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Research Article

Approximately Quadratic Mappings on Restricted Domains

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We introduce a generalized quadratic functional equation $f(rx+sy) = rf(x)+sf(y)-rsf(x-y)$, where r, s are nonzero real numbers with $r+s=1$. We show that this functional equation is quadratic if r, s are rational numbers. We also investigate its stability problem on restricted domains. These results are applied to study of an asymptotic behavior of these generalized quadratic mappings.

1. Introduction

Under what conditions does there exist a group homomorphism near an approximate group homomorphism? This question concerning the stability of group homomorphisms was posed by Ulam [1]. The case of approximately additive mappings was solved by Hyers [2] on Banach spaces. In 1950 Aoki [3] provided a generalization of the Hyers' theorem for additive mappings and in 1978 Th. M. Rassias [4] generalized the Hyers' theorem for linear mappings by allowing the Cauchy difference to be unbounded (see also [5]). The result of Rassias' theorem has been generalized by Găvruţa [6] who permitted the Cauchy difference to be bounded by a general control function. This stability concept is also applied to the case of other functional equations. For more results on the stability of functional equations, see [7–24]. We also refer the readers to the books in [25–29].

It is easy to see that the quadratic function $f(x) = x^2$ is a solution of each of the following functional equations:

$$f(x+y)+f(x-y) = 2f(x)+2f(y), \tag{1.1}$$

$$f(rx+sy)+rsf(x-y) = rf(x)+sf(y), \tag{1.2}$$

where r, s are nonzero real numbers with $r+s=1$. So, it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function $f : X \to Y$ between real vector spaces X and Y is quadratic if and only if there exists a unique symmetric biadditive function $B : X \times X \to Y$ such that $f(x) = B(x, x)$ for all $x \in X$ (see [13, 25, 27]).

We prove that the functional equations (1.1) and (1.2) are equivalent if r, s are nonzero rational numbers. The functional equation (1.1) is a spacial case of (1.2). Indeed, for the case $r = s = 1/2$ in (1.2), we get (1.1).

In 1983 Skof [30] was the first author to solve the Hyers-Ulam problem for additive mappings on a restricted domain (see also [31–33]). In 1998 Jung [34] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains (see also [35–37]). J. M. Rassias [38] investigated the Hyers-Ulam stability of mixed type mappings on restricted domains.

2. Solutions of (1.2)

In this section we show that the functional equation (1.2) is equivalent to the quadratic equation (1.1). That is, every solution of (1.2) is a quadratic function. We recall that r, s are nonzero real numbers with $r + s = 1$.

Theorem 2.1. *Let X and Y be real vector spaces and $f : X \to Y$ be an odd function satisfying* (1.2). *If r is a rational number, then $f \equiv 0$.*

Proof. Since f is odd, $f(0) = 0$. Letting $x = 0$ (resp., $y = 0$) in (1.2), we get

$$f(sy) = s(1+r)f(y), \qquad f(rx) = r^2 f(x) \tag{2.1}$$

for all $x, y \in X$. Replacing y by $-y$ in (1.2) and adding the obtained functional equation to (1.2), we get

$$f(rx+sy) + f(rx-sy) = 2rf(x) - rs[f(x+y) + f(x-y)] \tag{2.2}$$

for all $x, y \in X$. Replacing y by ry in (2.2) and using (2.1), we have

$$r[f(x+sy) + f(x-sy)] = 2f(x) - s[f(x+ry) + f(x-ry)] \tag{2.3}$$

for all $x, y \in X$. Again if we replace x by sx in (2.3) and use (2.1), we get

$$r(1+r)[f(x+y) + f(x-y)] = 2(1+r)f(x) - [f(sx+ry) + f(sx-ry)] \tag{2.4}$$

for all $x, y \in X$. Applying (1.2) and using the oddness of f, we have

$$f(sx+ry) + f(sx-ry) = 2sf(x) + rs[f(x+y) + f(x-y)] \tag{2.5}$$

for all x, y in X. So it follows from (2.4) and (2.5) that

$$f(x+y) + f(x-y) = 2f(x) \tag{2.6}$$

for all x, y in X. It easily follows from (2.6) that f is additive, that is, $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. So if r is a rational number, then $f(rx) = rf(x)$ for all x in X. Therefore, it follows from (2.1) that $(r^2 - r)f(x) = 0$ for all x in X. Since r, s are nonzero, we infer that $f \equiv 0$. □

Theorem 2.2. *Let X and Y be real vector spaces and $f : X \to Y$ be an even function satisfying* (1.2). *Then f satisfies* (1.1).

