

Research Article

Superstability and Stability of the Pexiderized Multiplicative Functional Equation

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We obtain the superstability of the Pexiderized multiplicative functional equation $f(xy) = g(x)h(y)$ and investigate the stability of this equation in the following form: $1/(1 + \psi(x, y)) \leq f(xy)/g(x)h(y) \leq 1 + \psi(x, y)$.

1. Introduction

The superstability of the functional equation $f(x + y) = f(x)f(y)$ was studied by Baker et al. [1]. They proved that if f is a functional on a real vector space W satisfying $|f(x + y) - f(x)f(y)| \leq \delta$ for some fixed $\delta > 0$ and all $x, y \in W$, then f is either bounded or else $f(xy) = f(x)f(y)$ for all $x, y \in W$. This result was generalized with a simplified proof by Baker [2] as follows.

Theorem 1.1 (Baker [2]). *Let $\delta > 0$, S be a semigroup and $f : S \rightarrow \mathbb{C}$ satisfying*

$$|f(xy) - f(x)f(y)| \leq \delta \tag{1.1}$$

for all $x, y \in S$. Put $\beta := (1 + \sqrt{1 + 4\delta})/2$. Then $|f(x)| \leq \beta$ for all $x \in S$ or else $f(xy) = f(x)f(y)$ for all $x, y \in S$.

A different generalization of the result of Baker et al. was given by Székelyhidi [3]. It involves an interesting generalization of the class of bounded functions on a group or semigroup and may be stated as follows.

Theorem 1.2 (Székelyhidi [3]). *Let G be a commutative group with identity 1 and let $f, m : S \rightarrow \mathbb{C}$ be functions such that there exist functions $M_1, M_2 : G \rightarrow [0, \infty)$ with*

$$|f(xy) - f(x)m(y)| \leq \min\{M_1(x), M_2(y)\} \quad (1.2)$$

for all $x, y \in G$. Then f is bounded or m is an exponential and $f = f(1)m$.

In this paper, we prove the superstability of the Pexiderized multiplicative functional equation (PMFE)

$$f(xy) = g(x)h(y). \quad (1.3)$$

That is, we prove that if f, g, h are functional on a semigroup S with identity 1 satisfying $g(1) = 1$ and

$$|f(xy) - g(x)h(y)| \leq \varphi(x, y) \quad (1.4)$$

for all $x, y \in S$ and for a function $\varphi : S \times S \rightarrow [0, \infty)$ with some conditions, then g is bounded or else g is an exponential and $h = h(1)g$. This is a generalization of the result of Székelyhidi. Also we investigate the stability of the Pexiderized multiplicative functional equation (1.3) in the sense of Ger [4].

2. Superstability of the PMFE

In this section, let (S, \cdot) be a semigroup with identity 1 and $\varphi : S \times S \rightarrow [0, \infty)$ a function with

$$\Phi_w(x) := \sum_{k=0}^{\infty} \frac{\varphi(w, x^{k+2}) + \varphi(wx, x^{k+1})}{2^k} < \infty \quad (2.1)$$

for all $x, w \in S$ and

$$\lim_{k \rightarrow \infty} \varphi(wx, zy^k) \text{ exists} \quad (2.2)$$

for all $x, y, z, w \in S$.

Example 2.1. The following functions satisfy conditions (2.1) and (2.2) above.

- (a) $\varphi(x, y) = \delta$, for every $x, y \in \mathbb{R}$ and $\delta \geq 0$.
- (b) $\varphi(x, y) = |t(x)|$, for every $x, y \in S$ and t is a functional on S .
- (c) $\varphi(x, y) = |x| + 1/(1 + |y|)$, for every $x, y \in \mathbb{R}$.
- (d) $\varphi(x, y) = 1/(1 + |x| + |y|)$, for every $x, y \in \mathbb{R}$.

Example 2.2. Let $(S, \cdot) = ([0, \infty), +)$ and also $g(x) = e^x, h(x) = e^{x+c}$,

$$f(x) = e^{x+c} + \frac{1}{1+x} \quad (2.3)$$

for all $x \in S$ and for some $c \in S$. Let $\varphi(x, y) = 1/(1+x+y)$. Then f, g, h, φ satisfy the conditions (1.4), (2.1), (2.2) and

$$|f(x+y) - g(x)h(y)| = \frac{1}{1+x+y}. \quad (2.4)$$

In particular, we know that $g(0) = 1, g(x+y) = g(x)g(y), h = h(0)g$, and $f(x+y) \neq f(x)f(y)$.

