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Research Article

Nonlinear Boundary Value Problem of First-Order Impulsive Functional Differential Equations

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This paper investigates the nonlinear boundary value problem for a class of first-order impulsive functional differential equations. By establishing a comparison result and utilizing the method of upper and lower solutions, some criteria on the existence of extremal solutions as well as the unique solution are obtained. Examples are discussed to illustrate the validity of the obtained results.

1. Introduction

It is now realized that the theory of impulsive differential equations provides a general framework for mathematical modelling of many real world phenomena. In particular, it serves as an adequate mathematical tool for studying evolution processes that are subjected to abrupt changes in their states. Some typical physical systems that exhibit impulsive behaviour include the action of a pendulum clock, mechanical systems subject to impacts, the maintenance of a species through periodic stocking or harvesting, the thrust impulse maneuver of a spacecraft, and the function of the heart. For an introduction to the theory of impulsive differential equations, refer to [1].

It is also known that the method of upper and lower solutions coupled with the monotone iterative technique is a powerful tool for obtaining existence results of nonlinear differential equations [2]. There are numerous papers devoted to the applications of this method to nonlinear differential equations in the literature, see [3–9] and references therein. The existence of extremal solutions of impulsive differential equations is considered in papers [3–11]. However, only a few papers have implemented the technique in nonlinear boundary value problem of impulsive differential equations [5, 12]. In this paper, we will investigate nonlinear boundary value problem of a class of first-order impulsive functional differential equations. Such equations include the retarded impulsive differential equations as special cases [5, 12–14].

The rest of this paper is organized as follows. In Section 2, we establish a new comparison principle and discuss the existence and uniqueness of the solution for first order impulsive functional differential equations with linear boundary condition. We then obtain existence results for extremal solutions and unique solution in Section 3 by using the method of upper and lower solutions coupled with monotone iterative technique. To illustrate the obtained results, two examples are discussed in Section 4.

2. Preliminaries

Let $J = [0,T]$, $T > 0$, $J' = J - \{t_1, t_2, \ldots, t_p\}$ with $0 < t_1 < t_2 < \cdots < t_p < T$. We define that $PC(J) = \{x : J \to \mathbb{R} : x$ is continuous for any $t \in J'$; $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)\}$, $PC^1(J) = \{x : J \to \mathbb{R} : x$ is continuously differentiable for any $t \in J'$; $x(t_k^+)$, $x(t_k^-)$ exist and $x'(t_k^-) = x'(t_k)\}$. It is clear that $PC(J)$ and $PC^1(J)$ are Banach spaces with respective norms

$$\|x\|_{PC} = \sup_{t \in J} |x(t)|, \qquad \|x\|_{PC^1} = \|x\|_{PC} + \left\|x'\right\|_{PC}. \tag{2.1}$$

Let us consider the following nonlinear boundary value problem (NBVP):

$$\begin{gathered} x'(t) = f\left(t, x(t), \left[\varphi x\right](t)\right), \quad t \in J' = J - \left\{t_1, t_2, \ldots, t_p\right\}, \\ \Delta x(t) = I_k\left(t, x(t), \left[\varphi x\right](t)\right), \quad t = t_k, \ k = 1, 2, \ldots, p, \\ g(x(0), x(T)) = 0, \end{gathered} \tag{2.2}$$

where $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is continuous in the second and the third variables, and for fixed $x, y \in \mathbb{R}$, $f(\cdot, x, y) \in PC(J)$, $g \in C(\mathbb{R}^2, \mathbb{R})$, $I_k \in C(\mathbb{R}^3, \mathbb{R})$, $k = 1, 2, \ldots, p$ and $\varphi : PC(J) \to PC(J)$ is continuous.

A function $x \in PC^1(J)$ is called a solutions of NBVP (2.2) if it satisfies (2.2).

Remark 2.1. (i) If $[\varphi x](t) = x(t)$ and the impulses I_k depend only on $x(t_k)$, the equation of NBVP (2.2) reduces to the simpler case of impulsive differential equations:

$$\begin{gathered} x'(t) = f(t, x(t)), \quad t \in J', \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, p \end{gathered} \tag{2.3}$$

which have been studied in many papers. In some situation, the impulse I_k depends also on some other parameters (e.g., the control of the amount of drug ingested by a patient at certain moments in the model for drug distribution [1, 3]).

(ii) If $[\varphi x](t) = x(\theta(t))$, where $\theta \in C(J, J)$, the equation of NBVP (2.2) can be regarded as retarded differential equation which has been considered in [5, 12–14].

We will need the following lemma.

