d\mathcal{H}^{n-1}_{\mathcal{M}}\right)^{1/p}}. \tag{5.28}$$

Computation yields

$$\begin{aligned}
|\nabla_2\phi(a,x)|^2 &= |\nabla_{\mathcal{A}}\phi(a,x)|^2 + |\nabla_{S^{n-1}(t)}\phi(a,x)|^2 \\
&= |\nabla_{\mathcal{A}}\phi(a,x)|^2 + \frac{1}{t^2}\left|\nabla_{S^{n-1}(1)}\phi\left(a,\frac{x}{|x|}\right)\right|^2. \\
d\mathcal{H}^{n-1}_{\mathcal{M}} &= d\sigma_{\mathcal{A}}\, dS^{n-1}(t),
\end{aligned} \tag{5.29}$$

where $d\sigma_{\mathcal{A}}$ is an element of $k$-dimensional area on $\mathcal{A}$. Therefore,

$$\begin{aligned}
&\lambda_p(\Sigma_h(t)) \\
&= \frac{1}{h'(t)} \inf \frac{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(t)} \left(|\nabla_{\mathcal{A}}\phi(a,x)|^2 + |\nabla_{S^{n-1}(t)}\phi(a,x)|^2\right)^{p/2} dS^{n-1}(t)\right)^{1/p}}{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(t)} \phi^p(a,x) dS^{n-1}(t)\right)^{1/p}} \\
&= \frac{1}{h'(t)} \inf \frac{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} \left(|\nabla_{\mathcal{A}}\phi(a,x/|x|)|^2 + (1/t^2)|\nabla_{S^{n-1}(t)}\phi(a,x/|x|)|^2\right)^{p/2} dS^{n-1}(1)\right)^{1/p}}{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} \phi^p(a,x/|x|)\ dS^{n-1}(1)\right)^{1/p}}
\end{aligned} \tag{5.30}$$

and we obtain

$$\lambda_p(\Sigma_h(t)) = \frac{1}{h'(t)} \inf \frac{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} \left(|\nabla_{\mathcal{A}}\psi|^2 + (1/t^2)|\nabla_{S^{n-1}(1)}\psi|^2\right)^{p/2} dS^{n-1}(1)\right)^{1/p}}{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} \psi^p dS^{n-1}(1)\right)^{1/p}}, \tag{5.31}$$

where the infimum is taken over all functions $\psi = \psi(a,x)$ with

$$\psi(a,x) \in W^1_p\left(\mathcal{A}\times S^{n-1}(1)\right), \quad \psi(a,x)|_{a\in\partial\mathcal{A}} = 0, \quad \forall x \in S^{n-1}(1). \tag{5.32}$$

In the particular case $n = 1$, Theorem 5.3 has a particularly simple content. Here $h(x)$ is a function of one variable, and $\Sigma_h(t) = \mathcal{A}\times S^0(t)$ is isometric to $\Sigma_h(1)$. Therefore, $h'(t) \equiv 1$ and by (5.31) we have

$$\lambda_p(\Sigma_h(t)) \equiv \lambda_p(\Sigma_h(1)) \equiv \lambda_p(\mathcal{A}) \quad \forall t \in R^1. \tag{5.33}$$

In the same way, (5.16) can be written in the form

$$\liminf_{t\to\infty} \max_{|x|=t} |f(a,x)| \exp\left\{-\frac{c_3}{p}\,\lambda_n(\mathcal{A})\right\} > 0. \tag{5.34}$$

Let $n \geq 2$. We do not know of examples where the quantity (5.31) had been exactly computed. Some idea about the rate of growth of the quantity $M(\tau)$ in the Phragmén-Lindelöf alternative can be obtained from the following arguments. Simplifying the numerator of (5.31) by ignoring the second summand, we get the estimate

$$\lambda_p(\Sigma_h(t)) \geq \frac{1}{h'(t)} \inf_{\psi} \frac{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} |\nabla_{\mathcal{A}}\psi(a,x)|^p dS^{n-1}(1)\right)^{1/p}}{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} \psi^p(a,x) dS^{n-1}(1)\right)^{1/p}}. \tag{5.35}$$

For each fixed $x \in S^{n-1}(1)$, the function $\psi(a,x)$ is finite on $\mathcal{A}$, because from the definition of the fundamental frequency it follows that

$$\left(\int_{\mathcal{A}} |\nabla_{\mathcal{A}}\psi(a,x)|^p \, d\sigma_{\mathcal{A}}\right)^{1/p} \geq \lambda_p(\mathcal{A})\left(\int_{\mathcal{A}} \psi^p(a,x) d\sigma_{\mathcal{A}}\right)^{1/p}. \tag{5.36}$$

