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Research Article

A Regularity Criterion for the Nematic Liquid Crystal Flows

Yong Zhou[1] and Jishan Fan[2, 3]

[1] *Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China*
[2] *Department of Applied Mathematics, Nanjing Forestry University, Nanjing, Jiangsu 210037, China*
[3] *Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan*

Correspondence should be addressed to Yong Zhou, yzhoumath@zjnu.edu.cn

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A logarithmically improved regularity criterion for the 3D nematic liquid crystal flows is established.

1. Introduction

We consider the following hydrodynamical systems modeling the flow of nematic liquid crystal materials ([1, 2]):

$$u_t + u \cdot \nabla u + \nabla \pi - \mu \Delta u = -\lambda \nabla \cdot \left(\nabla d \odot \nabla d + \left(\Delta d - f(d) \right) \otimes d \right), \tag{1.1}$$

$$d_t + u \cdot \nabla d - d \cdot \nabla u = \gamma \left(\Delta d - f(d) \right), \tag{1.2}$$

$$\operatorname{div} u = 0, \tag{1.3}$$

$$(v, d)|_{t=0} = (v_0, d_0) \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

$u(x,t) \in \mathbb{R}^3$ is the velocity field of the flow. $d(x,t) \in \mathbb{R}^3$ is the (averaged) macroscopic/continuum molecular orientations vector in $\mathbb{R}^3$. $\pi(x,t)$ is a scalar function representing the pressure (including both the hydrostatic part and the induced elastic part from the orientation field). μ is a positive viscosity constant. The constant λ represents the competition between kinetic energy and potential energy. The constant γ is the microscopic elastic relaxation time (Deborah number) for the molecular orientation field. $f(d) = (1/\epsilon^2)(|d|^2 - 1)d$. For simplicity, we will take $\mu = \lambda = \gamma = \epsilon = 1$. The 3×3 matrix is defined by $(\nabla \odot \nabla d)_{ij} = (\partial_i d \cdot \partial_j d)$. $\otimes$ is the usual Kronecker multiplication, for example, $(a \otimes b)_{ij} = a_i b_j$ for $a, b \in \mathbb{R}^3$.

Very recently, results for the local existence of classical solutions for the problems (1.1)–(1.4) were presented in [3]. The aim of this paper is to establish a regularity criterion for it. We will prove the following.

Theorem 1.1. *Let* $(u_0, d_0) \in H^2 \times H^3$ *with* $\operatorname{div} u_0 = 0$ *in* $\mathbb{R}^3$. *Suppose that a local smooth solution* (u, d) *satisfies*

$$\int_0^T \frac{\|\nabla u(t)\|_{L^p}^r}{1 + \ln(e + \|\nabla u(t)\|_{L^p})} dt < \infty, \quad \text{with } \frac{2}{r} + \frac{3}{p} = 2,\ 2 \leq p \leq 3. \tag{1.5}$$

Then (u, d) *can be extended beyond* T.

Remark 1.2. Equation (1.5) can be regarded as a logarithmically improved regularity criterion of the form $\nabla u \in L^r(0, T; L^p(\mathbb{R}^3))$ with $(2/r) + (3/p) = 2$. Condition (1.5) only involves the velocity field u, which plays a dominant role in regularity theorem. Similar phenomenon already appeared in the studies of MHD equations (see [4–6] for details).

Remark 1.3. When $\lambda = 0$ in (1.1), then (1.1) and (1.2) are the well-known Navier-Stokes equations. Similar conditions to (1.5) have been established in [7–10]. But previous methods can not be used here.

Remark 1.4. A natural region for p in (1.5) should be $3/2 \leq p \leq \infty$, but we only can prove it for $2 \leq p \leq 3$ here. We are unable to establish any other regularity criterion in terms of u or π.

2. Proof of Theorem 1.1

Since we deal with the regularity conditions of the local smooth solutions, we only need to establish the needed a priori estimates. We mainly will follow the method introduced in [9].