Lemma 2.2 (see [1]). *Asumme that*

(B_0) *the sequence* $\{t_k\}$ *satisfies* $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ *with* $\lim_{k\to\infty} t_k = +\infty$,

(B_1) $m \in PC^1(\mathbb{R}^+)$ *is left continous at* t_k *for* $k = 1, 2, \ldots$,

(B_2) *for* $k = 1, 2, \ldots$, $t \ge t_0$,

$$\begin{gathered} m'(t) \le p(t)m(t) + q(t), \quad t \ne t_k \\ m(t_k^+) \le d_k m(t_k) + b_k, \end{gathered} \tag{2.4}$$

where $p, q \in C(\mathbb{R}^+, \mathbb{R})$, $d_k \ge 0$ *and* b_k *are real constants.*

Then

$$\begin{aligned} m(t) \le{}& m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)ds\right) \\ &+ \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{s}^{t} p(\sigma)d\sigma\right) q(s)ds \\ &+ \sum_{t_0<t_k<t} \prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)ds\right) b_k. \end{aligned} \tag{2.5}$$

In order to establish a comparison result and some lemmas, we will make the following assumptions on the function φ.

(H1) There exists a constant $R > 0$ such that

$$[\varphi x](t) \ge R \inf_{t\in J} x(t), \quad \text{for any } x \in PC(J),\ \forall t \in J. \tag{2.6}$$

(H2) The function φ satisfies Lipschitz condition, that is, there exists a $L > 0$ such that

$$\|\varphi x - \varphi y\|_{PC} \le L\|x - y\|_{PC}, \quad \forall x, y \in PC(J). \tag{2.7}$$

Inspired by the ideas in [5, 6], we shall establish the following comparison result.

Theorem 2.3. *Let* $m \in PC^1(J)$ *such that*

$$\begin{gathered} m'(t) \le -Mm(t) - N[\varphi m](t), \quad t \in J', \\ \Delta m(t_k) \le -L_k m(t_k), \quad k = 1, 2, \ldots, p, \\ m(0) \le \mu m(T), \end{gathered} \tag{2.8}$$

where $M > 0$, $N \ge 0$, $0 \le L_k < 1$, $k = 1, 2, \ldots, p$, *and* $0 < \mu e^{-MT} \le 1$.

Suppose in addition that condition (H1) holds and

$$\frac{NR(e^{MT}+\mu)}{\mu}\int_0^T \prod_{s<t_k<T}(1-L_k)e^{Ms}ds \le \sum_{k=1}^{p}(1-L_k)^2 \tag{2.9}$$

then $m(t)\le 0$, *for* $t\in J$.

Proof. For simplicity, we let $c_k = 1-L_k$, $k=1,2,\ldots,p$. Set $v(t)=m(t)e^{Mt}$, then we have

$$\begin{aligned} v'(t) &\le -Ne^{Mt}[\varphi m](t), \quad t\in J', \\ v(t_k^+) &\le c_k v(t_k), \quad k=1,2,\ldots,p, \\ v(0) &\le \mu e^{-MT}v(T). \end{aligned} \tag{2.10}$$

Obviously, $v(t)\le 0$ implies $m(t)\le 0$.

To show $v(t)\le 0$, we suppose, on the contrary, that $v(t)>0$ for some $t\in J$. It is enough to consider the following cases.

(i) there exists a $\bar{t}\in J$, such that $v(\bar{t})>0$, and $v(t)\ge 0$ for all $t\in J$;

(ii) there exist $t_*, t^*\in J$, such that $v(t_*)<0$, $v(t^*)>0$.

Case (i). By (2.10), we have $v'(t)\le 0$ for $t\neq t_k$ and $\Delta v(t_k)\le 0$, $k=1,2,\ldots,m$, hence $v(t)$ is nonincreasing in J, that is, $v(T)\le v(0)$. If $\mu<e^{MT}$, then $v(0)<v(T)$, which is a contradiction. If $\mu=e^{MT}$, then $v(0)\le v(T)$ which implies $v(t)\equiv C>0$. But from (2.10), we get $v'(t)<0$ for $t\in J'$. Hence, $v(T)<v(0)$. It is again a contradiction.

Case (ii). Let $\inf_{t\in J}v(t)=-\lambda$, then $\lambda>0$. For some $i\in\{1,2,\ldots,p\}$, there exists $t_*\in(t_i,t_{i+1}]$ such that $v(t_*)=-\lambda$ or $v(t_*^+)=-\lambda$. We only consider $v(t_*)=-\lambda$, as for the case $v(t_*^+)=-\lambda$, the proof is similar.

From (2.10) and condition (H1), we get

$$\begin{aligned} v'(t) &\le -Ne^{Mt}[\varphi m](t) = -Ne^{Mt}\left[\varphi\left(v(t)e^{-Mt}\right)\right](t) \\ &\le -NRe^{Mt}\inf_{t\in J}\left\{v(t)e^{-Mt}\right\} \le -NRe^{Mt}\inf_{t\in J}\{v(t)\} \\ &\le \lambda NRe^{Mt}, \quad t\in J'. \end{aligned} \tag{2.11}$$

Consider the inequalities

$$\begin{aligned} v'(t) &\le \lambda NRe^{Mt}, \quad t\in J', \\ v(t_k^+) &\le c_k v(t_k), \quad k=1,2,\ldots,p. \end{aligned} \tag{2.12}$$

