Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 579509, 6 pages
doi:10.1155/2010/579509

Research Article

Multivariate Twisted p-Adic q-Integral on $\mathbb{Z}_p$ Associated with Twisted q-Bernoulli Polynomials and Numbers

Seog-Hoon Rim, Eun-Jung Moon, Sun-Jung Lee, and Jeong-Hee Jin

Department of Mathematics, Kyungpook National University, Daegu 702-701, Republic of Korea

Correspondence should be addressed to Seog-Hoon Rim, shrim@knu.ac.kr

Received 19 June 2010; Accepted 2 October 2010

Academic Editor: Ulrich Abel

Recently, many authors have studied twisted q-Bernoulli polynomials by using the p-adic invariant q-integral on $\mathbb{Z}_p$. In this paper, we define the twisted p-adic q-integral on $\mathbb{Z}_p$ and extend our result to the twisted q-Bernoulli polynomials and numbers. Finally, we derive some various identities related to the twisted q-Bernoulli polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < 1$.

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n} = C_{p^\infty}, \tag{1.1}$$

where $C_{p^n} = \{\zeta \in \mathbb{C}_p \mid \zeta^{p^n} = 1 \text{ for some } n \geq 0\}$ is the cyclic group of order p^n.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$.

For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \tag{1.2}$$

compare with [1–3].

It is well known that the twisted q-Bernoulli polynomials of order k are defined as

$$e^{xt}\left(\frac{t}{e^t\zeta q - 1}\right)^k = \sum_{n=0}^{\infty} \beta_{n,\zeta,q}^{(k)}(x) \frac{t^n}{n!}, \quad \zeta \in T_p, \tag{1.3}$$

and $\beta_{n,\zeta,q}^k = \beta_{n,\zeta,q}^k(0)$ are called the twisted q-Bernoulli numbers of order k. When $k = 1$, the polynomials and numbers are called the twisted q-Bernoulli polynomials and numbers, respectively. When $k = 1$ and $q = 1$, the polynomials and numbers are called the twisted Bernoulli polynomials and numbers, respectively. When $k = 1$, $q = 1$, and $\zeta = 1$, the polynomials and numbers are called the ordinary Bernoulli polynomials and numbers, respectively.

Many authors have studied the twisted q-Bernoulli polynomials by using the properties of the p-adic invariant q-integral on $\mathbb{Z}_p$ (cf. [4]). In this paper, we define the twisted p-adic q-integral on $\mathbb{Z}_p$ and extend our result to the twisted q-Bernoulli polynomials and numbers. Finally, we derive some various identities related to the twisted q-Bernoulli polynomials.

2. Multivariate Twisted p-Adic q-Integral on $\mathbb{Z}_p$ Associated with Twisted q-Bernoulli Polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < 1$. For $\zeta \in T_p$, we define the (q,ζ)-numbers as

$$[k]_{q,\zeta} = \frac{1 - q^k\zeta}{1 - q}, \quad \text{for } k \in \mathbb{Z}_p. \tag{2.1}$$

Note that $[k]_q = [k]_{q,1} = (1 - q^k)/(1 - q)$.

Let us define

$$\binom{n}{k}_{q,\zeta} = \frac{[n]_{q,\zeta}!}{[k]_{q,\zeta}![n-k]_{q,\zeta}!}, \tag{2.2}$$

where $[k]_{q,\zeta}! = [k]_{q,\zeta}[k-1]_{q,\zeta} \cdots [1]_{q,\zeta}$. Note that $\binom{n}{k} = \binom{n}{k}_{1,1} = n!/k!(n-k)!$.