Theorem 2.3. Let (S, \cdot) be a semigroup with identity 1. If $f, g, h : S \rightarrow \mathbb{C}$ are functions with $|g(m)| \geq \max\{2, 2\Phi_1(m)/|h(m)|\}$ for some $m \in S$ satisfying $g(1) = 1$ and condition (1.4), that is,

$$|f(xy) - g(x)h(y)| \leq \varphi(x, y), \quad (2.5)$$

then

$$g(xy) = g(x)g(y) \quad (2.6)$$

for all $x, y \in S$ and $h = h(1)g$.

Proof. If we replace x by m and also y by m in (1.4), we get

$$|f(m^2) - g(m)h(m)| \leq \varphi(m, m). \quad (2.7)$$

Also we replace x by 1 in (1.4), then we have

$$|f(y) - h(y)| \leq \varphi(1, y) \quad (2.8)$$

for all $y \in S$. An induction argument implies that for all $n \geq 2$,

$$|f(m^n) - g(m)^{n-1}h(m)| \leq \varphi(m, m^{n-1}) + \sum_{k=1}^{n-2} |g(m)|^k (\varphi(1, m^{n-k}) + \varphi(m, m^{n-k-1})). \quad (2.9)$$

Indeed, if inequality (2.9) holds, using inequality (1.4) and (2.8) we have

$$\begin{aligned}
& \left| f(m^{n+1}) - g(m)^n h(m) \right| \\
& \leq |f(mm^n) - g(m)h(m^n)| + |g(m)| |h(m^n) - f(m^n)| \\
& \quad + |g(m)| \left| f(m^n) - g(m)^{n-1} h(m) \right| \\
& \leq \varphi(m, m^n) + |g(m)| \varphi(1, m^n) \\
& \quad + |g(m)| \left(\varphi(m, m^{n-1}) + \sum_{k=1}^{n-2} |g(m)|^k (\varphi(1, m^{n-k}) + \varphi(m, m^{n-k-1})) \right) \\
& = \varphi(m, m^n) + \sum_{k=1}^{n-1} |g(m)|^k (\varphi(1, m^{n+1-k}) + \varphi(m, m^{n-k}))
\end{aligned} \tag{2.10}$$

for all $n \geq 2$. By (2.9), we have

$$\begin{aligned}
& \left| \frac{f(m^n)}{g(m)^{n-1} h(m)} - 1 \right| \\
& \leq \frac{1}{|h(m)| |g(m)|} \left(\frac{\varphi(m, m^{n-1})}{|g(m)|^{n-2}} + \sum_{k=1}^{n-2} \frac{1}{|g(m)|^{n-k-2}} (\varphi(1, m^{n-k}) + \varphi(m, m^{n-k-1})) \right) \\
& \leq \frac{1}{|h(m)| |g(m)|} \left((\varphi(1, m^2) + \varphi(m, m)) + \frac{1}{2} (\varphi(1, m^3) + \varphi(m, m^2)) + \dots \right. \\
& \quad \left. + \frac{1}{2^{n-3}} (\varphi(1, m^{n-1}) + \varphi(m, m^{n-2})) + \left(\frac{1}{2^{n-2}} \varphi(m, m^{n-1}) \right) \right) \\
& \leq \frac{1}{|h(m)| |g(m)|} \left(\sum_{k=0}^{n-3} \frac{1}{2^k} (\varphi(1, m^{k+2}) + \varphi(m, m^{k+1})) + \frac{1}{2^{n-2}} \varphi(m, m^{n-1}) + \varphi(1, m^n) \right) \\
& \leq \frac{1}{|h(m)| |g(m)|} \sum_{k=0}^{\infty} \frac{1}{2^k} (\varphi(1, m^{k+2}) + \varphi(m, m^{k+1})) \\
& = \frac{\Phi_1(m)}{|h(m)| |g(m)|} \leq \frac{1}{2}.
\end{aligned} \tag{2.11}$$