By Lemma 2.2, we have

$$v(t) \le v(t_*)\left(\prod_{t_*<t_k<t} c_k\right) + \int_{t_*}^{t}\left(\prod_{s<t_k<t} c_k\right)\lambda N R e^{Ms} ds, \tag{2.13}$$

that is

$$v(t) \le -\lambda\left(\prod_{t_*<t_k<t} c_k\right) + \lambda N R\int_{t_*}^{t}\left(\prod_{s<t_k<t} c_k\right)e^{Ms} ds. \tag{2.14}$$

First, we assume that $t^* > t_*$. Let $t = t^*$ in (2.14), then

$$v(t^*) \le -\lambda\left(\prod_{t_*<t_k<t^*} c_k\right) + \lambda N R\int_{t_*}^{t^*}\left(\prod_{s<t_k<t^*} c_k\right)e^{Ms} ds. \tag{2.15}$$

Noting that $v(t^*) > 0$, we have

$$\prod_{t_*<t_k<t^*} c_k < N R\int_{t_*}^{t^*}\left(\prod_{s<t_k<t^*} c_k\right)e^{Ms} ds. \tag{2.16}$$

Hence

$$\left(\prod_{k=1}^{p} c_k\right)^2 \le \prod_{k=1}^{p} c_k < N R\int_{0}^{T}\left(\prod_{s<t_k<T} c_k\right)e^{Ms} ds \tag{2.17}$$

which is a contradiction.

Next, we assume that $t^* < t_*$. By Lemma 2.2 and (2.10), we have

$$\begin{aligned} 0 < v(t^*) &\le v(0)\left(\prod_{0<t_k<t^*} c_k\right) + \int_{0}^{t^*}\left(\prod_{s<t_k<t^*} c_k\right)\lambda N R e^{Ms} ds \\ &\le \mu e^{-MT} v(T)\left(\prod_{0<t_k<t^*} c_k\right) + \lambda N R\int_{0}^{t^*}\left(\prod_{s<t_k<t^*} c_k\right)e^{Ms} ds, \end{aligned} \tag{2.18}$$

then

$$0 < \mu e^{-MT} v(T)\left(\prod_{k=1}^{p} c_k\right) + \lambda N R\int_{0}^{T}\left(\prod_{s<t_k<T} c_k\right)e^{Ms} ds. \tag{2.19}$$

Setting $t = T$ in (2.14), we have

$$\begin{aligned} v(T) &\le v(t_*)\left(\prod_{t_*<t_k<T} c_k\right) + \int_{t_*}^{T}\left(\prod_{s<t_k<T} c_k\right)\lambda NRe^{Ms}ds \\ &= -\lambda\left(\prod_{t_*<t_k<T} c_k\right) + \lambda NR\int_{t_*}^{T}\left(\prod_{s<t_k<T} c_k\right)e^{Ms}ds \\ &\le -\lambda\prod_{k=1}^{p} c_k + \lambda NR\int_{0}^{T}\left(\prod_{s<t_k<T} c_k\right)e^{Ms}ds \end{aligned} \tag{2.20}$$

with (2.19), we obtain that

$$\begin{aligned} 0 &< \mu e^{-MT}v(T)\left(\prod_{k=1}^{p} c_k\right) + \lambda NR\int_{0}^{T}\left(\prod_{s<t_k<T} c_k\right)e^{Ms}ds \\ &\le \left[-\lambda\left(\prod_{k=1}^{p} c_k\right) + \lambda NR\int_{0}^{T}\left(\prod_{s<t_k<T} c_k\right)e^{Ms}ds\right]\mu e^{-MT}\left(\prod_{k=1}^{p} c_k\right) + \lambda NR\int_{0}^{T}\left(\prod_{s<t_k<T} c_k\right)e^{Ms}ds \\ &= -\mu\lambda e^{-MT}\left(\prod_{k=1}^{p} c_k\right)^2 + \mu\lambda NRe^{-MT}\left(\prod_{k=1}^{p} c_k\right)\int_{0}^{T}\left(\prod_{s<t_k<T} c_k\right)e^{Ms}ds \\ &\quad + \lambda NR\int_{0}^{T}\left(\prod_{s<t_k<T} c_k\right)e^{Ms}ds, \end{aligned} \tag{2.21}$$

that is,

$$\begin{aligned} \mu e^{-MT}\left(\prod_{k=1}^{p} c_k\right)^2 &\le \left[\mu NRe^{-MT}\left(\prod_{k=1}^{p} c_k\right) + NR\right]\int_{0}^{T}\left(\prod_{s<t_k<T} c_k\right)e^{Ms}ds \\ &< NR\left(\mu e^{-MT} + 1\right)\int_{0}^{T}\left(\prod_{s<t_k<T} c_k\right)e^{Ms}ds. \end{aligned} \tag{2.22}$$

Therefore,

$$\sum_{k=1}^{p}(1 - L_k)^2 < \frac{NR(e^{MT} + \mu)}{\mu}\int_{0}^{T}\prod_{s<t_k<T}(1 - L_k)e^{Ms}ds \tag{2.23}$$

which is a contradiction. The proof of Theorem 2.3 is complete. □

The following corollary is an easy consequence of Theorem 2.3.