From this we get

$$\lambda_p(\Sigma_h(t)) \geq \frac{1}{h'(t)}\lambda_p(\mathcal{A}). \tag{5.37}$$

Thus

$$\int_{\tau_0}^{\tau} \lambda_p(\Sigma_h(r))dr \geq \int_{\tau_0}^{\tau} \lambda_p(\mathcal{A})\frac{dh(r)}{h'(r)} = \lambda_p(\mathcal{A})\int_{\tau_0}^{\tau} r'(h)dh = \lambda_p(\mathcal{A})(r(\tau) - r(\tau_0)). \tag{5.38}$$

Here $r(h)$ is the inverse function of $h(r)$. Because

$$\max_{h(|x|)=\tau} |f(a,x)| \exp\left\{-\frac{c_3}{p}\lambda_p(\mathcal{A})r(\tau)\right\} = \max_{|x|=r(\tau)} |f(a,x)| \exp\left\{-\frac{c_3}{p}\lambda_p(\mathcal{A})r(\tau)\right\}, \tag{5.39}$$

the relation (5.16) can be written in the form (5.34).

*Example 5.5.* Let $U \subset S^{n-1}$ be an arbitrary domain with nonempty boundary. We consider a warped Riemannian product $\mathcal{M} = (r_1, r_2) \times U$ equipped with the metric (4.30) of the domain $D$. We now analyze Theorem 5.3 in this case.

The function $h(r)$, given by (4.36) under the requirement (4.35), is a special exhaustion function on $\mathcal{M}$. We compute the quantity $\lambda_p(\Sigma_h(\tau))$ as follows:

$$\begin{gathered}
|\nabla h(|x|)|_{\Sigma_h(\tau)} = h'(r(\tau)) = \frac{\alpha(r(\tau))}{\beta^{n-1}(r(\tau))}, \\
\left|\nabla_2 \phi\right|_{\Sigma_h(\tau)} = \frac{\left|\nabla_{S^{n-1}(1)}\phi\right|}{\beta(r(\tau))}, \\
d\mathcal{H}_{\mathcal{M}}^{n-1} = \beta^{n-1}(r(\tau))\,dS^{n-1}(1), \qquad r(\tau) = h^{-1}(\tau).
\end{gathered} \tag{5.40}$$

Therefore, observing that

$$\frac{1}{h'(r(\tau))} = r'(\tau), \tag{5.41}$$

we have

$$\begin{aligned}
\lambda_p(\Sigma_h(\tau)) &= \frac{1}{h'(r(\tau))} \inf_{\phi} \frac{\left(\int_{\Sigma_h(\tau)} \left|\nabla_2 \phi\right|^p d\mathcal{H}_{\mathcal{M}}^{n-1}\right)^{1/p}}{\left(\int_{\Sigma_h(\tau)} \phi^p d\mathcal{H}_{\mathcal{M}}^{n-1}\right)^{1/p}} \\
&= \frac{r'(\tau)}{\beta(r(\tau))} \inf \frac{\left(\int_U \left|\nabla_{S^{n-1}(1)}\phi\right|^p dS^{n-1}(1)\right)^{1/p}}{\left(\int_U \phi^p dS^{n-1}(1)\right)^{1/p}}.
\end{aligned} \tag{5.42}$$

Thus

$$\lambda_p(\Sigma_h(\tau)) = \frac{r'(\tau)}{\beta(r(\tau))} \lambda_p(U). \tag{5.43}$$

Further we get

$$\begin{gathered}
\int_{\tau_0}^{\tau} \lambda_h(\Sigma_h(\tau))\,d\tau = \lambda_p(U) \int_{r(\tau_0)}^{r(\tau)} \frac{dr}{\beta(r)}, \\
\max_{h(|x|)=\tau} \left|f(x)\right| \exp\left\{-\frac{c_3}{p}\lambda_p(U)\int^{r(\tau)} \frac{dr}{\beta(r)}\right\} = \max_{|x|=r(\tau)} \left|f(x)\right| \exp\left\{-\frac{c_3}{p}\lambda_p(U)\int^{r(\tau)} \frac{dr}{\beta(r)}\right\}.
\end{gathered} \tag{5.44}$$

Thus the relation (5.16) attains the form

$$\liminf_{r\longrightarrow\infty} \max_{|x|=r} \left|f(x)\right| \exp\left\{-\frac{c_3}{p}\lambda_p(U)\int^{r} \frac{dr}{\beta(r)}\right\} > 0. \tag{5.45}$$

### *5.2. Proof of Theorem 1.1*

We assume that

$$\limsup_{\tau \to \infty} \min_{m \in \Sigma_h(\tau)} u(f(m)) = K < \infty. \tag{5.46}$$

Consider the set

$$\mathcal{O} = \{m \in \mathcal{X} : u(f(m)) > qK\}, \quad q < 1. \tag{5.47}$$

It is clear that for a suitable choice of $q$, the set $\mathcal{O}$ is not empty.