Thus we can easily show that $|f(m^n)| \rightarrow \infty$ from $|g(m)^{n-1} h(m)| \rightarrow \infty$ as $n \rightarrow \infty$ and thus $|h(m^n)| \rightarrow \infty$ as $n \rightarrow \infty$. By (1.4),

$$\left| \frac{f(xm^n)}{h(m^n)} - g(x) \right| \leq \frac{\varphi(x, m^n)}{|h(m^n)|}, \tag{2.12}$$

and thus we have

$$g(x) = \lim_{n \rightarrow \infty} \frac{f(xm^n)}{h(m^n)} \quad (2.13)$$

for all $x \in S$. Then, by (2.2),

$$\begin{aligned} |g(xy) - g(x)g(y)| &= \lim_{n \rightarrow \infty} \frac{1}{|h(m^n)|} |f(xym^n) - g(x)f(y m^n)| \\ &= \lim_{n \rightarrow \infty} \frac{1}{|h(m^n)|} (|f(xym^n) - g(x)h(y m^n)| + |g(x)||h(y m^n) - f(y m^n)|) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|h(m^n)|} (\varphi(x, y m^n) + |g(x)|\varphi(1, y m^n)) = 0, \end{aligned} \quad (2.14)$$

and so

$$g(xy) = g(x)g(y) \quad (2.15)$$

for all $x, y \in S$. Thus we have $|g(m^n)| = |g(m)^n| \rightarrow \infty$ as $n \rightarrow \infty$. Since

$$\left| \frac{f(xm^n)}{g(m^n)} - h(x) \right| \leq \frac{\varphi(x, m^n)}{|g(m^n)|} \rightarrow 0 \quad (2.16)$$

as $n \rightarrow \infty$, we can define h by

$$h(x) = \lim_{n \rightarrow \infty} \frac{f(xm^n)}{g(m^n)} \quad (2.17)$$

for all $x \in S$. Then

$$h(1)g(x) = \lim_{n \rightarrow \infty} \frac{f(m^n)}{g(m^n)} \cdot \frac{f(xm^n)}{h(m^n)} = g(1)h(x) = h(x) \quad (2.18)$$

for all $x \in S$. □

Corollary 2.4. *Let $(S, +)$ be a semigroup with identity 0 and $f, g, h : S \rightarrow \mathbb{C}$ functions satisfying the inequality*

$$|f(x + y) - g(x)h(y)| \leq \varphi(x, y) \quad (2.19)$$

for all $x, y \in S$. If $g(0) = 1$, then g is bounded or else g is exponential and $h = h(0)g$.

Theorem 2.5. Let (S, \cdot) be a semigroup with identity 1 and $f, g, h : S \rightarrow C$ functions satisfying condition (1.4), that is,

$$|f(xy) - g(x)h(y)| \leq \varphi(x, y). \quad (2.20)$$

If g satisfies that $g(s) \neq 0$ for some $s \in S$ and $|g(sm)| \geq \max\{2|g(s)|, 2\Phi_s(m)/|h(m)|\}$ for some $m \in S$, then

$$g(sxy) = \frac{1}{g(s)}g(sx)g(sy) \quad (2.21)$$

for all $x, y \in S$ and $h(x) = (h(1)/g(s))g(sx)$.

Proof. Let $\bar{f}(x) = f(sx)$, $\bar{g}(x) = g(sx)/g(s)$ and $\bar{h}(x) = g(s)h(x)$ for every $x \in S$ and $\bar{\varphi}(x, y) = \varphi(sx, y)$. Then

$$|\bar{f}(xy) - \bar{g}(x)\bar{h}(y)| \leq \bar{\varphi}(x, y) \quad (2.22)$$

for all $x, y \in S$, $|\bar{g}(m)| \geq \max\{2, 2\bar{\Phi}_1(m)/|\bar{h}(m)|\}$ and $\bar{g}(1) = 1$ where $\bar{\Phi}_1(m) = \Phi_s(m)$. By Theorem 2.3, we complete the proof. \square

Corollary 2.6. Let (S, \cdot) be a semigroup with identity 1. If $f, g, h : S \rightarrow C$ are nonzero functions satisfying condition (1.4), that is,

$$|f(xy) - g(x)h(y)| \leq \varphi(x, y), \quad (2.23)$$

then either g is bounded, or else

$$g(sxy) = \frac{1}{g(s)}g(sx)g(sy) \quad (2.24)$$

for all $x, y \in S$ and $h(x) = (h(1)/g(s))g(sx)$.