First, it has been proved in [3] that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \left(|u|^2(x,t) + |\nabla d|^2(x,t) + \left(|d|^2 - 1\right)^2(x,t)\right) dx \\ &\quad + \int_{\mathbb{R}^3} \left(|\nabla u|^2(x,t) + |\Delta d - f(d)|^2(x,t)\right) dx = 0. \end{aligned} \tag{2.1}$$

Hence

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \leq C. \tag{2.2}$$

Multiplying (1.3) by d, integration by parts yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|d|^2(x,t)dx+\int_{\mathbb{R}^3}\left(|\nabla d|^2(x,t)+|d|^4(x,t)\right)dx \\
&\quad=\int_{\mathbb{R}^3}\left(|d|^2(x,t)+(d\cdot\nabla)u\cdot d(x,t)\right)dx \\
&\quad\le\frac{1}{2}\int_{\mathbb{R}^3}|d|^4(x,t)dx+\int_{\mathbb{R}^3}\left(|d|^2(x,t)+\frac{1}{2}|\nabla u|^2(x,t)\right)dx.
\end{aligned}\tag{2.3}$$

Thanks to (2.1), (2.2), and the Gronwall inequality, we get

$$\|d\|_{L^\infty(0,T;H^1)}+\|d\|_{L^2(0,T;H^2)}\le C.\tag{2.4}$$

Let $u=(u_1,u_2,u_3)^T$ and $d=(d_1,d_2,d_3)^T$, then the ith $(i=1,2,3)$ component of u satisfies

$$\partial_t u_i+u\cdot\nabla u_i+\partial_i\pi-\Delta u_i=-\sum_{j=1}^{3}\partial_j\left(\sum_k\partial_i d_k\partial_j d_k+\left(\Delta d_i-\left(|d|^2-1\right)d_i\right)d_j\right).\tag{2.5}$$

Multiplying (2.5) by $-\Delta u_i$, after integration by parts, summing over i, and using (1.2), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla u|^2(x,t)dx+\int_{\mathbb{R}^3}|\Delta u|^2(x,t)dx \\
&\quad=-\sum_{i,j,k}\int_{\mathbb{R}^3}\partial_k u_j\cdot\partial_j u_i\cdot\partial_k u_i dx-\sum_{i,k}\int_{\mathbb{R}^3}\Delta d_k\cdot\partial_i\nabla d_k\cdot\nabla u_i dx \\
&\qquad-\sum_{i,k}\int_{\mathbb{R}^3}\partial_i d_k\cdot\nabla\Delta d_k\cdot\nabla u_i dx+\sum_{i,j}\int_{\mathbb{R}^3}\partial_j\left(d_j\Delta d_i\right)\cdot\Delta u_i dx \\
&\qquad-\sum_{i,j}\int_{\mathbb{R}^3}\partial_j\left(\left(|d|^2-1\right)d_i d_j\right)\cdot\Delta u_i dx \\
&\quad=: I_1+I_2+I_3+I_4+I_5.
\end{aligned}\tag{2.6}$$

Applying Δ on (1.3), multiplying it by Δd, and using (1.2), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\Delta d|^2(x,t)dx+\int_{\mathbb{R}^3}\Big(|\nabla\Delta d|^2(x,t)+\Delta f(d)\cdot\Delta d(x,t)\Big)dx\\
&\quad=\sum_{i,k}\int_{\mathbb{R}^3}\partial_i d_k\cdot\nabla\Delta d_k\cdot\nabla u_i dx-\sum_{i,j,k}\int_{\mathbb{R}^3}\partial_i\partial_j d_k\cdot\partial_j\nabla d_k\cdot\nabla u_i dx\\
&\qquad+\sum_{i,j}\int_{\mathbb{R}^3}(d_j\Delta d_i)\cdot\partial_j\Delta u_i dx-\sum_{i,j}\int_{\mathbb{R}^3}\Delta d_j\Delta d_i\cdot\partial_j u_i dx\\
&\qquad-2\sum_{i,j}\int_{\mathbb{R}^3}\nabla d_j\cdot\partial_j u_i\cdot\nabla\Delta d_i dx\\
&\quad=: I_6+I_7+I_8+I_9+I_{10}.
\end{aligned}\tag{2.7}$$

Combining (2.6) and (2.7) together, noting that $I_3+I_6=0$, $I_4+I_8=0$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\Big(|\nabla u|^2(x,t)+|\Delta d|^2(x,t)\Big)dx+\int_{\mathbb{R}^3}|\Delta u|^2(x,t)dx\\
&\quad+\int_{\mathbb{R}^3}\Big(|\nabla\Delta d|^2(x,t)+\Delta f(d)\cdot\Delta d(x,t)\Big)dx=I_1+I_2+I_5+I_7+I_9+I_{10}.
\end{aligned}\tag{2.8}$$