Corollary 2.4. *Assume that there exist* $M > 0$*,* $N \geq 0$*,* $0 \leq L_k < 1$*, for* $k = 1, 2, \ldots, p$ *such that* $m \in PC^1(J)$ *satisfies* (2.8) *with* $0 < \mu e^{-MT} \leq 1$ *and*

$$\frac{RN(e^{MT} + \mu)e^{MT}}{\mu} \leq \frac{\sum_{k=1}^{p}(1 - L_k)^2}{\int_0^T \sum_{s<t_k<T}(1 - L_k)ds} \tag{2.24}$$

then $m(t) \leq 0$*, for* $t \in J$*.*

Remark 2.5. Setting $\mu \equiv 1$, Corollary 2.4 reduces to the Theorem 2.3 of Li and Shen [6]. Therefore, Theorem 2.3 and Corollary 2.4 develops and generalizes the result in [6].

Remark 2.6. We show some examples of function φ satisfying (H1).

(i) $[\varphi x](t) = x(\theta(t))$, where $\theta \in C(J \times J)$, satisfies (H1) with $R = 1$,

$$[\varphi x](t) = x(\theta(t)) \geq \inf_{t\in J} x(t), \quad \text{for } t \in J. \tag{2.25}$$

(ii) $[\varphi x](t) = \int_0^{t+T} x(s)ds$, satisfies (H1) with $R = T$,

$$[\varphi x](t) = \int_0^{t+T} x(s)ds \geq (t + T)\inf_{t\in J} x(t) \geq T\inf_{t\in J} x(t), \quad \text{for } t \in J. \tag{2.26}$$

Consider the linear boundary value problem (LBVP)

$$\begin{gathered} y'(t) + My(t) + N[\varphi y](t) = \sigma(t), \quad t \in J', \\ \Delta y(t_k) = -L_k y(t_k) + I_k(t_k, \eta(t_k), [\varphi\eta](t_k)) + L_k\eta(t_k), \quad k = 1, 2, \ldots, p, \\ g(\eta(0), \eta(T)) + M_1(y(0) - \eta(0)) - M_2(y(T) - \eta(T)) = 0, \end{gathered} \tag{2.27}$$

where $M > 0$, $N \geq 0$, $0 \leq L_k < 1$, $k = 1, 2, \ldots, p$, and $\eta, \sigma \in PC(J)$.

By direct computation, we have the following result.

Lemma 2.7. $y \in PC^1(J)$ *is a solution of LBVP* (2.27) *if and only if* y *is a solution of the impulsive integral equation*

$$\begin{aligned} y(t) = Ce^{-Mt}B\eta &+ \int_0^T G(t,s)\{\sigma(s) - N[\varphi y](s)\}ds \\ &+ \sum_{0<t_k<T} G(t,t_k)\{-L_k y(t_k) + I_k(t_k, \eta(t_k), [\varphi\eta](t_k)) + L_k\eta(t_k)\}, \quad t \in J, \end{aligned} \tag{2.28}$$

where $B\eta = -g(\eta(0), \eta(T)) + M_1\eta(0) - M_2\eta(T)$*,* $C = (M_1 - M_2e^{-MT})^{-1}$*,* $M_1 \neq M_2e^{-MT}$ *and*

$$G(t,s) = \begin{cases} CM_2e^{M(s-t-T)} + e^{M(s-t)}, & 0 \leq s < t \leq T \\ CM_2e^{M(s-t-T)}, & 0 \leq t \leq s \leq T \end{cases} \tag{2.29}$$

Lemma 2.8. *Let (H2) hold. Suppose further*

$$\left(NLT+\sum_{k=1}^{p}|L_k|\right)r<1,\quad r=\max\{|CM_1|,|CM_2|\},\quad C=\left(M_1-M_2e^{-MT}\right)^{-1},\tag{2.30}$$

where $M>0$, $N\geq 0$, $M_1\neq M_2e^{-MT}$, then LBVP (2.27) *has a unique solution.*

By Lemma 2.7 and Banach fixed point theorem, the proof of Lemma 2.8 is apparent, so we omit the details.

3. Main Results

In this section, we use monotone iterative technique to obtain the existence results of extremal solutions and the unique solution of NBVP (2.2). We shall need the following definition.

Definition 3.1. A function $\alpha\in PC^1(J)$ is said to be a lower solution of NBVP (2.2) if it satisfies

$$\begin{gathered}\alpha'(t)\leq f(t,\alpha(t),[\varphi\alpha](t)),\quad t\in J',\\ \Delta\alpha(t_k)\leq I_k(t_k,\alpha(t_k),[\varphi\alpha](t_k)),\quad k=1,2,\ldots,p,\\ g(\alpha(0),\alpha(T))\leq 0.\end{gathered}\tag{3.1}$$

Analogously, $\beta\in PC^1(J)$ is an upper solution of NBVP (2.2) if

$$\begin{gathered}\beta'(t)\geq f(t,\beta(t),[\varphi\beta](t)),\quad t\in J',\\ \Delta\beta(t_k)\geq I_k(t_k,\beta(t_k),[\varphi\beta](t_k)),\quad k=1,2,\ldots,p,\\ g(\beta(0),\beta(T))\geq 0.\end{gathered}\tag{3.2}$$

For convenience, let us list the following conditions.