By assumptions, the function $u$ satisfies (3.4) with (3.2), (3.3) and structure constants $p = n$, $\nu_1$, $\nu_2$. Since $f$ is quasiregular, by Lemma 14.38 of [12] the function $u(f(m))$ is a subsolution of another equation of the form (3.4) with structure constants $\nu_1' = \nu_1/K_O$, $\nu_2' = \nu_2 K_I$, where $K_O$ and $K_I$ are outer and inner dilatations of $f$. In view of the maximum principle for subsolutions, the set $\mathcal{O}$ does not have relatively compact components. Without restricting generality, we may assume that $\mathcal{O}$ is connected. Because for sufficiently large $\tau$, the condition

$$\mathcal{O} \cap \Sigma_h(\tau) \neq \emptyset \tag{5.48}$$

holds; we see that

$$\lambda_n(\mathcal{O} \cap \Sigma_h(\tau)) \geq \lambda_n(\Sigma_h(\tau); 1). \tag{5.49}$$

Therefore, the condition (1.1) on the manifold $\mathcal{X}$ implies the following property:

$$\int^{\infty} \lambda_n(\mathcal{O} \cap \Sigma_h(\tau)) d\tau = \infty. \tag{5.50}$$

Observing that

$$\max_{m \in \Sigma_h(\tau)} u(f(m)) \geq \max_{m \in \Sigma_h(\tau) \cap \mathcal{O}} u(f(m)), \tag{5.51}$$

we see that by (1.2)

$$\liminf_{\tau \to \infty} \max_{\Sigma_h(\tau) \cap \mathcal{O}} u(f(m)) \exp\left\{-C \int^{\tau} \lambda_n(\mathcal{O} \cap \Sigma_h(t)) dt\right\} = 0 \tag{5.52}$$

with the constant $C$ of Theorem 1.1.

It is easy to see that $C = c_3/n$. Using (5.16) with $p = n$ for the function $u(f(m))$ in the domain $\mathcal{O}$, we see that $u(f(m)) \equiv qK$ on $\mathcal{O}$. This contradicts with the definition of the domain $\mathcal{O}$.

*Example 5.6.* As the first corollary, we shall now prove a generalization of Wiman's theorem for the case of quasiregular mappings $f : \mathcal{M} \to \mathbf{R}^n$ where $\mathcal{M}$ is a warped Riemannian product.

For $0 \le r_1 < r_2 \le \infty$, let

$$D = \left\{ m = (r, \theta) \in \mathbf{R}^n : r_1 < r < r_2, \theta \in S^{n-1}(1) \right\} \tag{5.53}$$

be a ring domain in $\mathbf{R}^n$ and let $\mathcal{M} = (r_1, r_2) \times S^{n-1}(1)$ be an $n$-dimensional Riemannian manifold on $D$ with the metric

$$ds^2_{\mathcal{M}} = \alpha^2(r)dr^2 + \beta^2(r)dl^2_{n-1}, \tag{5.54}$$

where $\alpha(r), \beta(r) > 0$ are continuously differentiable on $[r_1, r_2)$ and $dl_{n-1}$ is an element of length on $S^{n-1}(1)$.

As we have proved in Example 4.9, under condition (4.35), the function

$$h(r) = \int_{r_1}^{r} \frac{\alpha(t)}{\beta(t)} dt \tag{5.55}$$

is a special exhaustion function on $\mathcal{M}$.

Let $f : \mathcal{M} \to \mathbf{R}^n$ be a quasiregular mapping. We set $u(y) = \log^+|y|$. This function is a subsolution of (3.4) with $p = n$ and also satisfies all the other requirements imposed on a growth function.