Proof. Let $g(s) \neq 0$ for some $s \in S$. If g is unbounded, then there exists m such that $|g(sm)| \geq \max\{2|g(s)|, 2\Phi_s(m)/|h(m)|\}$. By Theorem 2.5, we complete the proof. \square

3. Stability of the PMFE

In 1940, Ulam gave a wide-ranging talk in the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems [5]. One of those was the question concerning the stability of homomorphisms.

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In the next year, Hyers [6] answered the Ulam's question for the case of the additive mapping on the Banach spaces G_1, G_2 . Thereafter, the result of Hyers has been generalized by Rassias [7]. Since then, the stability problems of various functional equations have been investigated by many authors (see [6, 8–18]).

Ger [4] suggested another type of stability for the exponential equation in the following type:

$$\left| \frac{f(x+y)}{f(x)f(y)} - 1 \right| \leq \delta. \quad (3.1)$$

In this section, the stability problem for the Pexiderized multiplicative functional equation in the following form:

$$\frac{1}{1 + \varphi(x, y)} \leq \frac{f(xy)}{g(x)h(y)} \leq 1 + \varphi(x, y) \quad (3.2)$$

will be investigated.

Throughout this section, we denote by (S, \cdot) a commutative semigroup and by $\varphi : S \times S \rightarrow [0, \infty)$ a function such that

$$\Psi(x, y, z, w) = \sum_{n=0}^{\infty} \frac{1}{2^n} \ln(1 + \varphi(xz^{2^n}, yw^{2^n})) < \infty \quad (3.3)$$

for all $x, y, z, w \in S$. Also we let

$$u(x, y) := \ln(1 + \varphi(x, y))(1 + \varphi(y, x))(1 + \varphi(x, x))(1 + \varphi(y, y)) \quad (3.4)$$

for all $x, y \in S$. Inequality (3.3) implies that

(a) for all $x, z \in S$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} u(xz^{2^n}, xz^{2^n}) = \sum_{n=0}^{\infty} \frac{1}{2^n} \ln(1 + \varphi(xz^{2^n}, xz^{2^n}))^4 = 4\Psi(x, x, z, z) < \infty, \quad (3.5)$$

(b) for all $x, z \in S$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{2^n} u(x^2, z^{2^n}) &= \sum_{n=0}^{\infty} \frac{1}{2^n} \ln(1 + \varphi(x^2, z^{2^n}))(1 + \varphi(z^{2^n}, x^2)) \\ &\quad \cdot (1 + \varphi(x^2, x^2))(1 + \varphi(z^{2^n}, z^{2^n})) \\ &= \Psi(x^2, 1, 1, z) + \Psi(1, x^2, z, 1) + \Psi(x^2, x^2, 1, 1) + \Psi(1, 1, z, z) < \infty, \end{aligned} \quad (3.6)$$

(c) for all $x, z \in S$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} u(x^4, z^{2^n}) = \Psi(x^4, 1, 1, z) + \Psi(1, x^4, z, 1) + \Psi(x^4, x^4, 1, 1) + \Psi(1, 1, z, z) < \infty, \quad (3.7)$$

(d) for all $x, y \in S$ for

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{2^n} u(x^{2^n}, y^{2^n}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \ln \left((1 + \varphi(x^{2^n}, y^{2^n})) (1 + \varphi(y^{2^n}, x^{2^n})) \cdot (1 + \varphi(x^{2^n}, x^{2^n})) (1 + \varphi(y^{2^n}, y^{2^n})) \right) = 0, \end{aligned} \quad (3.8)$$

because

$$\Psi(1, 1, x, y) + \Psi(1, 1, y, x) + \Psi(1, 1, x, x) + \Psi(1, 1, y, y) < \infty. \quad (3.9)$$

Example 3.1. The following functions satisfy condition (3.3) above.

(a) $\varphi(x, y) = \delta$, for every $x, y \in R$ and $\delta \geq 0$.

(b) $\varphi(x, y) = 1/(1 + |x| + |y|)$, for every $x, y \in R$.