(H3) There exist constants $M>0$, $N\geq 0$ such that

$$f(t,x,\varphi x)-f(t,\overline{x},\varphi\overline{x})\geq -M(x-\overline{x})-N(\varphi x-\varphi\overline{x})\tag{3.3}$$

wherever $\alpha_0(t)\leq\overline{x}\leq x\leq\beta_0(t)$.

(H4) There exist constants $0\leq L_k<1$ for $k=1,2,\ldots,p$ such that

$$I_k(t_k,x,\varphi x)-I_k(t_k,\overline{x},\varphi\overline{x})\geq -L_k(x-\overline{x}),\quad k=1,2,\ldots,p\tag{3.4}$$

wherever $\alpha_0(t_k)\leq\overline{x}\leq x\leq\beta_0(t_k)$.

(H5) The function φ satisfies

$$\varphi x-\varphi\overline{x}\geq\varphi(x-\overline{x}),\quad \text{for } x,\overline{x}\in PC(J),\ x\geq\overline{x}.\tag{3.5}$$

(H6) There exist constants M_1, M_2 with $0 < M_2 e^{-MT} \le M_1$ such that

$$g(x,y) - g(\overline{x},\overline{y}) \le M_1(x-\overline{x}) - M_2(y-\overline{y}) \tag{3.6}$$

wherever $\alpha_0(0) \le \overline{x} \le x \le \beta_0(0)$, and $\alpha_0(T) \le \overline{y} \le y \le \beta_0(T)$.

Let $[\alpha_0, \beta_0] = \{x \in PC^1(J) : \alpha_0(t) \le x(t) \le \beta_0(t), \text{ for all } t \in J\}$. Now we are in the position to establish the main results of this paper.

Theorem 3.2. *Let* (H_1)*–*(H_6) *and inequalities* (2.9) *and* (2.30) *hold. Assume further that there exist lower and upper solutions* α_0 *and* β_0 *of NBVP* (2.2)*, respectively, such that* $\alpha_0 \le \beta_0$ *on* J*. Then there exist monotone sequences* $\{\alpha_n\}, \{\beta_n\} \subset PC^1(J)$ *with* $\alpha_0 \le \cdots \le \alpha_n \le \cdots \le \beta_n \le \cdots \le \beta_0$*, such that* $\lim_{n\to\infty}\alpha_n = x_*(t)$*,* $\lim_{n\to\infty}\beta_n = x^*(t)$ *uniformly on* J*. Moreover,* $x_*(t)$*,* $x^*(t)$ *are minimal and maximal solutions of NBVP* (2.2) *in* $[\alpha_0, \beta_0]$*, respectively.*

Proof. For any $\eta \in [\alpha_0, \beta_0]$, consider LVBP (2.27) with

$$\sigma(t) = f(t, \eta(t), [\varphi\eta](t)) + M\eta(t) + N[\varphi\eta](t). \tag{3.7}$$

By Lemma 2.8, we know that LBVP (2.27) has a unique solution $y \in PC^1(J)$. Define an operator $A : PC(J) \to PC(J)$ by $y = A\eta$, then the operator A has the following properties:

(a) $\alpha_0 \le A\alpha_0$, $A\beta_0 \le \beta_0$,

(b) $A\eta_1 \le A\eta_2$, if $\alpha_0 \le \eta_1 \le \eta_2 \le \beta_0$.

To prove (a), let $\alpha_1 = A\alpha_0$ and $m(t) = \alpha_0(t) - \alpha_1(t)$.

$$\begin{aligned} m'(t) &= \alpha_0'(t) - \alpha_1'(t) \\ &= f(t, \alpha_0(t), [\varphi\alpha_0](t)) \\ &\quad - \{-M\alpha_1(t) - N[\varphi\alpha_1](t) + f(t, \alpha_0(t), [\varphi\alpha_0](t)) + M\alpha_0(t) + N[\varphi\alpha_0](t)\} \\ &\le -Mm(t) - N[\varphi m](t), \\ \Delta m(t_k) &= \Delta\alpha_0(t_k) - \Delta\alpha_1(t_k) \\ &\le I_k(t_k, \alpha_0(t_k), [\varphi\alpha_0](t_k)) - \{-L_k\alpha_1(t_k) + I_k(t_k, \alpha_0(t_k), [\varphi\alpha_0](t_k)) + L_k\alpha_0(t_k)\} \\ &\le -L_k m(t_k), \\ m(0) &= \alpha_0(0) - \alpha_1(0) \\ &= \alpha_0(0) - \left\{-\frac{1}{M_1} g(\alpha_0(0), \alpha_0(T)) + \alpha_0(0) + \frac{M_2}{M_1}(\alpha_1(T) - \alpha_0(T))\right\} \\ &\le \frac{M_2}{M_1} m(T). \end{aligned} \tag{3.8}$$

By Theorem 2.3, we get $m(t) \le 0$ for $t \in J$, that is, $\alpha_0 \le A\alpha_0$. Similarly, we can show that $A\beta_0 \le \beta_0$.