We find

$$\lambda_n\left(S^{n-1}(\tau); 1\right) = \frac{1}{\beta(r(\tau))} \lambda_n\left(S^{n-1}(1); 1\right) \tag{5.56}$$

and further

$$\lambda_n(\Sigma_h(\tau); 1) = \frac{\lambda_n(S^{n-1}(1); 1)}{\beta(r(\tau))h'(r(\tau))}. \tag{5.57}$$

Therefore, the requirement (1.1) on the manifold will be fulfilled if

$$\int^{r_2} \frac{dr}{\beta(r)} = \infty \tag{5.58}$$

holds.

Because

$$\begin{aligned} &\max_{\Sigma_h(\tau)=\tau} \log^+|f(r,\theta)| \exp\left\{ -C \int^{\tau} \lambda_n(\Sigma_h(t); 1) dt \right\} \\ &\quad \le \max_{r=h^{-1}(\tau)} \log^+|f(r,\theta)| \exp\left\{ -C\lambda_n\left(S^{n-1}(1); 1\right) \int^{h^{-1}(\tau)} \frac{dr}{\beta(r)} \right\}, \end{aligned} \tag{5.59}$$

we see that, in view of (1.2), it suffices that

$$\liminf_{\tau\to r_2}\max_{\Sigma_h(\tau)}\log^+\left|f(r,\theta)\right|\exp\left\{-C\lambda_n\left(S^{n-1}(1);1\right)\int^{\tau}\frac{dt}{\beta(t)}\right\}=0. \tag{5.60}$$

In this way, we get the following corollary.

**Corollary 5.7.** *Let $f : \mathcal{M} \to \mathbf{R}^n$ be a nonconstant quasiregular mapping from the warped Riemannian product $\mathcal{M} = (r_1, r_2) \times S^{n-1}(1)$ and $h$ a special exhaustion function of $\mathcal{M}$. If the manifold $\mathcal{M}$ has property* (5.58) *and the mapping $f$ has property* (5.60), *then*

$$\limsup_{\tau\to r_2}\min_{\Sigma_h(\tau)}\left|f(r,\theta)\right|=\infty. \tag{5.61}$$

*Example 5.8.* Suppose that under the assumptions of Example 5.6, we have (in addition) $r_1 = 0$, $r_2 = \infty$, and the functions $\alpha(r) = \beta(r) \equiv 1$, that is, $\mathcal{M} = (0,\infty) \times S^{n-1}(1)$ with the metric $ds^2_{\mathcal{M}} = dr^2 + dl^2_{n-1}$ is an $n$-dimensional half-cylinder. As the special exhaustion function of the manifold $\mathcal{M}$, we can take $h \equiv r$. The condition (5.58) is obviously fulfilled for the manifold.

The condition (5.60) for the mapping $f$ attains the form

$$\liminf_{r\to\infty}\max_{\theta\in S^{n-1}(1)}\log^+\left|f(r,\theta)\right|e^{-C\lambda_n(S^{n-1}(1);1)r}=0. \tag{5.62}$$

**Corollary 5.9.** *If $\mathcal{M} = (0,\infty) \times S^{n-1}(1)$ is a half-cylinder and $f : \mathcal{M} \to \mathbf{R}^n$ is a nonconstant quasiregular mapping satisfying* (5.62), *then*

$$\limsup_{r\to\infty}\min_{\theta\in S^{n-1}(1)}\left|f(r,\theta)\right|=\infty. \tag{5.63}$$

*We assume that in Example 5.8 the quantities $r_1 = 0$, $r_2 = \infty$, and the functions $\alpha(r) \equiv 1$, $\beta(r) = r$, that is, the manifold is $\mathbf{R}^n$. As the special exhaustion function, we choose $h = \log|x|$. This function satisfies* (3.6) *with $p = n$ and $\nu_1 = \nu_2 = 1$. The condition* (5.58) *for the manifold is obviously fulfilled.*

*The condition* (5.62) *attains the form*

$$\liminf_{r\to\infty}\max_{|x|=r}\log^+\left|f(x)\right|r^{-C'\lambda_n(S^{n-1}(1);1)}=0, \tag{5.64}$$

*where*

$$C'=\left(n-1+n\left(K^2(f)-1\right)^{1/2}\right)^{-1}. \tag{5.65}$$

*We have the following corollary.*

**Corollary 5.10.** *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a nonconstant quasiregular mapping satisfying* (5.64). *Then*

$$\limsup_{r\to\infty}\min_{|x|=r}\left|f(x)\right|=\infty. \tag{5.66}$$

## 6. Asymptotic Tracts and Their Sizes

Wiman's theorem for the quasiregular mappings $f : \mathbf{R}^n \to \mathbf{R}^n$ asserts the existence of a sequence of spheres $S^{n-1}(r_k)$, $r_k \to \infty$, along which the mapping $f(x)$ tends to $\infty$. It is possible to further strengthen the theorem and to specify the sizes of the sets along which such a convergence takes place. For the formulation of this result it is convenient to use the language of asymptotic tracts discussed by MacLane [15].