We do estimates for I_i $(i=1,2,5,7,9,10)$ as follows:

$$\begin{aligned}
I_1&\le C\|\nabla u\|_{L^p}\|\nabla u\|^2_{L^{2p/(p-1)}}\\
&\le C\|\nabla u\|_{L^p}\|\nabla u\|^{2(1-(3/2p))}_{L^2}\|\Delta u\|^{3/p}_{L^2}\\
&\le \epsilon\|\Delta u\|^2_{L^2}+C\|\nabla u\|^{2p/(2p-3)}_{L^p}\|\nabla u\|^2_{L^2},\quad\text{for any }\ \epsilon>0.
\end{aligned}\tag{2.9}$$

Here we have used the following Gagliardo-Nirenberg inequality:

$$\|\nabla u\|_{L^{2p/(p-1)}}\le C\|\nabla u\|^{1-(3/2p)}_{L^2}\|\Delta u\|^{3/2p}_{L^2}.\tag{2.10}$$

Similarly, by using (2.10), we have

$$\begin{aligned}
I_2+I_7+I_9&\le C\|\nabla u\|_{L^p}\|\Delta d\|^2_{L^{2p/(p-1)}}\\
&\le C\|\nabla u\|_{L^p}\|\Delta d\|^{2(1-(3/2p))}_{L^2}\|\nabla\Delta d\|^{3/p}_{L^2}\\
&\le \epsilon\|\nabla\Delta d\|^2_{L^2}+C\|\nabla u\|^{2p/(2p-3)}_{L^p}\|\Delta d\|^2_{L^2},\quad\text{for any }\ \epsilon>0.
\end{aligned}\tag{2.11}$$

I_5 is simply bounded as follows:

$$\begin{aligned}
I_5 &\le C\int_{\mathbb{R}^3}\left(|d|+|d|^3\right)|\nabla d|\cdot|\Delta u|dx\\
&\le C\left(\|d\|_{L^6}\|\nabla d\|_{L^3}+\|d\|_{L^6}^3\|\nabla d\|_{L^\infty}\right)\|\Delta u\|_{L^2}\\
&\le C(\|\nabla d\|_{L^3}+\|\nabla d\|_{L^\infty})\|\Delta u\|_{L^2}\\
&\le C\left(\|\nabla d\|_{L^2}^{1/2}\|\Delta d\|_{L^2}^{1/2}+\|\nabla d\|_{L^2}^{1/4}\|\nabla\Delta d\|_{L^2}^{3/4}\right)\|\Delta u\|_{L^2}\\
&\le \epsilon\|\Delta u\|_{L^2}^2+C\|\Delta d\|_{L^2}+C\|\nabla\Delta d\|_{L^2}^{3/2}\\
&\le \epsilon\|\Delta u\|_{L^2}^2+C\|\Delta d\|_{L^2}^2+\epsilon\|\nabla\Delta d\|_{L^2}^2+C,
\end{aligned} \tag{2.12}$$

for any $\epsilon > 0$.

When $p = 2$ or 3, I_{10} can be estimated easily and hence omitted here. If $2 < p < 3$, we do estimates as follows:

$$\begin{aligned}
I_{10} &\le C\|\nabla u\|_{L^p}\|\nabla d\|_{L^{2p/(p-2)}}\|\nabla\Delta d\|_{L^2}\\
&\le C\|\nabla u\|_{L^p}\cdot\|\Delta d\|_{L^2}^{2-(3/p)}\cdot\|\nabla\Delta d\|_{L^2}^{3/p}\\
&\le \epsilon\|\nabla\Delta d\|_{L^2}^2+C\|\nabla u\|_{L^p}^{2p/(2p-3)}\cdot\|\Delta d\|_{L^2}^2,
\end{aligned} \tag{2.13}$$

for any $\epsilon > 0$. Here we have used the Gagliardo-Nirenberg inequality:

$$\|\nabla d\|_{L^{2p/(p-2)}} \le C\|\Delta d\|_{L^2}^{2-(3/p)}\|\nabla\Delta d\|_{L^2}^{(3/p)-1}. \tag{2.14}$$

Finally, we omit the trivial term

$$\int_{\mathbb{R}^3}\Delta f(d)\cdot\Delta d\,dx = -\sum_i\int_{\mathbb{R}^3}\partial_i f(d)\cdot\partial_i\Delta d\,dx. \tag{2.15}$$