To prove (b), set $m(t) = x_1(t) - x_2(t)$, where $x_1 = A\eta_1$ and $x_2 = A\eta_2$. Using (H3), (H4) and (H6), we get

$$\begin{aligned}
m'(t) &= x_1'(t) - x_2'(t) \\
&= M(\eta_1(t) - x_1(t)) + N([\varphi\eta_1](t) - [\varphi x_1](t)) + f(t, \eta_1(t), [\varphi\eta_1](t)) \\
&\quad - M(\eta_2(t) - x_2(t)) - N([\varphi\eta_2](t) - [\varphi x_2](t)) - f(t, \eta_2(t), [\varphi\eta_2](t)) \\
&\leq -Mm(t) - N[\varphi m](t), \\
\Delta m(t_k) &= \Delta x_1(t_k) - \Delta x_2(t_k) \\
&\leq L_k(\eta_1(t_k) - x_1(t_k)) + I_k(t_k, \eta_1(t_k), [\varphi\eta_1](t_k)) \\
&\quad - L_k(\eta_2(t_k) - x_2(t_k)) - I_k(t_k, \eta_2(t_k), [\varphi\eta_2](t_k)) \\
&\leq -L_k m(t_k), \\
m(0) &= x_1(0) - x_2(0) \\
&= -\frac{1}{M_1} g(\eta_1(0), \eta_1(T)) + \eta_1(0) + \frac{M_2}{M_1}(x_1(T) - \eta_1(T)) \\
&\quad + \frac{1}{M_1} g(\eta_2(0), \eta_2(T)) - \eta_2(0) - \frac{M_2}{M_1}(x_2(T) - \eta_2(T)) \\
&\leq \frac{M_2}{M_1} m(T).
\end{aligned} \tag{3.9}$$

By Theorem 2.3, we get $m(t) \leq 0$ for $t \in J$, that is, $A\eta_1 \leq A\eta_2$, then (b) is proved.

Let $\alpha_n = A\alpha_{n-1}$ and $\beta_n = A\beta_{n-1}$ for $n = 1, 2, 3, \ldots$. By the properties (a) and (b), we have

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0. \tag{3.10}$$

By the definition of operator A, we have that $\{\alpha_n'\}$ and $\{\beta_n'\}$ are uniformly bounded in $[\alpha_0, \beta_0]$. Thus $\{\alpha_n\}$ and $\{\beta_n\}$ are uniformly bounded and equicontinuous in $[\alpha_0, \beta_0]$. By Arzela-Ascoli Theorem and (3.10), we know that there exist x_*, x^* in $[\alpha_0, \beta_0]$ such that

$$\lim_{n\to\infty} \alpha_n(t) = x_*(t), \qquad \lim_{n\to\infty} \beta_n(t) = x^*(t) \quad \text{uniformly on } J \tag{3.11}$$

Moreover, $x_*(t)$, $x^*(t)$ are solutions of NBVP (2.2) in $[\alpha_0, \beta_0]$.

To prove that x_*, x^* are extremal solutions of NBVP (2.2), let $u(t) \in [\alpha_0, \beta_0]$ be any solution of NBVP (2.2), that is,

$$\begin{gathered}
u'(t) = f(t, u(t), [\varphi u](\mathrm{t})), \quad t \in J', \\
\Delta u(t_k) = I_k(t_k, u(t_k), [\varphi u](t_k)), \quad k = 1, 2, \ldots, p, \\
g(u(0), u(T)) = 0.
\end{gathered} \tag{3.12}$$

By Theorem 2.3 and Induction, we get $\alpha_n(t) \leq u(t) \leq \beta_n(t)$ with $t \in J$ and $n = 1, 2, 3, \ldots$ which implies that $x_*(t) \leq u(t) \leq x^*(t)$, that is, x_* and x^* are minimal and maximal solution of NBVP (2.2) in $[\alpha_0, \beta_0]$, respectively. The proof is complete. □

Theorem 3.3. *Let the assumptions of Theorem 3.2 hold and assume the following.*

(H7) *There exist constants* $\widetilde{M} > 0$, $\widetilde{N} \geq 0$ *such that*

$$f(t, x, \varphi x) - f(t, \overline{x}, \varphi\overline{x}) \leq -\widetilde{M}(x - \overline{x}) - \widetilde{N}(\varphi x - \varphi\overline{x}), \tag{3.13}$$

where $\alpha_0(t) \leq \overline{x} \leq x \leq \beta_0(t)$.