Example 3.2. Let $(S, \cdot) = ([0, \infty), +)$ and also $g(x) = e^x, h(x) = e^{x+c}$,

$$f(x) = e^{x+c} \left(1 + \frac{1}{1+x} \right) \quad (3.10)$$

for all $x \in S$ and for some $c \in S$. Let $\varphi(x, y) = 1/(1 + x + y)$. Then f, g, h, φ satisfy condition (3.3) and

$$\frac{1}{1 + \varphi(x, y)} \leq \frac{f(x+y)}{g(x)h(y)} \leq 1 + \varphi(x, y). \quad (3.11)$$

In particular, we know that if we let $T(x) = e^x$ then

$$\frac{e^{-c}}{2} \leq \frac{T(x)}{f(x)} \leq e^{-c}, \quad \frac{T(x)}{g(x)} = 1, \quad \frac{T(x)}{h(x)} = e^{-c}. \quad (3.12)$$

Theorem 3.3. *If $f, g, h : S \rightarrow (0, \infty)$ are functions such that*

$$\frac{1}{1 + \varphi(x, y)} \leq \frac{f(xy)}{g(x)h(y)} \leq 1 + \varphi(x, y) \quad (3.13)$$

for all $x, y \in S$, then there exists a function $T : S \rightarrow (0, \infty)$ and there exists a constant M such that $T(xy) = T(x)T(y)$ for all $x, y \in S$ and

$$e^{-M} \leq \frac{T(x)}{f(x)} \leq e^M \quad (3.14)$$

for all $x \in S$. Moreover, if φ is bounded, then

$$\begin{aligned} e^{-M_1} &\leq \frac{T(x)}{g(x)} \leq e^{M_1}, \\ e^{-M_1} &\leq \frac{T(x)}{h(x)} \leq e^{M_1} \end{aligned} \quad (3.15)$$

for all $x \in S$ and for some constant M_1 .

Proof. If we define functions $F, G, H : S \rightarrow R$ by

$$F(x) = \ln f(x), \quad G(x) = \ln g(x), \quad H(x) = \ln h(x) \quad (3.16)$$

for all $x \in S$, then equality (3.13) may be transformed into

$$\ln \frac{1}{1 + \varphi(x, y)} \leq F(xy) - G(x) - H(y) \leq \ln(1 + \varphi(x, y)), \quad (3.17)$$

and thus

$$|F(xy) - G(x) - H(y)| \leq \ln(1 + \varphi(x, y)), \quad (3.18)$$

for all $x, y \in S$. For the case of $x = y$, the above inequality implies

$$|F(x^2) - G(x) - H(x)| \leq \ln(1 + \varphi(x, x)) \quad (3.19)$$

and so

$$\begin{aligned} &|2F(xy) - F(x^2) - F(y^2)| \\ &\leq |F(xy) - G(x) - H(y)| + |F(xy) - G(y) - H(x)| \\ &\quad + |G(x) + H(x) - F(x^2)| + |G(y) + H(y) - F(y^2)| \\ &\leq \ln(1 + \varphi(x, y))(1 + \varphi(y, x)) \cdot (1 + \varphi(x, x))(1 + \varphi(y, y)) := u(x, y) \end{aligned} \quad (3.20)$$

for all $x, y \in S$. Putting xz instead of x and yz instead of y in (3.20), respectively, we get

$$|2F(xz^2y) - F(x^2z^2) - F(y^2z^2)| \leq u(xz, yz). \quad (3.21)$$

Letting x by x^2 and y by z^2 in (3.20), we have

$$\left| F(x^2 z^2) - \frac{1}{2}F(x^4) - \frac{1}{2}F(z^4) \right| \leq \frac{1}{2}u(x^2, z^2), \quad (3.22)$$

and also

$$\left| F(y^2 z^2) - \frac{1}{2}F(y^4) - \frac{1}{2}F(z^4) \right| \leq \frac{1}{2}u(y^2, z^2). \quad (3.23)$$

From (3.21), (3.22) and (3.23),

$$\left| 2F(xz^2 y) - \frac{1}{2}F(x^4) - \frac{1}{2}F(y^4) - F(z^4) \right| \leq u(xz, yz) + \frac{1}{2}u(x^2, z^2) + \frac{1}{2}u(y^2, z^2) \quad (3.24)$$

for all $x, y, z \in S$. Now replacing x by xz and y by yz , respectively, we have

$$\begin{aligned} & \left| 2F(xz^4 y) - \frac{1}{2}F(x^4 y^4) - \frac{1}{2}F(y^4 z^4) - F(z^4) \right| \\ & \leq u(xz^2, yz^4) + \frac{1}{2}u(x^2 z^2, z^2) + \frac{1}{2}u(y^2 z^2, z^2) \end{aligned} \quad (3.25)$$