Now, putting the above estimates for I_is into (2.8) and taking ϵ small enough, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\left(|\nabla u|^2+|\Delta d|^2\right)dx+\int_{\mathbb{R}^3}\left(|\Delta u|^2+|\nabla\Delta d|^2\right)dx\\
&\quad\le C\|\nabla u\|_{L^p}^{2p/(2p-3)}\left(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)+C\|\Delta d\|_{L^2}^2+C\\
&\quad\le C\left(1+\|\nabla u\|_{L^p}^{2p/(2p-3)}\right)\left(1+\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right).
\end{aligned} \tag{2.16}$$

Due to the integrability of (1.5), we conclude that for any small constant $\epsilon > 0$, there exists a time $T_* < T$ such that

$$\int_{T_*}^{T} \frac{1 + \|\nabla u(t)\|_{L^p}^{2p/(2p-3)}}{1 + \ln(e + \|\nabla u(t)\|_{L^p})} dt \leq \epsilon. \tag{2.17}$$

Easily, from (2.16) and (2.17) it follows that

$$\begin{aligned} &\frac{d}{dt}\left(1 + \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right) \\ &\quad \leq C \frac{1 + \|\nabla u\|_{L^p}^{2p/(2p-3)}}{1 + \ln(e + \|\nabla u\|_{L^p})} \ln(e + \|\Delta u\|_{L^2} + \|\nabla \Delta d\|_{L^2})\left(1 + \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right), \end{aligned} \tag{2.18}$$

which implies that for $t \in [T_*, T)$,

$$\|\nabla u(t)\|_{L^2}^2 + \|\Delta d(t)\|_{L^2}^2 \leq C\left(1 + \sup_{[T_*,t]}\|\Delta u(\cdot)\|_{L^2} + \sup_{[T_*,t]}\|\nabla \Delta d(\cdot)\|_{L^2}\right)^{C\epsilon}. \tag{2.19}$$

We are going to do the estimate for Δu and $\nabla \Delta d$. To this end, we introduce the following commutator estimates due to the work of Kato and Ponce [11]:

$$\|\Lambda^\alpha(fg) - f\Lambda^\alpha g\|_{L^p} \leq C\left(\left\|\Lambda^{\alpha-1} g\right\|_{L^{q_1}} \|\nabla f\|_{L^{p_1}} + \|\Lambda^\alpha f\|_{L^{p_2}} \|g\|_{L^{q_2}}\right), \tag{2.20}$$

$$\|\Lambda^\alpha(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}} \|\Lambda^\alpha g\|_{L^{q_1}} + \|\Lambda^\alpha f\|_{L^{p_2}} \|g\|_{L^{q_2}}), \tag{2.21}$$

where $\Lambda^\alpha = (-\Delta)^{\alpha/2}$, for $\alpha > 1$, and $1/p = (1/p_1) + (1/q_1) = (1/p_2) + (1/q_2)$.

Applying Δ to (2.5) and multiplying it by Δu_i, after integration by parts, and summing over i yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\Delta u|^2(x,t)dx + \int_{\mathbb{R}^3} |\nabla \Delta u|^2(x,t)dx \\ &\quad \leq \left|\int_{\mathbb{R}^3} (\Delta(u \cdot \nabla u) - (u \cdot \nabla) \cdot \Delta u) \cdot \Delta u \, dx\right| + \sum_{i,j}\left|\int_{\mathbb{R}^3} \partial_j \Delta(\partial_i d \cdot \partial_j d) \cdot \Delta u_i dx\right| \\ &\qquad + \sum_{i,j}\left|\int_{\mathbb{R}^3} \partial_j \Delta\left(\left(|d|^2 - 1\right) d_i d_j\right) \cdot \Delta u_i dx\right| + \sum_{i,j}\int_{\mathbb{R}^3} d_j \Delta^2 d_i \cdot \partial_j \Delta u_i dx \\ &\qquad + \sum_{i,j}\left|\int_{\mathbb{R}^3} \Delta d_i \cdot \Delta d_j \cdot \partial_j \Delta u_i dx\right| + 2\sum_{i,j}\int_{\mathbb{R}^3} \left|\nabla d_j \cdot \nabla \Delta d_i\right| \cdot \left|\partial_j \Delta u_i\right| dx \\ &\quad =: J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \end{aligned} \tag{2.22}$$