Now we construct the twisted p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$
\begin{aligned}
I_{q,\zeta}(f) &= \int_{\mathbb{Z}_p} f(x)\,d\mu_{q,\zeta}(x) \\
&= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{q,\zeta}\left(x+p^N\mathbb{Z}_p\right) \\
&= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x \zeta^x,
\end{aligned}
\tag{2.3}
$$

where $\mu_{q,\zeta}(x+p^N\mathbb{Z}_p) = q^x\zeta^x/[p^N]_q$. From the definition of the twisted p-adic q-integral on $\mathbb{Z}_p$, we can consider the twisted q-Bernoulli polynomials and numbers of order k as follows:

$$
\begin{aligned}
\beta_{n,q,\zeta}^{(k)}(x) &= \int_{\mathbb{Z}_p^k} [x_1+x_2+\cdots+x_k+x]_q^n\, d\mu_{q,\zeta}(x_1)\,d\mu_{q,\zeta}(x_2)\cdots d\mu_{q,\zeta}(x_k) \\
&= \lim_{N\to\infty} \frac{1}{[p^N]_q^k} \sum_{x_1,\ldots,x_k=0}^{p^N-1} [x_1+x_2+\cdots+x_k+x]_q^n q^{x_1+x_2+\cdots+x_k} \zeta^{x_1+x_2+\cdots+x_k} \\
&= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \lim_{N\to\infty} \frac{1}{[p^N]_q^k} \sum_{x_1,\ldots,x_k=0}^{p^N-1} q^{(l+1)x_1+\cdots+(l+1)x_k} \zeta^{x_1+\cdots+x_k} \\
&= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{(l+1)^k}{[l+1]_{q,\zeta}^k}.
\end{aligned}
\tag{2.4}
$$

In the special case $x=0$ in (2.4), $\beta_{n,q,\zeta}^{(k)}(0) = \beta_{n,q,\zeta}^{(k)}$ are called the twisted q-Bernoulli numbers of order k.

If we take $k=1$ and $\zeta=1$ in (2.4), we can easily see that

$$
\beta_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+1}{[l+1]_q}.
\tag{2.5}
$$

compare with [4].

Theorem 2.1. *For $k \in \mathbb{Z}_+$ and $\zeta \in T_p$, we have*

$$
\beta_{n,q,\zeta}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{(l+1)^k}{[l+1]_{q,\zeta}^k}.
\tag{2.6}
$$

Moreover, if we take $x = 0$ in Theorem 2.1, then we have the following identity for the twisted q-Bernoull numbers

$$\beta_{n,q,\zeta}^{(k)} = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{(l+1)^k}{[l+1]_{q,\zeta}^k}. \tag{2.7}$$

From the definition of multivariate twisted p-adic q-integral, we also see that

$$\begin{aligned}\beta_{n,q,\zeta}^{(k)}(x) &= \int_{\mathbb{Z}_p^k}[x_1+x_2+\cdots+x_k+x]_q^n d\mu_{q,\zeta}(x_1)d\mu_{q,\zeta}(x_2)\cdots d\mu_{q,\zeta}(x_k)\\ &= \sum_{l=0}^{n}\binom{n}{l}q^{lx}[x]_q^{n-l}\int_{\mathbb{Z}_p^k}[x_1+x_2+\cdots+x_k]_q^l d\mu_{q,\zeta}(x_1)d\mu_{q,\zeta}(x_2)\cdots d\mu_{q,\zeta}(x_k)\\ &= \sum_{l=0}^{n}\binom{n}{l}q^{lx}[x]_q^{n-l}\beta_{l,q,\zeta}^{(k)}.\end{aligned} \tag{2.8}$$

Corollary 2.2. *For $k \in \mathbb{Z}_+$ and $\zeta \in T_p$, one obtains*

$$\beta_{n,q,\zeta}^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}q^{lx}[x]_q^{n-l}\beta_{l,q,\zeta}^{(k)}. \tag{2.9}$$

Note that

$$q^{n(x_1+\cdots+x_k)} = \sum_{l=0}^{n}\binom{n}{l}(q-1)^l[x_1+\cdots+x_k]_q^l. \tag{2.10}$$

We have

$$\int_{\mathbb{Z}_p^k}q^{n(x_1+\cdots+x_k)}d\mu_{q,\zeta}(x_1)d\mu_{q,\zeta}(x_2)\cdots d\mu_{q,\zeta}(x_k) = \sum_{l=0}^{n}\binom{n}{l}(q-1)^l\beta_{l,q,\zeta}^{(k)}. \tag{2.11}$$