Proof. Letting $x = y = 0$ in (1.2), we get $f(0) = 0$. Replacing x by $x + y$ in (1.2), we get

$$f(rx+y) = rf(x+y) + sf(y) - rsf(x) \tag{2.7}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.7) and using the evenness of f, we get

$$f(rx-y) = rf(x-y) + sf(y) - rsf(x) \tag{2.8}$$

for all x, y in X. Adding (2.7) to (2.8), we obtain

$$f(rx+y) + f(rx-y) = r\left[f(x+y) + f(x-y)\right] + 2sf(y) - 2rsf(x) \tag{2.9}$$

for all $x, y \in X$. Replacing y by $x + ry$ in (2.7), we get

$$f(r(x+y)+x) = rf(2x+ry) + sf(x+ry) - rsf(x) \tag{2.10}$$

for all x, y in X. Using (2.7) in (2.10), by a simple computation, we get

$$f(2x+y) + 2f(x) + f(y) = 2f(x+y) + f(2x) \tag{2.11}$$

for all x, y in X. Putting $y = -x$ in (2.11), we get that $f(2x) = 4f(x)$ for all $x \in X$. Therefore, it follows from (2.11) that

$$f(2x+y) + f(y) = 2f(x+y) + 2f(x) \tag{2.12}$$

for all x, y in X. Replacing y by $y - x$ in (2.12), we get that $f(x+y) + f(y-x) = 2f(x) + 2f(x)$ for all $x, y \in X$. So f satisfies (1.1). □

Theorem 2.3. *Let $f : X \to Y$ be a function between real vector spaces X and Y. If r is a rational number, then f satisfies* (1.2) *if and only if f satisfies* (1.1).

Proof. Let f_o and f_e be the odd and the even parts of f. Suppose that f satisfies (1.2). It is clear that f_o and f_e satisfy (1.2). By Theorems 2.1 and 2.2, $f_o \equiv 0$ and f_e satisfies (1.1). Since $f = f_o + f_e$, we conclude that f satisfies (1.1).

Conversely, let f satisfy (1.1). Then there exists a unique symmetric biadditive function $B : X \times X \to Y$ such that $f(x) = B(x,x)$ for all $x \in X$ (see [13]). Therefore

$$\begin{aligned} rf(x) + sf(y) - rsf(x-y) &= rB(x,x) + sB(y,y) - rsB(x-y,x-y) \\ &= r^2B(x,x) + s^2B(y,y) + 2rsB(x,y) \quad (r,s \text{ are rational numbers}) \\ &= B(rx+sy, rx+sy) = f(rx+sy) \end{aligned} \tag{2.13}$$

for all $x, y \in X$. So f satisfies (1.2). □

Proposition 2.4. *Let $\mathcal{X}$ be a linear space with the norm $\|\cdot\|$. $\mathcal{X}$ is an inner product space if and only if there exists a real number $0 < r < 1$ such that*

$$\|rx+sy\|^2 + rs\|x-y\|^2 = r\|x\|^2 + s\|y\|^2 \tag{2.14}$$

for all $x, y \in \mathcal{X}$, where $s = 1 - r$.