### *6.1. Tracts*

Let $D$ be a domain in the complex plane $C$ and let $f$ be a holomorphic function on $D$. A collection of domains $\{\mathfrak{D}(s) : s > 0\}$ is called an *asymptotic tract* of $f$ if

(a) each of the sets $\mathfrak{D}(s)$ is a component of the set

$$\{z \in D : |f(z)| > s > 0\}; \tag{6.1}$$

(b) for all $s_2 > s_1 > 0$, we have $\mathfrak{D}(s_2) \subset \mathfrak{D}(s_1)$ and $\bigcap_{s>0} \overline{\mathfrak{D}}(s) = \emptyset$.

Two asymptotic tracts $\{\mathfrak{D}'(s)\}$ and $\{\mathfrak{D}''(s)\}$ are considered to be different if for some $s > 0$ we have $\mathfrak{D}'(s) \cap \mathfrak{D}''(s) = \emptyset$.

Below we shall extend this notion to quasiregular mappings $f : \mathcal{M} \to \mathcal{N}$ of Riemannian manifolds. We study the existence of an asymptotic tract and its size.

Let $\mathcal{M}, \mathcal{N}$ be $n$-dimensional connected noncompact Riemannian manifolds and let $u = u(y)$ be a growth function on $\mathcal{N}$, which is a positive subsolution of (3.4) with structure constants $p = n$, $\nu_1$, $\nu_2$.

A family $\{\mathcal{M}(s)\}$ is called an asymptotic tract of a quasiregular mapping $f : \mathcal{M} \to \mathcal{N}$ if

(a) each of the sets $\{\mathcal{M}(s)\}$ is a component of the set

$$\{m \in \mathcal{M} : u(f(m)) > s > 0\}; \tag{6.2}$$

(b) for all $s_2 > s_1 > 0$, we have $\mathcal{M}(s_2) \subset \mathcal{M}(s_1)$ and $\bigcap_{s>0} \overline{\mathcal{M}}(s) = \emptyset$.

Let $f : \mathcal{M} \to \mathbf{R}^n$ be a quasiregular mapping having a point $a \in \mathbf{R}^n$ as a Picard exceptional value, that is, $f(m) \neq a$ and $f(m)$ attains on $\mathcal{M}$ all values of $B(a, r) \setminus \{a\}$ for some $r > 0$.

The set $\{\infty\} \cup \{a\}$ has $n$-capacity zero in $\mathbf{R}^n$ and there is a solution $g(y)$ in $\mathbf{R}^n \setminus \{a\}$ of (3.4) such that $g(y) \to \infty$ as $y \to a$ or $y \to \infty$ (cf. [12, Chapter 10, polar sets] ). As the growth function on $\mathbf{R}^n \setminus \{a\}$, we choose the function $u(y) = \max(0, g(y))$. It is clear that this function is a subsolution of (3.4) in $\mathbf{R}^n \setminus \{a\}$.

The function $u(f(m))$ also is a subsolution of an equation of the form (3.4) on $\mathcal{M}$. Because the mapping $f(m)$ attains all values in the punctured ball $B(a, r)$, then among the components of the set

$$\{m \in \mathcal{M} : u(f(m)) > s\} \tag{6.3}$$

there exists at least one $\mathcal{M}(s)$ having a nonempty intersection with $f^{-1}(B(a,r))$. Then by the maximum principle for subsolutions, such a component cannot be relatively compact.

Letting $s \to \infty$, we find an asymptotic tract $\{\mathcal{M}(s)\}$, along which a quasiregular mapping tends to a Picard exceptional value $a \in \mathbf{R}^n$.

Because one can find in every asymptotic tract a curve $\Gamma$ along which $u(f(m)) \to \infty$, we obtain the following generalization of Iversen's theorem [16].

**Theorem 6.1.** *Every Picard exceptional value of a quasiregular mapping $f : \mathcal{M} \to \mathbf{R}^n$ is an asymptotic value.*

*The classical form of Iversen's theorem asserts that if $f$ is an entire holomorphic function of the plane, then there exists a curve $\Gamma$ tending to infinity such that*

$$f(z) \longrightarrow \infty \quad \text{as } z \longrightarrow \infty \quad \text{on } \Gamma. \tag{6.4}$$

*We prove a generalization of this theorem for quasiregular mappings $f : \mathcal{M} \to \mathcal{N}$ of Riemannian manifolds.*

The following result holds.