(H8) *There exist constants* $0 \leq \widetilde{L}_k < 1$, $k = 1, 2, \ldots, p$ *such that*

$$I_k(t_k, x, \varphi x) - I_k(t_k, \overline{x}, \varphi\overline{x}) \leq -\widetilde{L}_k(x - \overline{x}), \quad k = 1, 2, \ldots, p, \tag{3.14}$$

where $\alpha_0(t_k) \leq \overline{x} \leq x \leq \beta_0(t_k)$.

(H9) *There exist constants* $\widetilde{M}_1$, $\widetilde{M}_2$ *with* $0 < \widetilde{M}_2 e^{-\widetilde{M}T} < \widetilde{M}_1$ *such that*

$$g(x, y) - g(\overline{x}, \overline{y}) \geq \widetilde{M}_1(x - \overline{x}) - \widetilde{M}_2(y - \overline{y}\,) \tag{3.15}$$

whenever $\alpha_0(0) \leq \overline{x} \leq x \leq \beta_0(0)$, *and* $\alpha_0(T) \leq \overline{y} \leq y \leq \beta_0(T)$.

Then NBVP (2.2) has a unique solution in $[\alpha_0, \beta_0]$.

Proof. By Theorem 3.2, we know that there exist $x_*, x^* \in [\alpha_0, \beta_0]$, which are minimal and maximal solutions of NBVP (2.2) with $x_*(t) \leq x^*(t)$, $t \in J$.

Let $m(t) = x^*(t) - x_*(t)$. Using (H7), (H8), and (H9), we get

$$\begin{aligned}
m'(t) &= (x^*(t))' - (x_*(t))' = f(t, x^*(t), [\varphi x^*](t)) - f(t, x_*(t), [\varphi x_*](t)) \\
&\leq -\widetilde{M}m(t) - \widetilde{N}[\varphi m](t), \\
\Delta m(t_k) &= \Delta x^*(t_k) - \Delta x_*(t_k) = I_k(t_k, x^*(t_k), [\varphi x^*](t_k)) - I_k(t_k, x_*(t_k), [\varphi x_*](t_k)) \\
&\leq \widetilde{L}_k m(t_k), \\
m(0) &= x^*(0) - x_*(0) \leq \frac{\widetilde{M}_2}{\widetilde{M}_1}(x^*(T) - x_*(T)) = \frac{\widetilde{M}_2}{\widetilde{M}_1}m(T).
\end{aligned} \tag{3.16}$$

By Theorem 2.3, we have that $m(t) < 0$, $t \in J$, that is, $x^*(t) \leq x_*(t)$. Hence $x^*(t) = x_*(t)$, this completes the proof. □

4. Examples

To illustrate our main results, we shall discuss in this section some examples.

Example 4.1. Consider the problem

$$x'(t) = -\frac{1}{10}(-|\sin t| + x - 2)^5 - \frac{e^{-2\pi}}{3}\left(\int_t^{t+1} x(s)ds - \sin\frac{t}{4}\right)^2 + \frac{4}{3}e^{-2\pi}, \quad t \in [0,T],\ t \neq t_k,$$
$$\Delta x(t_k) = -\frac{1}{2}e^{-\pi/2}(x(t_k) - 3) + \left(\int_{t_k}^{t_k+1} x(s)ds - \sin\frac{t_k}{4}\right)^{1/7} + \frac{4}{17}e^{\pi/2}\cos t_k, \quad k = 1, \tag{4.1}$$
$$x(0) - \frac{1}{2}x(T) - \frac{1}{6\pi}\int_0^T x(s)ds = 0,$$

where $T = 2\pi$, $k = 1$, $t_1 = \pi$.

Let

$$[\varphi x](t) = \int_t^{t+1} x(s)ds - \sin\frac{t}{4},$$
$$f(t,x,y) = -\frac{1}{10}(-|\sin t| + x - 2)^5 - \frac{e^{-2\pi}}{3}y^2 + \frac{4}{3}e^{-2\pi}, \tag{4.2}$$
$$I_1(t,x,y) = -\frac{1}{2}e^{-\pi/2}(x - 3) + y^{1/7} + \frac{4}{17}e^{\pi/2}\cos t.$$

Setting $\alpha_0(t) \equiv 2$ and $\beta_0(t) \equiv 3$, it is easy to verify that $\alpha_0(t)$ is a lower solution, and $\beta_0(t)$ is an upper solution with $\alpha_0(t) \leq \beta_0(t)$.