Applying Λ^3 to (1.3), multiplying it by $\Lambda^3 d$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\Lambda^3 d|^2(x,t)dx+\int_{\mathbb{R}^3}|\Lambda^4 d|^2(x,t)dx\\
&\quad\le\left|\int_{\mathbb{R}^3}\left(\Lambda^3(u\cdot\nabla d)-u\cdot\nabla\Lambda^3 d\right)\cdot\Lambda^3 d\,dx\right|\\
&\qquad+\left|\int_{\mathbb{R}^3}\Lambda^3 f(d)\cdot\Lambda^3 d\,dx\right|-\sum_{i,j}\int_{\mathbb{R}^3}d_j\Delta^2 d_i\cdot\partial_j\Delta u_i dx\\
&\qquad-\sum_{i,j}\int_{\mathbb{R}^3}\partial_j u_i\Delta d_j\cdot\Delta^2 d_i dx-2\sum_{i,j}\int_{\mathbb{R}^3}\nabla d_j\cdot\nabla\partial_j u_i\cdot\Delta^2 d_i dx\\
&\quad=: J_7+J_8+J_9+J_{10}+J_{11}.
\end{aligned}\tag{2.23}$$

Summing up (2.22) and (2.23), using $J_4+J_9=0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left(|\Delta u|^2(x,t)+|\Lambda^3 d|^2(x,t)\right)dx+\int_{\mathbb{R}^3}\left(|\nabla\Delta u|^2(x,t)+|\Lambda^4 d|^2(x,t)\right)dx\\
&\quad\le J_1+J_2+J_3+J_5+J_6+J_7+J_8+J_{10}+J_{11}.
\end{aligned}\tag{2.24}$$

Now we estimate each term J_i as follows.

By using (2.20), we estimate J_1 as

$$\begin{aligned}
J_1&\le C\|\nabla u\|_{L^3}\|\Delta u\|_{L^3}^2\le C\|\nabla u\|_{L^2}^{3/4}\|\nabla\Delta u\|^{1/4}\cdot\|\nabla u\|_{L^2}^{1/2}\|\nabla\Delta u\|_{L^2}^{3/2}\\
&\le\epsilon\|\nabla\Delta u\|_{L^2}^2+C\|\nabla u\|_{L^2}^{10},\quad\text{for any }\epsilon>0;
\end{aligned}\tag{2.25}$$

here we used the following Gagliardo-Nirenberg inequalities:

$$\|\nabla u\|_{L^3}\le C\|\nabla u\|_{L^2}^{3/4}\|\nabla\Delta u\|_{L^2}^{1/4},\qquad\|\Delta u\|_{L^3}\le C\|\nabla u\|_{L^2}^{1/4}\|\nabla\Delta u\|_{L^2}^{3/4}.\tag{2.26}$$

Using (2.21), we estimate J_2 as

$$\begin{aligned}
J_2&\le C\|\nabla d\|_{L^\infty}\|\Lambda^4 d\|_{L^2}\|\Delta u\|_{L^2}\\
&\le C\|\Delta d\|_{L^2}^{3/4}\|\Lambda^4 d\|_{L^2}^{5/4}\cdot\|\nabla u\|_{L^2}^{1/2}\|\nabla\Delta u\|_{L^2}^{1/2}\\
&\le\epsilon\|\nabla\Delta u\|_{L^2}^2+\epsilon\|\Lambda^4 d\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|\Delta d\|_{L^2}^6,
\end{aligned}\tag{2.27}$$

for any $\epsilon>0$. Here we have used the following Gagliardo-Nirenberg inequalities:

$$\|\nabla d\|_{L^\infty}\le C\|\Delta d\|_{L^2}^{3/4}\|\Lambda^4 d\|_{L^2}^{1/4},\qquad\|\Delta u\|_{L^2}\le C\|\nabla u\|_{L^2}^{1/2}\|\nabla\Delta u\|_{L^2}^{1/2}.\tag{2.28}$$

J_3 only involves lower derivatives of d and is easy to handle, so we omit it here:

$$\begin{aligned} J_5 &\le C\|\Delta d\|_{L^4}^2 \|\nabla\Delta u\|_{L^2} \\ &\le C\|\Delta d\|_{L^2}^{5/4} \|\Lambda^4 d\|_{L^2}^{3/4} \|\nabla\Delta u\|_{L^2} \\ &\le \epsilon\|\nabla\Delta u\|_{L^2}^2 + \epsilon\|\Lambda^4 d\|_{L^2}^2 + C\|\Delta d\|_{L^2}^{10}, \end{aligned} \tag{2.29}$$