Proof. Let $f : \mathcal{X} \to \mathbb{R}$ be a function defined by $f(x) = \|x\|^2$. If $\mathcal{X}$ is an inner product space, then f satisfies (2.14) for all $r \in \mathbb{R}$. Conversely, let $r \in (0,1)$ and the (even) function f satisfy (2.14). So f satisfies (1.2). By Theorem 2.3, the function f satisfies (1.1), that is,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{2.15}$$

for all $x, y \in \mathcal{X}$. Therefore $\mathcal{X}$ is an inner product space (see [14]). □

Proposition 2.5. *Let $p, q, u, v \in \mathbb{R} \setminus \{0\}$ and $\mathcal{X}$ be a linear space with the norm $\|\cdot\|$. Suppose that*

$$\|rx+sy\|^p + rs\|x-y\|^q = r\|x\|^u + s\|y\|^v \tag{2.16}$$

for all x, y in $\mathcal{X}$, where $0 < r < 1$ and $s = 1 - r$. Then $p = q = u = v = 2$.

Proof. Setting $y = 0$ in (2.16), we get

$$|r|^p\|x\|^p + rs\|x\|^q = r\|x\|^u \tag{2.17}$$

for all x in $\mathcal{X}$. If we take $x \in \mathcal{X}$ with $\|x\| = 1$ in (2.17), we get that $p = 2$. Letting $y = x$ in (2.16), we get

$$\|x\|^2 = r\|x\|^u + s\|x\|^v \tag{2.18}$$

for all x in $\mathcal{X}$. Letting $x = 0$ in (2.16), we get

$$r\|y\|^q = \|y\|^v - s\|y\|^2 \tag{2.19}$$

for all y in $\mathcal{X}$. Since $p = 2$, it follows from (2.17) and (2.19) that

$$r\|x\|^u - s\|x\|^v = (r-s)\|x\|^2 \tag{2.20}$$

for all $x \in \mathcal{X}$. Using (2.18) and (2.20), we get $\|x\|^u = \|x\|^v$ for all $x \in \mathcal{X}$. Hence $u = v$ and (2.18) implies that $u = v = 2$. Finally, $q = 2$ follows from (2.19). □

Corollary 2.6. *Let $\mathcal{X}$ be a linear space with the norm $\|\cdot\|$. $\mathcal{X}$ is an inner product space if and only if there exists a real number $0 < r < 1$ and $p, q, u, v \in \mathbb{R} \setminus \{0\}$ such that*

$$\|rx + sy\|^p + rs\|x - y\|^q = r\|x\|^u + s\|y\|^v \tag{2.21}$$

for all $x, y \in \mathcal{X}$, where $s = 1 - r$.

3. Stability of (1.2) on Restricted Domains

In this section, we investigate the Hyers-Ulam stability of the functional equation (1.2) on a restricted domain. As an application we use the result to the study of an asymptotic behavior of that equation. It should be mentioned that Skof [39] was the first author who treats the Hyers-Ulam stability of the quadratic equation. Czerwik [8] proved a Hyers-Ulam-Rassias stability theorem on the quadratic equation. As a particular case he proved the following theorem.

Theorem 3.1. *Let $\delta \geq 0$ be fixed. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta \tag{3.1}$$

for all $x, y \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ such that $\|f(x) - Q(x)\| \leq \delta/2$ for all $x \in X$. Moreover, if f is measurable or if $f(tx)$ is continuous in t for each fixed $x \in X$, then $Q(tx) = t^2 Q(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

We recall that r, s are nonzero real numbers with $r + s = 1$.

Theorem 3.2. *Let $d > 0$ and $\delta \geq 0$ be given. Assume that an even mapping $f : X \to Y$ satisfies the inequality*

$$\|f(rx + sy) + rsf(x - y) - rf(x) - sf(y)\| \leq \delta \tag{3.2}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists $K > 0$ such that f satisfies

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \frac{4(2 + |r| + |s|)}{|rs|}\delta \tag{3.3}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq K$.

Proof. Let $x, y \in X$ with $\|x\| + \|y\| \geq 2d$. Then, since $\|x+y\| + \|y\| \geq \max\{\|x\|, 2\|y\| - \|x\|\}$, we get $\|x+y\| + \|y\| \geq d$. So it follows from (3.2) that

$$\|f(rx+y) + rsf(x) - rf(x+y) - sf(y)\| \leq \delta \tag{3.4}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq 2d$. So

$$\|f(ry+x) + rsf(y) - rf(x+y) - sf(x)\| \leq \delta \tag{3.5}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq 2d$.