For $t \in J$, and $2 \leq \overline{x}(t) \leq x(t) \leq 3$, we have

$$\varphi x \geq 1 \geq \frac{1}{3}\inf_{t\in J} x(t),$$
$$\varphi x - \varphi\overline{x} \geq \varphi(x - \overline{x}), \tag{4.3}$$
$$\|\varphi x - \varphi\overline{x}\| = \left\|\int_t^{t+1}[x(s) - \overline{x}(s)]ds\right\| \leq \|x - \overline{x}\|.$$

Setting $M = 1/2$, $N = e^{-2\pi}/6$, $L = 1$, $R = 1/3$, $L_1 = (1/2)e^{-\pi/2}$ and $M_1 = 1$, $M_2 = 1/2$, then conditions (H1)–(H6) are all satisfied:

$$\begin{gathered}
\int_0^{2\pi} \prod_{s<t_k<2\pi} (1-L_k)e^{Ms}ds = \int_0^{\pi}(1-L_1)e^{Ms}ds + \int_{\pi}^{2\pi} e^{Ms}ds \approx 43.4893, \\
\prod_{k=1}^{p}(1-L_k)^2 = (1-L_1)^2 = \left(1-\frac{1}{2}e^{-\pi/2}\right)^2 \approx 0.8029, \\
\frac{NR(e^{MT}+(M_2/M_1))}{M_2/M_1} = \frac{2e^{\pi}+1}{18e^{2\pi}} \approx 0.0049, \\
\frac{NR(e^{MT}+(M_2/M_1))}{M_2/M_1}\int_0^{2\pi} \prod_{s<t_k<2\pi} (1-L_k)e^{Ms}ds \approx 0.2131 < 0.8029, \\
\left(NLT + \sum_{k=1}^{p}|L_k|\right)r = (NLT + L_1)CM_1 \approx 0.1082 < 1,
\end{gathered} \tag{4.4}$$

then inequalities (2.9) and (2.30) are satisfied. By Theorem 3.2, problem (4.1) has extremal solutions $x_*, x^* \in [\alpha_0, \beta_0]$.

Example 4.2. Consider the problem

$$\begin{gathered}
x'(t) = -\frac{1}{2}x(t) - \frac{e^{-3\pi}}{e^{\pi}+1}\left(e^{2x}-1\right) + \frac{3}{2}, \quad t \in [0,T],\ t \neq t_k, \\
\Delta x(t_k) = -2e^{-\pi}(x(t_k)-2) + \left(e^{2x(t_k)}-1\right)^{1/50} + \cos t_k, \quad k = 1, \\
x(0) = \frac{1}{2}x(T),
\end{gathered} \tag{4.5}$$

where $T = 2\pi$, $k = 1$, $t_1 = \pi$.

Let

$$\begin{gathered}
\varphi x = e^{2x} - 1, \\
f(t,x,y) = -\frac{1}{10}(-|\sin t| + x - 2)^5 - \frac{e^{-3\pi}}{16(e^{\pi}+1)}y^2 + \frac{(e^4-1)^2}{16e^{3\pi}(e^{\pi}+1)}, \\
I_1(t,x,y) = -2e^{-\pi}(x-2) + y^{1/50} + \cos t.
\end{gathered} \tag{4.6}$$

Setting $\alpha_0(t) \equiv 2$ and $\beta_0(t) \equiv 3$, then $\alpha_0(t)$ is a lower solution, and $\beta_0(t)$ is an upper solution with $\alpha_0(t) \le \beta_0(t)$.

For $t \in J$, and $2 \le \overline{x}(t) \le x(t) \le 3$, we have $\varphi x = e^{2x} - 1 \ge x$, $\varphi x - \varphi\overline{x} \ge \varphi(x - \overline{x})$, and $|\varphi x - \varphi\overline{x}| = |e^{2x} - e^{2\overline{x}}| = |e^x + e^{\overline{x}}| \cdot |e^x - e^{\overline{x}}| \le 2e^6|e^{x-\overline{x}} - 1| \le 14e^6|x - \overline{x}|$. Setting $M = 1/2$,

$N = e^{-3\pi}/(e^{\pi}+1)$, $L = 14e^6$, $R = 1$, $L_1 = 2e^{-\pi}$ and $M_1 = 1$, $M_2 = 1/2$, then conditions (H1)–(H6) are all satisfied:

$$\frac{\prod_{k=1}^{p}(1-L_k)^2}{\int_0^{2\pi}\prod_{s<t_k<2\pi}(1-L_k)ds} = \frac{(e^{\pi}-2)^2}{2\pi e^{\pi}(e^{\pi}-1)} \approx 0.1388,$$
$$\frac{NR\left(e^{MT}+(M_2/M_1)\right)e^{MT}}{M_2/M_1} = \frac{2e^{\pi}+1}{e^{2\pi}(e^{\pi}+1)} \approx 0.0037 < 0.1388, \tag{4.7}$$
$$\left(NLT + \sum_{k=1}^{p}|L_k|\right)r = (NLT + L_1)CM_1 \approx 0.2095 < 1,$$

then inequalities (2.24) and (2.30) are satisfied. By Corollary 2.4 and Theorem 3.2, problem (4.5) has extremal solutions $x_*, x^* \in [\alpha_0, \beta_0]$.

Moreover, let $\widetilde{M} = 1/2$, $\widetilde{N} = e^{-3\pi}/(e^{\pi}+1)$, $\widetilde{L}_1 = 2e^{-\pi}$ and $\widetilde{M}_1 = 1$, $\widetilde{M}_2 = 1/2$. It is easy to see that conditions (H7)–(H9) are satisfied. By Corollary 2.4 and Theorem 3.3, problem (4.5) has an unique solution in $[\alpha_0, \beta_0]$.

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