for all $x, y, z \in S$. Replacing x by xz and y by yz in (3.21), (3.22), and (3.23), respectively, one obtains

$$\begin{aligned} & \left| 2F(xz^4 y) - F(x^2 z^4) - F(y^2 z^4) \right| \leq u(xz^2, yz^2), \\ & \left| F(x^2 z^4) - \frac{1}{2}F(x^4 z^4) - \frac{1}{2}F(z^4) \right| \leq \frac{1}{2}u(x^2 z^2, z^2), \\ & \left| F(y^2 z^4) - \frac{1}{2}F(y^4 z^4) - \frac{1}{2}F(z^4) \right| \leq \frac{1}{2}u(y^2 z^2, z^2) \end{aligned} \quad (3.26)$$

for all $x, y, z \in S$. Also from (3.22) and (3.23), we have

$$\begin{aligned} & \left| \frac{1}{2}F(x^4 z^4) - \frac{1}{4}F(x^8) - \frac{1}{4}F(z^8) \right| \leq \frac{1}{4}u(x^4, z^4), \\ & \left| \frac{1}{2}F(y^4 z^4) - \frac{1}{4}F(y^8) - \frac{1}{4}F(z^8) \right| \leq \frac{1}{4}u(y^4, z^4) \end{aligned} \quad (3.27)$$

for all $x, y, z \in S$. Thus we have

$$\begin{aligned}
& \left| 2F(xz^4y) - \frac{1}{4}F(x^8) - \frac{1}{4}F(y^8) - \frac{1}{2}F(z^8) - F(z^4) \right| \\
& \leq \left| 2F(xz^4y) - F(x^2z^4) - F(y^2z^4) \right| + \left| F(x^2z^4) - \frac{1}{2}F(x^4z^4) - \frac{1}{2}F(z^4) \right| \\
& \quad + \left| F(y^2z^4) - \frac{1}{2}F(y^4z^4) - \frac{1}{2}F(z^4) \right| + \left| \frac{1}{2}F(x^4z^4) - \frac{1}{4}F(x^8) - \frac{1}{4}F(z^8) \right| \\
& \quad + \left| \frac{1}{2}F(y^4z^4) - \frac{1}{4}F(y^8) - \frac{1}{4}F(z^8) \right| \\
& \leq u(xz^2, yz^2) + \frac{1}{2}u(x^2z^2, z^2) + \frac{1}{2}u(y^2z^2, z^2) + \frac{1}{4}u(x^4, z^4) + \frac{1}{4}u(y^4, z^4),
\end{aligned} \tag{3.28}$$

for all $x, y, z \in S$. For arbitrary positive integer n , putting z^{2^n} instead of z in (3.24) and $z^{2^{n-1}}$ instead of z in (3.28), respectively, we see that

$$\begin{aligned}
& \left| 2F(xz^{2^{n+1}}y) - F(z^{2^{n+2}}) \right| \\
& \leq u(xz^{2^n}, yz^{2^n}) + \frac{1}{2}u(x^2, z^{2^{n+1}}) + \frac{1}{2}u(y^2, z^{2^{n+1}}) + \frac{1}{2}|F(x^4)| + \frac{1}{2}|F(y^4)|, \\
& \left| 2F(xz^{2^{n+1}}y) - \frac{1}{2}F(z^{2^{n+2}}) - F(z^{2^{n+1}}) \right| \\
& \leq u(xz^{2^n}, yz^{2^n}) + \frac{1}{2}u(x^2z^{2^n}, z^{2^n}) + \frac{1}{2}u(y^2z^{2^n}, z^{2^n}) \\
& \quad + \frac{1}{4}u(x^4, z^{2^{n+1}}) + \frac{1}{4}u(y^4, z^{2^{n+1}}) + \frac{1}{4}|F(x^8)| + \frac{1}{4}|F(y^8)|
\end{aligned} \tag{3.29}$$