Let $x, y \in X$ with $\|x\| + \|y\| \geq 4d(1/|r| + |1 - 1/|r||)$. We have two cases.

Case 1. $\|y\| > 2d/|r|$. Then $\|x\| + \|x + ry\| \geq |r|\|y\| \geq 2d$.

Case 2. $\|y\| \leq 2d/|r|$. Then we have $\|x\| \geq 2d(1/|r| + 2|1 - 1/|r||)$. So

$$\|x\| + \left\|x + ry\right\| \geq 2\|x\| - |r|\left\|y\right\| \geq 2d\left(\frac{2}{|r|} + 4\left|1 - \frac{1}{|r|}\right| - 1\right) \geq 2d. \tag{3.6}$$

Therefore we get that $\|x\| + \|x + ry\| \geq 2d$ from Cases 1 and 2. Hence by (3.4) we have

$$\|f(r(x+y)+x) + rsf(x) - rf(2x+ry) - sf(x+ry)\| \leq \delta \tag{3.7}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq 4d(1/|r| + |1 - 1/|r||)$. Set $M := 4d(1/|r| + |1 - 1/|r||)$. Then

$$\left\|x+y\right\| + \|x\| \geq \frac{M}{2} \geq 2d, \qquad \|2x\| + \left\|y\right\| \geq M \geq 4d \tag{3.8}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq M$. From (3.4) and (3.5), we get the following inequalities:

$$\begin{gathered}\|f(r(x+y)+x) + rsf(x+y) - rf(2x+y) - sf(x)\| \leq \delta, \\ \left\|rf(ry+2x) + r^2sf(y) - r^2f(2x+y) - rsf(2x)\right\| \leq \delta|r|, \\ \left\|sf(ry+x) + rs^2f(y) - rsf(x+y) - s^2f(x)\right\| \leq \delta|s|.\end{gathered} \tag{3.9}$$

Using (3.7) and the above inequalities, we get

$$\|f(2x+y) + 2f(x) + f(y) - 2f(x+y) - f(2x)\| \leq \frac{2+|r|+|s|}{|rs|}\delta \tag{3.10}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq M$. If $x, y \in X$ with $\|x\| + \|y\| \geq 2M$, then $\|x\| + \|y - x\| \geq M$. So it follows from (3.10) that

$$\|f(x+y) + 2f(x) + f(y-x) - 2f(y) - f(2x)\| \leq \frac{2+|r|+|s|}{|rs|}\delta. \tag{3.11}$$

Letting $y = 0$ in (3.11), we get

$$\|4f(x) - f(2x) - 2f(0)\| \leq \frac{2 + |r| + |s|}{|rs|}\delta \tag{3.12}$$

for all $x, y \in X$ with $\|x\| \geq 2M$. Letting $x = 0$ (and $y \in X$ with $\|y\| \geq 2M$) in (3.11), we get $\|f(0)\| \leq ((2 + |r| + |s|)/|rs|)\delta$. Therefore it follows from (3.11) and (3.12) that

$$\begin{aligned} &\|f(x+y) + f(y-x) - 2f(x) - 2f(y)\| \\ &\quad\leq \|f(x+y) + 2f(x) + f(y-x) - 2f(y) - f(2x)\| \\ &\qquad + \|4f(x) - f(2x) - 2f(0)\| + 2\|f(0)\| \\ &\quad\leq \frac{4(2 + |r| + |s|)}{|rs|}\delta \end{aligned} \tag{3.13}$$

for all $x, y \in X$ with $\|x\| \geq 2M$. Since f is even, the inequality (3.13) holds for all $x, y \in X$ with $\|y\| \geq 2M$. Therefore

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \frac{4(2 + |r| + |s|)}{|rs|}\delta \tag{3.14}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq 4M$. This completes the proof by letting $K := 4M$. □

Theorem 3.3. *Let $d > 0$ and $\delta \geq 0$ be given. Assume that an even mapping $f : X \to Y$ satisfies the inequality* (3.2) *for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then f satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \frac{19(2 + |r| + |s|)}{|rs|}\delta \tag{3.15}$$

for all $x, y \in X$.