**Theorem 6.2.** *Let $f : \mathcal{M} \to \mathcal{N}$ be a nonconstant quasiregular mapping between $n$-dimensional noncompact Riemannian manifolds without boundaries. If there exists a growth function $u$ on $\mathcal{N}$ which is a positive subsolution of* (3.4) *with $p = n$ and on $\mathcal{M}$ a special exhaustion function, then the mapping $f$ has at least one asymptotic tract and, in particular, at least one curve $\Gamma$ on $\mathcal{M}$ along which $u(f(m)) \to \infty$.*

*Proof.* Let $h : \mathcal{M} \to (0,\infty)$ be a special exhaustion function of the manifold $\mathcal{M}$. Set

$$\liminf_{\tau\to\infty} \min_{h(m)=\tau} u\big(f(m)\big) = K. \tag{6.5}$$

If $K = \infty$, then $u(f(m))$ tends uniformly on $\mathcal{M}$ to $\infty$ for $h(m) \to \infty$. The asymptotic tract $\{\mathcal{M}(s)\}$ generates mutual inclusion of the components of the set $\{m \in \mathcal{M} : h(m) > s\}$.

Let $K < \infty$. For an arbitrary $s > K$, we consider the set

$$\mathcal{O}(s) = \big\{m \in \mathcal{M} : u\big(f(m)\big) > s\big\}. \tag{6.6}$$

Because $u(f(m))$ is a subsolution, the nonempty set $\mathcal{O}(s)$ does not have relatively compact components. By a standard argument, we choose for each $s > K$, as $\mathcal{M}(s)$, a component of the set $\mathcal{O}(s)$ having property (b) of the definition of an asymptotic tract. We now easily complete the proof for the theorem. □

## 6.2. *Proof of Theorem 1.2*

We fix a growth function $u$ and a special exhaustion function $h$ as in Section 4. Let $f : \mathcal{M} \to \mathcal{N}$ be a nonconstant quasiregular mapping. We set

$$M(\tau) = \max_{h(m)=\tau} u\big(f(m)\big). \tag{6.7}$$

Let $K$ be the quantity defined in (6.5). The case $K=\infty$ is degenerate and has no interest in the present case.

Suppose now that $K<\infty$. For $s>K$, we consider the set $\mathcal{M}(s)$, defined in the proof of the preceding theorem. Define

$$\tau_0=\tau_0(s)>\inf_{m\in\mathcal{M}(s)}h(m). \tag{6.8}$$

Because $u(f(m))$ is a subsolution of an equation of the form (3.4) on $\mathcal{M}$ by [3, Theorem 5.59], we have for an arbitrary $\tau>\tau_0$

$$\int_{B_h(\tau_0)\cap\mathcal{M}(s)}\left|\nabla u\left(f(m)\right)\right|^n*\mathbb{1}\le\exp\left\{-\nu_1\int_{\tau_0}^{\tau}\varepsilon(t)dt\right\}\int_{B_h(\tau)\cap\mathcal{M}(s)}\left|\nabla u\left(f(m)\right)\right|^n*\mathbb{1}. \tag{6.9}$$

Using the inequality (4.5) of [10] for the quantity $\varepsilon(t)$, we get

$$\begin{aligned}&\int_{B_h(\tau_0)\cap\mathcal{M}(s)}\left|\nabla u\left(f(m)\right)\right|^n*\mathbb{1}\\&\quad\le\exp\left\{-\frac{\nu_1}{c}\int_{\tau_0}^{\tau}\lambda_n(\Sigma_h(t)\cap\mathcal{M}(s))dt\right\}\int_{B_h(\tau)\cap\mathcal{M}(s)}\left|\nabla u\left(f(m)\right)\right|^n*\mathbb{1},\end{aligned} \tag{6.10}$$

where

$$c=\sqrt{\nu_2^{-2}-\nu_1^{-2}}+\frac{n-1}{n}\nu_1. \tag{6.11}$$

By [3, 5.71], we have

$$\begin{aligned}\left(\frac{\nu_1}{\nu_2}\right)^n\int_{B_h(\tau)}\left|\nabla u\left(f(m)\right)\right|^n*\mathbb{1}&\le n^n\int_{B_h(\tau+1)\setminus B_h(\tau)}|\nabla h|^n\left|u\left(f(m)\right)\right|^n*\mathbb{1}\\&\le n^nM^n(\tau+1)V(\tau),\end{aligned} \tag{6.12}$$

where

$$V(\tau)=\int_{B_h(\tau+1)\setminus B_h(\tau)}|\nabla_{\mathcal{M}}h|^n*\mathbb{1}. \tag{6.13}$$