for all $x, y, z \in S$. By (3.29) with $x = y$,

$$\begin{aligned}
& \left| \frac{F(z^{2^{n+2}})}{2^{n+2}} - \frac{F(z^{2^{n+1}})}{2^{n+1}} \right| \\
& \leq \frac{1}{2^{n+1}} \left| \frac{1}{2}F(z^{2^{n+2}}) - F(z^{2^{n+1}}) \right| \\
& \leq \frac{1}{2^{n+1}} \left(\left| F(z^{2^{n+2}}) - 2F(xz^{2^{n+1}}x) \right| + \left| 2F(xz^{2^{n+1}}x) - \frac{1}{2}F(z^{2^{n+2}}) - F(z^{2^{n+1}}) \right| \right) \\
& \leq \frac{1}{2^{n+1}} \left(2u(xz^{2^n}, xz^{2^n}) + u(x^2, z^{2^{n+1}}) + u(x^2z^{2^n}, z^{2^n}) \right. \\
& \quad \left. + \frac{1}{2}u(x^4, z^{2^{n+1}}) + |F(x^4)| + \frac{1}{2}|F(x^8)| \right)
\end{aligned} \tag{3.30}$$

for all $x, z \in S$. By (3.30), for every positive integer k, m with $k \geq m \geq 2$, we have

$$\begin{aligned}
& \left| \frac{F(z^{2^{m+k}})}{2^{m+k}} - \frac{F(z^{2^m})}{2^m} \right| \\
& \leq \sum_{i=1}^k \left| \frac{F(z^{2^{m+i}})}{2^{m+i}} - \frac{F(z^{2^{m+i-1}})}{2^{m+i-1}} \right| \\
& \leq \sum_{i=1}^{\infty} \frac{1}{2^{m+i-1}} \left(2u(xz^{2^{m+i-2}}, xz^{2^{m+i-2}}) + u(x^2, z^{2^{m+i-1}}) + u(x^2 z^{2^{m+i-2}}, z^{2^{m+i-2}}) \right. \\
& \quad \left. + \frac{1}{2}u(x^4, z^{2^{m+i-1}}) + |F(x^4)| + \frac{1}{2}|F(x^8)| \right) \\
& \leq \sum_{i=m-1}^{\infty} \frac{1}{2^i} u(xz^{2^i}, xz^{2^i}) + \sum_{i=m}^{\infty} \frac{1}{2^i} u(x^2, xz^{2^i}) + \sum_{i=m}^{\infty} \frac{1}{2^i} u(x^2 z^{2^{i-1}}, z^{2^i}) \\
& \quad + \frac{1}{2} \sum_{i=m}^{\infty} \frac{1}{2^i} u(x^4, z^{2^{i-1}}) + \left(|F(x^4)| + \frac{1}{2}|F(x^8)| \right) \sum_{i=m}^{\infty} \frac{1}{2^i} \rightarrow 0,
\end{aligned} \tag{3.31}$$

as $m \rightarrow \infty$. This proves that $\{F(z^{2^n})/2^n\}$ is a Cauchy sequence in R . Thus we can define a function $L : S \rightarrow R$ by

$$L(z) = \lim_{n \rightarrow \infty} \frac{F(z^{2^n})}{2^n} \tag{3.32}$$

for all $z \in S$. Then, by (3.20) and (3.31), we have

$$\begin{aligned}
|L(xy) - L(x) - L(y)| & \leq \lim_{n \rightarrow \infty} \left| \frac{F(x^{2^n} y^{2^n})}{2^n} - \frac{F(x^{2^n})}{2^n} - \frac{F(y^{2^n})}{2^n} \right| \\
& \leq \lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \left| 2F(x^{2^n} y^{2^n}) - F(x^{2^{n+1}}) - F(y^{2^{n+1}}) \right| \\
& \quad + \lim_{n \rightarrow \infty} \left| \frac{F(x^{2^{n+1}})}{2^{n+1}} - \frac{F(x^{2^n})}{2^n} \right| + \lim_{n \rightarrow \infty} \left| \frac{F(y^{2^{n+1}})}{2^{n+1}} - \frac{F(y^{2^n})}{2^n} \right| \\
& \leq \lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} u(x^{2^n}, y^{2^n}) + 0 + 0 = 0
\end{aligned} \tag{3.33}$$

for all $x, y \in S$. Thus

$$L(xy) = L(x) + L(y) \tag{3.34}$$

for all $x, y \in S$. Now replacing x by xz and then y by yz in (3.20), respectively, we obtain