Proof. By Theorem 3.2 there exists $K > 0$ such that f satisfies (3.3) for all $x, y \in X$ with $\|x\| + \|y\| \geq K$ and $\|f(0)\| \leq ((2 + |r| + |s|)/|rs|)\delta$ (see the proof of Theorem 3.2). Using Theorem 2 of [38], we get that

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| &\leq \frac{18(2 + |r| + |s|)}{|rs|}\delta + \|f(0)\| \\ &\leq \frac{19(2 + |r| + |s|)}{|rs|}\delta \end{aligned} \tag{3.16}$$

all $x, y \in X$. □

Theorem 3.4. *Let $d > 0$ and $\delta \geq 0$ be given. Assume that an even mapping $f : X \to Y$ satisfies the inequality* (3.2) *for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that $Q(x) = \lim_{n\to\infty} 4^{-n} f(2^n x)$ and*

$$\|f(x) - Q(x)\| \leq \frac{19(2 + |r| + |s|)}{2|rs|}\delta \tag{3.17}$$

for all $x \in X$.

Proof. The result follows from Theorems 3.1 and 3.3. □

Skof [39] has proved an asymptotic property of the additive mappings and Jung [34] has proved an asymptotic property of the quadratic mappings (see also [36]). We prove such a property also for the quadratic mappings.

Corollary 3.5. *An even mapping $f : X \to Y$ satisfies* (1.2) *if and only if the asymptotic condition*

$$\|f(rx + sy) + rsf(x - y) - rf(x) - sf(y)\| \longrightarrow 0, \quad \text{as } \|x\| + \|y\| \longrightarrow \infty \tag{3.18}$$

holds true.

Proof. By the asymptotic condition (3.18), there exists a sequence $\{\delta_n\}$ monotonically decreasing to 0 such that

$$\|f(rx + sy) + rsf(x - y) - rf(x) - sf(y)\| \leq \delta_n \tag{3.19}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq n$. Hence, it follows from (3.19) and Theorem 3.4 that there exists a unique quadratic mapping $Q_n : X \to Y$ such that

$$\|f(x) - Q_n(x)\| \leq \frac{19(2 + |r| + |s|)}{2|rs|}\delta_n \tag{3.20}$$

for all $x \in X$. Since $\{\delta_n\}$ is a monotonically decreasing sequence, the quadratic mapping Q_m satisfies (3.20) for all $m \geq n$. The uniqueness of Q_n implies $Q_m = Q_n$ for all $m \geq n$. Hence, by letting $n \to \infty$ in (3.20), we conclude that f is quadratic. □

Corollary 3.6. *Let r be rational. An even mapping $f : X \to Y$ is quadratic if and only if the asymptotic condition* (3.18) *holds true.*

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References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[5] D. G. Bourgin, "Classes of transformations and bordering transformations," *Bulletin of the American Mathematical Society*, vol. 57, pp. 223–237, 1951.

[6] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[7] P. W. Cholewa, "Remarks on the stability of functional equations," *Aequationes Mathematicae*, vol. 27, no. 1-2, pp. 76–86, 1984.

[8] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.

[9] V. A. Faĭziev, Th. M. Rassias, and P. K. Sahoo, "The space of (ψ, γ)-additive mappings on semigroups," *Transactions of the American Mathematical Society*, vol. 354, no. 11, pp. 4455–4472, 2002.

[10] G. L. Forti, "An existence and stability theorem for a class of functional equations," *Stochastica*, vol. 4, no. 1, pp. 23–30, 1980.

[11] G. L. Forti, "Hyers-Ulam stability of functional equations in several variables," *Aequationes Mathematicae*, vol. 50, no. 1-2, pp. 143–190, 1995.