But $h$ is a special exhaustion function and as in the proof of (4.37) we get

$$V(\tau)\le J\equiv\text{const} \tag{6.14}$$

for all sufficiently large $\tau$. Hence

$$\int_{B_h(\tau)}\left|\nabla u\left(f(m)\right)\right|^n*\mathbb{1}\le JM^n(\tau+1) \tag{6.15}$$

and further

$$\int_{B_h(\tau_0)\cap \mathcal{M}(s)} |\nabla u(f(m))|^n * \mathbb{1} \le J M^n(\tau+1)\exp\left\{-C\int_{\tau_0}^{\tau} \lambda_n(\Sigma_h(t)\cap \mathcal{M}(s))dt\right\}, \tag{6.16}$$

where $C = \nu_1/c$ and $c$ is defined in Lemma 5.2.

Under these circumstances, from the condition (1.5) for the growth of $M(\tau)$, it follows that for all $\varepsilon > 0$ and for all sufficiently large $\tau$, we have

$$\int_{B_h(\tau_0)\cap \mathcal{M}(s)} |\nabla u(f(m))|^n * \mathbb{1} \le J\varepsilon \exp\left\{\int_{\tau_0}^{\tau} \left(n\gamma\lambda_n(\Sigma_h(t);1) - C\lambda_n(\Sigma_h(t)\cap \mathcal{M}(s))\right)dt\right\}. \tag{6.17}$$

If we assume that for all $\tau > \tau_0$

$$\int_{\tau_0}^{\tau} \left(n\gamma\lambda_n(\Sigma_h(t);1) - C\lambda_n(\Sigma_h(t)\cap \mathcal{M}(s))\right)dt \le 0, \tag{6.18}$$

then because $\varepsilon > 0$ was arbitrary, it would follow from (6.17) that $|\nabla u(f(m))| \equiv 0$ on $B_h(\tau_0)\cap \mathcal{M}(s)$ which is impossible.

Hence there exists $\tau = \tau(K) > \tau_0(K)$ for which

$$\lambda_n(\Sigma_h(\tau)\cap \mathcal{M}(s)) < \frac{n\gamma}{C}\lambda_n(\Sigma_h(\tau);1). \tag{6.19}$$

Letting $K \to \infty$, we see that $\tau_0 \to \infty$. Using each time the relation (6.17), we get Theorem 1.2.

In the formulation of the theorem, we used only a part of the information about the sizes of the sets $\mathcal{M}(s)$ which is contained in (6.17). In particular, the relation (6.17) to some extent characterizes also the linear measure of those $t > \tau_0$ for which the intersection of the sets $\mathcal{M}(s)$ with the $h$-spheres $\Sigma_h(t)$ is not too narrow.

We consider the case of warped Riemannian product $\mathcal{M} = (r_1, r_2) \times S^{n-1}(1)$ with the metric $ds^2_{\mathcal{M}}$ described in Example 5.6. Let $h$ be a special exhaustion function of the manifold $\mathcal{M}$ of the type (4.36) with $p = n$, satisfying condition (4.35).

Here, as in Example 5.6,

$$\lambda_n(\Sigma_h(\tau);1) = \frac{\lambda_n(S^{n-1}(1);1)}{\beta(r(\tau))h'(r(\tau))}, \quad \lambda_n(U) = \frac{\lambda_n(U^*)}{\beta(r(\tau))h'(r(\tau))}, \tag{6.20}$$

where $r(\tau) = h^{-1}(\tau)$ and $U^* \subset S^{n-1}(1)$ is the image of the set $U$ under the similarity mapping

$$x \longmapsto \frac{x}{\beta(r(\tau))} \tag{6.21}$$

of $\mathbf{R}^n$.