$$\begin{aligned} |2F(xyz) - F(x^2z^2) - F(y^2)| &\leq u(xz, y), \\ |2F(xyz) - F(x^2) - F(y^2z^2)| &\leq u(x, yz) \end{aligned} \quad (3.35)$$

and so

$$|F(x^2) - F(y^2) + F(y^2z^2) - F(x^2z^2)| \leq u(xz, y) + u(x, yz) \quad (3.36)$$

for all $x, y, z \in S$. By (3.22), (3.23), and (3.36), we have

$$\left| F(x^2) - F(y^2) - \frac{1}{2}F(x^4) + \frac{1}{2}F(y^4) \right| \leq u(xz, y) + u(x, yz) + \frac{1}{2}u(x^2, z^2) + \frac{1}{2}u(y^2, z^2), \quad (3.37)$$

and thus

$$\left| F(x^4) - 2F(x^2) \right| \leq 2u(xz, y) + 2u(x, yz) + u(x^2, z^2) + u(y^2, z^2) + \left| F(y^4) - 2F(y^2) \right| < \infty \quad (3.38)$$

for all $x \in S$ and for fixed $y, z \in S$. By (3.3) and (3.30),

$$\begin{aligned} \left| L(z^4) - F(z^4) \right| &= 4 \left| L(z) - \frac{F(z^4)}{4} \right| = 4 \lim_{n \rightarrow \infty} \left| \frac{F(z^{2^n})}{2^n} - \frac{F(z^4)}{4} \right| \\ &= 4 \lim_{n \rightarrow \infty} \sum_{i=1}^{n-2} \left| \frac{F(z^{2^{i+2}})}{2^{i+2}} - \frac{F(z^{2^{i+1}})}{2^{i+1}} \right| < \infty \end{aligned} \quad (3.39)$$

for all $s \in S$. By (3.20), (3.38) and (3.39), for all $z \in S$ with $x = zw$, there exists a constant M such that

$$\begin{aligned} |L(x) - F(x)| &= |L(zw) - F(zw)| \\ &= \left| F(zw) - \frac{1}{2}F(z^2) - \frac{1}{2}F(w^2) \right| + \frac{1}{4} |L(z^4) - F(z^4)| \\ &\quad + \frac{1}{4} |L(w^4) - F(w^4)| + \frac{1}{4} |F(w^4) - 2F(w^2)| \\ &\quad + \frac{1}{4} |F(z^4) - 2F(z^2)| \\ &\leq M. \end{aligned} \quad (3.40)$$

Now we define a function $T : S \rightarrow (0, \infty)$ by

$$T(x) := e^{L(x)} \quad (3.41)$$

for all $x \in S$. Then

$$T(xy) = e^{L(xy)} = e^{L(x)+L(y)} = T(x)T(y) \quad (3.42)$$

for all $x, y \in S$. By (3.40), we have

$$-M \leq L(x) - \ln f(x) \leq M, \quad (3.43)$$

and thus for all $x \in S$

$$e^{-M} \leq \frac{T(x)}{f(x)} \leq e^M. \quad (3.44)$$

If φ is bounded, there exist constants M_0, M_1 such that

$$\begin{aligned} |G(x) - H(x)| &\leq |G(x) + H(y_0) - F(xy_0)| + |F(xy_0) - G(y_0) - H(x)| + |G(y_0) - H(y_0)| \\ &\leq \ln(1 + \varphi(x, y_0)) + \ln(1 + \varphi(y_0, x)) + |G(y_0) - H(y_0)| \leq M_0, \end{aligned} \quad (3.45)$$

and so

$$\begin{aligned} |L(x) - G(x)| &\leq \frac{1}{2} |L(x^2) - F(x^2)| + \frac{1}{2} |F(x^2) - H(x) - G(x)| + \frac{1}{2} |H(x) - G(x)| \\ &\leq \frac{1}{2} (M + \ln(1 + \varphi(x, x)) + M_0) \leq M_1, \end{aligned} \quad (3.46)$$

and by the same method above, we have

$$|L(x) - H(x)| \leq M_1 \quad (3.47)$$

for all $x \in S$. Therefore, we have

$$e^{-M_1} \leq \frac{T(x)}{g(x)} \leq e^{M_1}, \quad e^{-M_1} \leq \frac{T(x)}{h(x)} \leq e^{M_1} \quad (3.48)$$

for all $x \in S$. □

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