It is easy to see that

$$\begin{aligned}&\int_{\mathbb{Z}_p^k}q^{n(x_1+\cdots+x_k)}d\mu_{q,\zeta}(x_1)d\mu_{q,\zeta}(x_2)\cdots d\mu_{q,\zeta}(x_k)\\ &\quad= \lim_{N\to\infty}\frac{1}{[p^N]_q^k}\sum_{x_1,\ldots,x_k=0}^{p^N-1}q^{n(x_1+\cdots+x_k)}q^{x_1+\cdots+x_k}\zeta^{x_1+\cdots+x_k} = \frac{(n+1)^k}{[n+1]_{q,\zeta}^k}.\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$ and $\zeta \in T_p$, one has*

$$\sum_{l=0}^{n} \binom{n}{l} (q-1)^l \beta_{l,q,\zeta}^{(k)} = \frac{(n+1)^k}{[n+1]_{q,\zeta}^k}. \tag{2.13}$$

Now we consider the modified extension of the twisted q-Bernoulli polynomials of order k as follows:

$$B_{n,q,\zeta}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} q^{ix} \int_{\mathbb{Z}_p^k} q^{\sum_{l=1}^{k}(k-l+i)x_i} d\mu_{q,\zeta}(x_1) \cdots d\mu_{q,\zeta}(x_k). \tag{2.14}$$

In the special case $x = 0$, we write $B_{n,q,\zeta}^{(k)} = B_{n,q,\zeta}^{(k)}(0)$, which are called the modified extension of the twisted q-Bernoulli numbers of order k.

From (2.14), we derive that

$$\begin{aligned} B_{n,q,\zeta}^{(k)}(x) &= \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(i+k)\cdots(i+1)}{[i+k]_{q,\zeta}\cdots[i+1]_{q,\zeta}} q^{ix} \\ &= \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\binom{i+k}{k} k!}{\binom{i+k}{k}_{q,\zeta} [k]_{q,\zeta}!} q^{ix}. \end{aligned} \tag{2.15}$$

Therefore, we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$ and $\zeta \in T_p$, one has*

$$B_{n,q,\zeta}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\binom{i+k}{k} k!}{\binom{i+k}{k}_{q,\zeta} [k]_{q,\zeta}!} q^{ix}. \tag{2.16}$$

Now, we define $B_{n,q,\zeta}^{(-k)}(x)$ as follows:

$$B_{n,q,\zeta}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \frac{(-1)^i \binom{n}{i} q^{ix}}{\int_{\mathbb{Z}_p^k} q^{\sum_{l=1}^{k}(k-l+i)x_i} d\mu_{q,\zeta}(x_1) \cdots d\mu_{q,\zeta}(x_k)}. \tag{2.17}$$

By (2.17), we can see that

$$B_{n,q,\zeta}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\binom{i+k}{k}_{q,\zeta} [k]_{q,\zeta}!}{\binom{i+k}{k} k!} q^{ix}. \tag{2.18}$$

Therefore, we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$ and $\zeta \in T_p$, one has*

$$B_{n,q,\zeta}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{i+k}{k}_{q,\zeta} \frac{\binom{n+k}{n-i} [k]_{q,\zeta}!}{\binom{n+k}{k} k!} q^{ix}. \tag{2.19}$$

In (2.19), we can see the relations between the binomial coefficients and the modified extension of the twisted q-Bernoulli polynomials of order k.

Acknowledgments

The authors would like to thank the anonymous referee for his/her excellent detail comments and suggestions. This Research was supported by Kyungpook National University Research Fund, 2010.

References

[1] L.-C. Jang, "Multiple twisted q-Euler numbers and polynomials associated with p-adic q-integrals," *Advances in Difference Equations*, Article ID 738603, 11 pages, 2008.

[2] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[3] T. Kim, "q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients," *Russian Journal of Mathematical Physics*, vol. 15, no. 1, pp. 51–57, 2008.

[4] T. Kim, "Sums of products of q-Bernoulli numbers," *Archiv der Mathematik*, vol. 76, no. 3, pp. 190–195, 2001.