Let $f : \mathcal{M} \to \mathbf{R}^n$ be a nonconstant quasiregular mapping. We choose as a growth function $u$ the function $u = \log^+|y|$. This function satisfies (3.6) with $p = n$ and $\nu_1 = \nu_2 = 1$. The condition (1.5) can be written as follows:

$$\liminf_{\tau \to r_2} \max_{r=\tau} \log^+ \left| f(r,\theta) \right| \exp\left\{ -\gamma \lambda_n \left( S^{n-1}(1); 1 \right) \int^r \frac{dt}{\beta(t)} \right\} = 0. \tag{6.22}$$

Hence we obtain

**Corollary 6.3.** *If a quasiregular mapping $f : \mathcal{M} \to \mathbf{R}^n$ has the property* (6.22) *for some $\gamma > 0$, then for each $k = 1, 2, \ldots$ there are spheres $S^{n-1}(t_k)$, $t_k \in (r_1, r_2)$, $t_k \to r_2$, and open sets $U \subset S^{n-1}(t_k)$ for which*

$$\left| f(m) \right| > k \quad \forall m \in U, \qquad \lambda_n(U) < \frac{n\gamma}{C'} \lambda_n \left( S^{n-1}(1); 1 \right), \tag{6.23}$$

*where as above*

$$C' = \left( n - 1 + n \left( K^2(f) - 1 \right)^{1/2} \right)^{-1}. \tag{6.24}$$

Corresponding estimates of the quantities $\lambda_n(U^*)$ and $\lambda_n(S^{n-1}(1); 1)$ were given in [7] in terms of the $(n-1)$-dimensional surface area and in terms of the best constant in the embedding theorem of the Sobolev space $W_n^1$ into the space $C$ on open subsets of the sphere. This last constant can be estimated without difficulties in terms of the maximal radius of balls contained in the given subset.

## References


[1] A. Wiman, "Sur une extension d'un théoréme de M. Hadamard," *Arkiv för Matematik,Astronomi och Fysik*, vol. 2, no. 14, pp. 1–5, 1905.

[2] K. Arima, "On maximum modulus of integral functions," *Journal of the Mathematical Society of Japan*, vol. 4, pp. 62–66, 1952.

[3] O. Martio, V. Miklyukov, and M. Vuorinen, "Ahlfors theorems for differential forms," Tech. Rep., Department of Mathematics, University of Helsinki, Helsinki, Finland, 2005, http://arxiv.org/abs/0710.4648.

[4] O. Martio, V. Miklyukov, and M. Vuorinen, "Generalized Wiman and Arima theorems for $n$-subharmonic functions on cones," *The Journal of Geometric Analysis*, vol. 13, no. 4, pp. 605–630, 2003.

[5] D. Franke, O. Martio, V. M. Miklyukov, M. Vuorinen, and R. Wisk, "Quasiregular mappings and $\mathcal{WT}$-classes of differential forms on Riemannian manifolds," *Pacific Journal of Mathematics*, vol. 202, no. 1, pp. 73–92, 2002.

[6] E. Phragmén and E. Lindelöf, "Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier," *Acta Mathematica*, vol. 31, no. 1, pp. 381–406, 1908.

[7] V. M. Mikljukov, "Asymptotic properties of subsolutions of quasilinear equations of elliptic type and mappings with bounded distortion," *Matematicheskiĭ Sbornik. Novaya Seriya*, vol. 111, no. 1, pp. 42–66, 1980 (Russian), English translation: Mathematics USSR-Sbornik vol. 39, pp. 37–60, 1981.

[8] S. Rickman and M. Vuorinen, "On the order of quasiregular mappings," *Annales Academiae Scientiarum Fennicae. Series A*, vol. 7, no. 2, pp. 221–231, 1982.

[9] S. Granlund, P. Lindqvist, and O. Martio, "Phragmén-Lindelöf's and Lindelöf's theorems," *Arkiv för Matematik*, vol. 23, no. 1, pp. 103–128, 1985.

[10] O. Martio, V. M. Miklyukov, and M. Vuorinen, "Critical points of $A$-solutions of quasilinear elliptic equations," *Houston Journal of Mathematics*, vol. 25, no. 3, pp. 583–601, 1999.

[11] O. Martio, V. M. Miklyukov, and M. Vuorinen, "Morrey's lemma on Riemannian manifolds," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 43, no. 1-2, pp. 183–210, 1998.

[12] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, The Clarendon Press, New York, NY, USA, 1993.

[13] H. Federer, *Geometric Measure Theory*, Die Grundlehren der Mathematischen Wissenschaften, Vol 153, Springer, Berlin, Germany, 1969.

[14] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematics Studies, no. 27, Princeton University Press, Princeton, NJ, USA, 1951.

[15] G. R. MacLane, "Asymptotic values of holomorphic functions," *Rice University Studies*, vol. 49, no. 1, p. 83, 1963.

[16] F. Iversen, *Recherches sur les fonctions inverses des fonctions meromorphes*, Thesis Helsinki, 1914.