[12] A. Grabiec, "The generalized Hyers-Ulam stability of a class of functional equations," *Publicationes Mathematicae Debrecen*, vol. 48, no. 3-4, pp. 217–235, 1996.

[13] P. Kannappan, "Quadratic functional equation and inner product spaces," *Results in Mathematics*, vol. 27, no. 3-4, pp. 368–372, 1995.

[14] P. Jordan and J. von Neumann, "On inner products in linear, metric spaces," *Annals of Mathematics*, vol. 36, no. 3, pp. 719–723, 1935.

[15] D. H. Hyers and Th. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.

[16] G. Isac and Th. M. Rassias, "Stability of ψ-additive mappings: applications to nonlinear analysis," *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 2, pp. 219–228, 1996.

[17] K.-W. Jun and Y.-H. Lee, "On the Hyers-Ulam-Rassias stability of a Pexiderized quadratic inequality," *Mathematical Inequalities & Applications*, vol. 4, no. 1, pp. 93–118, 2001.

[18] A. Najati, "Hyers-Ulam stability of an n-Apollonius type quadratic mapping," *Bulletin of the Belgian Mathematical Society. Simon Stevin*, vol. 14, no. 4, pp. 755–774, 2007.

[19] A. Najati and C. Park, "Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 763–778, 2007.

[20] A. Najati and C. Park, "The Pexiderized Apollonius-Jensen type additive mapping and isomorphisms between C^*-algebras," *Journal of Difference Equations and Applications*, vol. 14, no. 5, pp. 459–479, 2008.

[21] C.-G. Park, "On the stability of the linear mapping in Banach modules," *Journal of Mathematical Analysis and Applications*, vol. 275, no. 2, pp. 711–720, 2002.

[22] Th. M. Rassias, "On a modified Hyers-Ulam sequence," *Journal of Mathematical Analysis and Applications*, vol. 158, no. 1, pp. 106–113, 1991.

[23] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.

[24] Th. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.

[25] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1989.

[26] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.

[27] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 1998.

[28] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[29] Th. M. Rassias, Ed., *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[30] F. Skof, "Sull' approssimazione delle applicazioni localmente δ-additive," *Atti della Accademia delle Scienze di Torino*, vol. 117, pp. 377–389, 1983.

[31] D. H. Hyers, G. Isac, and Th. M. Rassias, "On the asymptoticity aspect of Hyers-Ulam stability of mappings," *Proceedings of the American Mathematical Society*, vol. 126, no. 2, pp. 425–430, 1998.

[32] S.-M. Jung, "Hyers-Ulam-Rassias stability of Jensen's equation and its application," *Proceedings of the American Mathematical Society*, vol. 126, no. 11, pp. 3137–3143, 1998.

[33] S.-M. Jung, M. S. Moslehian, and P. K. Sahoo, "Stability of a generalized Jensen equation on restricted domains," *Journal of Mathematical Inequalities*, vol. 4, pp. 191–206, 2010.

[34] S.-M. Jung, "On the Hyers-Ulam stability of the functional equations that have the quadratic property," *Journal of Mathematical Analysis and Applications*, vol. 222, no. 1, pp. 126–137, 1998.

[35] S.-M. Jung, "Stability of the quadratic equation of Pexider type," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 70, pp. 175–190, 2000.

[36] S.-M. Jung and B. Kim, "On the stability of the quadratic functional equation on bounded domains," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 69, pp. 293–308, 1999.

[37] S.-M. Jung and P. K. Sahoo, "Hyers-Ulam stability of the quadratic equation of Pexider type," *Journal of the Korean Mathematical Society*, vol. 38, no. 3, pp. 645–656, 2001.

[38] J. M. Rassias, "On the Ulam stability of mixed type mappings on restricted domains," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 2, pp. 747–762, 2002.

[39] F. Skof, "Proprieta' locali e approssimazione di operatori," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, no. 1, pp. 113–129, 1983.