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Research Article

A New Nonlinear Retarded Integral Inequality and Its Application

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The main objective of this paper is to establish a new retarded nonlinear integral inequality with two variables, which provide explicit bound on unknown function. This inequality given here can be used as tool in the study of integral equations.

1. Introduction

Being important tools in the study of differential equations, integral equations and integro-differential equations, various generalizations of Gronwall inequality and their applications have attracted great interests of many mathematicians. Some recent works can be found, for example, in [1–7] and some references therein. Agarwal et al. [1] studied the inequality

$$u(t) \leq a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} g_i(t,s) w_i(u(s)) ds, \quad t_0 \leq \mathrm{t} < t_1. \tag{1.1}$$

Agarwal et al. [2] obtained the explicit bound to the unknown function of the following retarded integral inequality

$$\varphi(u(t)) \leq c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s) \left[f_i(s) \varphi_1(u(s)) + g_i(s) \varphi_2(\log(u(s))) \right] ds. \tag{1.2}$$

Cheung [3] investigated the inequality in two variables

$$\begin{aligned} u^p(x,y) \le a &+ \frac{p}{p-q}\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} g_1(s,t)u^q(s,t)dt\,ds \\ &+ \frac{p}{p-q}\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} g_2(s,t)u^q(s,t)\varphi(u(s,t))dt\,ds. \end{aligned} \tag{1.3}$$

Chen et al. [4] discussed the following inequality in two variables

$$\begin{aligned} \psi(u(x,y)) \le c &+ \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} g(s,t)w(u(s,t))dt\,ds \\ &+ \int_{\gamma(x_0)}^{\gamma(x)}\int_{\delta(y_0)}^{\delta(y)} f(s,t)w(u(s,t))\varphi(u(s,t))dt\,ds. \end{aligned} \tag{1.4}$$

Pachpatte [8] obtained an upper bound in the following inequality:

$$u^2(t) \le \left(c_1^2 + 2\int_0^t f(s)u(s)ds\right)\left(c_2^2 + 2\int_0^t h(s)u(s)ds\right). \tag{1.5}$$

Pachpatte [9] firstly got the estimation of the unknown function of the following inequality:

$$u(t) \le \left(c_1 + \int_0^t f(s)u(s)ds\right)\left(c_2 + \int_0^t h(s)u(s)ds\right), \tag{1.6}$$

then, the estimation was used to study the boundedness, asymptotic behavior, slowly growth of the solutions of the integral equation

$$u(t) = k\left(c_1(t) - \int_0^t f_1(t-s)u(s)ds\right)\left(c_2(t) + \int_0^t f_2(t-s)u(s)ds\right), \tag{1.7}$$

(1.7) was studied by Gripenberg in [10].

However, the bound given on such inequality in [8] is not directly applicable in the study of certain retarded differential and integral equations. It is desirable to establish new inequalities of the above type, which can be used more effectively in the study of certain classes of retarded differential and integral equations.

In this paper, we establish a new integral inequality

$$\psi(u(x,y)) \le \left(c_1(x,y) + \int_{\alpha_1(x_0)}^{\alpha_1(x)} \int_{\beta_1(y_0)}^{\beta_1(y)} f_1(s,t)\varphi_1(u(s,t))dt\,ds \right) \times \left(c_2(x,y) + \int_{\alpha_2(x_0)}^{\alpha_2(x)} \int_{\beta_2(y_0)}^{\beta_2(y)} f_2(s,t)\varphi_2(u(s,t))dt\,ds \right). \quad (1.8)$$

We will prove importance of (1.8) in achieving a desired goal.

2. Main Result

Throughout this paper, $x_0, x_1, y_0, y_1 \in \mathbb{R}$ are given numbers, and $x_0 < x_1$, $y_0 < y_1$. $I := [x_0, x_1)$, $J := [y_0, y_1)$, $\Delta := I \times J$, $\mathbb{R}_+ := [0, \infty)$. For functions $h(x)$, $g(x,y)$, $h'(x)$ denotes the derivative of $h(x)$, and $g_x(x,y)$ denotes the partial derivative $g(x,y)$ on x. Consider (1.8), and suppose that

(H_1) $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a strictly increasing function with $\psi(0) = 0$ and $\psi(t) \to \infty$ as $t \to \infty$;

(H_2) $c_1, c_2 : \Delta \to (0, \infty)$ are nondecreasing in each variable;

(H_3) $\varphi_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing with $\varphi_i(r) > 0$ for $r > 0, i = 1, 2$;

(H_4) $\alpha_i \in C^1(I, I)$ and $\beta_i \in C^1(J, J)$ are nondecreasing such that $\alpha_i(x) \le x$ and $\beta_i(y) \le y, i = 1, 2$;

(H_5) $f_i \in C(\Delta, \mathbb{R}_+), i = 1, 2$.