for any $\epsilon > 0$. Here we have used

$$\begin{aligned} \|\Delta d\|_{L^4} &\le C\|\Delta d\|_{L^2}^{5/8} \|\Lambda^4 d\|_{L^2}^{3/8}, \\ J_6 &\le C\|\nabla d\|_{L^6} \|\nabla\Delta d\|_{L^3} \|\nabla\Delta u\|_{L^2} \\ &\le C\|\Delta d\|_{L^2} \cdot \|\Delta d\|_{L^2}^{1/4} \|\Lambda^4 d\|_{L^2}^{3/4} \|\nabla\Delta u\|_{L^2} \\ &\le \epsilon\|\nabla\Delta u\|_{L^2}^2 + \epsilon\|\Lambda^4 d\|_{L^2}^2 + C\|\Delta d\|_{L^2}^{10}, \end{aligned} \tag{2.30}$$

for any $\epsilon > 0$. Where we have used the following inequality

$$\|\nabla\Delta d\|_{L^3} \le C\|\Delta d\|_{L^2}^{1/4} \|\Lambda^4 d\|_{L^2}^{3/4}. \tag{2.31}$$

By using (2.20), we estimate J_7 as follows:

$$\begin{aligned} J_7 &\le C\|\nabla u\|_{L^2} \|\Lambda^3 d\|_{L^4}^2 + C\|\Lambda^3 u\|_{L^2} \|\nabla d\|_{L^4} \|\Lambda^3 d\|_{L^4} \\ &\le C\|\nabla u\|_{L^2} \|\Delta d\|_{L^2}^{1/4} \|\Lambda^4 d\|_{L^2}^{7/4} + C\|\Lambda^3 u\|_{L^2} \|\nabla d\|_{L^4} \|\Delta d\|_{L^2}^{1/8} \|\Lambda^4 d\|_{L^2}^{7/8} \\ &\le \epsilon\|\Lambda^3 u\|_{L^2}^2 + \epsilon\|\Lambda^4 d\|_{L^2}^2 + C\|\Delta d\|_{L^2}^2 \|\nabla u\|_{L^2}^8 + C\|\Delta d\|_{L^2}^2 \|\nabla d\|_{L^4}^{16}, \end{aligned} \tag{2.32}$$

for any $\epsilon > 0$. Here we have used

$$\|\Lambda^3 d\|_{L^4} \le C\|\Delta d\|_{L^2}^{1/8} \|\Lambda^4 d\|_{L^2}^{7/8}. \tag{2.33}$$

The term J_8 is trivial, and we omit it here:

$$\begin{aligned} J_{10} &\le C\|\Delta d\|_{L^\infty} \|\nabla u\|_{L^2} \|\Lambda^4 d\|_{L^2} \\ &\le C\|\nabla u\|_{L^2} \cdot \|\Delta d\|_{L^2}^{1/4} \cdot \|\Lambda^4 d\|_{L^2}^{7/4} \\ &\le \epsilon\|\Lambda^4 d\|_{L^2}^2 + C\|\nabla u\|_{L^2}^8 \|\Delta d\|_{L^2}^2, \end{aligned} \tag{2.34}$$

for any $\epsilon > 0$. Where we have used the following inequality:

$$\|\Delta d\|_{L^\infty} \le C\|\Delta d\|_{L^2}^{1/4} \|\Lambda^4 d\|_{L^2}^{3/4}. \tag{2.35}$$

Finally, using (2.26), J_{11} can be bounded as follows:

$$\begin{aligned} J_{11} &\le C\|\nabla d\|_{L^6}\|\Delta u\|_{L^3}\|\Lambda^4 d\|_{L^2} \\ &\le C\|\Delta d\|_{L^2}\cdot\|\nabla u\|_{L^2}^{1/4}\cdot\|\Lambda^3 u\|_{L^2}^{3/4}\|\Lambda^4 d\|_{L^2} \\ &\le \epsilon\|\Lambda^3 u\|_{L^2}^2+\epsilon\|\Lambda^4 d\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\|\Delta d\|_{L^2}^8, \end{aligned} \tag{2.36}$$

for any $\epsilon > 0$. Now, inserting the above estimates for J_is into (2.24), using (2.19), and taking ϵ be small enough, we get

$$\begin{gathered} \|u\|_{L^\infty(0,T;H^2)}+\|u\|_{L^2(0,T;H^3)}\le C, \\ \|d\|_{L^\infty(0,T;H^3)}+\|d\|_{L^2(0,T;H^4)}\le C. \end{gathered} \tag{2.37}$$

This completes the proof.

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