We define functions Φ, Ψ, and φ by

$$\Phi(r) := \int_0^r \frac{ds}{\varphi(\psi^{-1}(s))}, \qquad \Psi(r) := \int_0^r \frac{ds}{\varphi(\psi^{-1}(\Phi^{-1}(s)))}, \quad r > 0, \qquad \varphi(r) := \max\{\varphi_1(r), \varphi_2(r)\}. \quad (2.1)$$

Theorem 2.1. *Suppose that (H_1)–(H_5) hold and $u(x,y)$ is a nonnegative and continuous function on Δ satisfying* (1.8). *Then one has*

$$u(x,y) \le \psi^{-1}\left(\Phi^{-1}\left(\Psi^{-1}(E(x,y))\right)\right), \quad (2.2)$$

for all $(x,y) \in [x_0, X_1) \times [y_0, Y_1)$, *where*

$$E(x,y) = \Psi(G(x,y)) + \sum_{i=1}^{2} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \left[\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dt \right) \times \int_{\alpha_{3-i}(x_0)}^{\alpha_{3-i}(s)} \int_{\beta_{3-i}(y_0)}^{\beta_{3-i}(y)} f_{(3-i)}(\sigma,t)dt\, d\sigma \right] ds,$$
$$G(x,y) = \Phi(c_1(x,y)c_2(x,y)) + \sum_{i=1}^{2} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} c_{(3-i)}(s,t) f_i(s,t)dt\, ds, \tag{2.3}$$

φ^{-1}, Φ^{-1}, *and* Ψ^{-1} *denote the inverse function of* φ, Φ *and* Ψ, *respectively, and* $(X_1, Y_1) \in \Delta$ *is arbitrarily given on the boundary of the planar region*

$$\mathcal{R} := \left\{ (x,y) \in \Delta : E(x,y) \in \mathrm{Dom}\left(\Psi^{-1}\right), \Psi^{-1}(E(x,y)) \in \mathrm{Dom}\left(\Phi^{-1}\right) \right\}. \tag{2.4}$$

Proof. From the inequality (1.8), for all $(x,y) \in [x_0, X] \times J$, we have

$$\varphi(u(x,y)) \le \left(c_1(X,y) + \int_{\alpha_1(x_0)}^{\alpha_1(x)} \int_{\beta_1(y_0)}^{\beta_1(y)} f_1(s,t)\varphi_1(u(s,t))dt\, ds \right) \times \left(c_2(X,y) + \int_{\alpha_2(x_0)}^{\alpha_2(x)} \int_{\beta_2(y_0)}^{\beta_2(y)} f_2(s,t)\varphi_2(u(s,t))dt\, ds \right), \tag{2.5}$$

where $x_0 \le X \le X_1$ is chosen arbitrarily, using the assumption H_2. For convenience, we define a function $\theta(x,y)$ by the right-hand side of (1.8), that is,

$$\theta(x,y) = \left(c_1(X,y) + \int_{\alpha_1(x_0)}^{\alpha_1(x)} \int_{\beta_1(y_0)}^{\beta_1(y)} f_1(s,t)\varphi_1(u(s,t))dt\, ds \right) \times \left(c_2(X,y) + \int_{\alpha_2(x_0)}^{\alpha_2(x)} \int_{\beta_2(y_0)}^{\beta_2(y)} f_2(s,t)\varphi_2(u(s,t))dt\, ds \right). \tag{2.6}$$

□

By the assumptions (H_2)–(H_5), $\theta(x,y)$ is a positive and nondecreasing function in each variable, $\theta(x_0,y) = c_1(X,y)c_2(X,y) > 0$. Differentiating both sides of (2.6) and using the fact that $u(x,y) \leq \psi^{-1}(\theta(x,y))$, we obtain

$$\begin{aligned}\theta_x(x,y) &= \sum_{i=1}^{2}\left(\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)\varphi_i(u(\alpha_i(x),t))dt\right)\\&\quad\times\left(c_{3-i}(X,y)+\int_{\alpha_{3-i}(x_0)}^{\alpha_{3-i}(x)}\int_{\beta_{3-i}(y_0)}^{\beta_{3-i}(y)} f_{3-i}(s,t)\varphi_{3-i}(u(s,t))dt\,ds\right)\\&\leq \varphi\left(\psi^{-1}(\theta(x,y))\right)\sum_{i=1}^{2}\left(\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)dt\right)\\&\quad\times\left(c_{3-i}(X,y)+\int_{\alpha_{3-i}(x_0)}^{\alpha_{3-i}(x)}\int_{\beta_{3-i}(y_0)}^{\beta_{3-i}(y)} f_{3-i}(s,t)\varphi_{3-i}\left(\psi^{-1}(\theta(s,t))\right)dt\,ds\right),\end{aligned} \tag{2.7}$$

for all $(x,y) \in [x_0,X]\times J$. From (2.7), we get

$$\begin{aligned}\frac{\theta_x(x,y)}{\varphi(\psi^{-1}(\theta(x,y)))} &\leq \sum_{i=1}^{2}\left(\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)dt\right)\\&\quad\times\left(c_{3-i}(X,y)+\int_{\alpha_{3-i}(x_0)}^{\alpha_{3-i}(x)}\int_{\beta_{3-i}(y_0)}^{\beta_{3-i}(y)} f_{3-i}(s,t)\varphi_{3-i}\left(\psi^{-1}(\theta(s,t))\right)dt\,ds\right).\end{aligned} \tag{2.8}$$

By taking $s = x$ in (2.8) and then integrating it from x_0 to x, we get

$$\begin{aligned}\Phi(\theta(x,y)) &\leq \Phi(\theta(x_0,y))+\sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} c_{(3-i)}(X,y)f_i(s,t)dt\,ds\\&\quad+\sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\left[\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dt\right)\right.\\&\quad\left.\times\int_{\alpha_{3-i}(x_0)}^{\alpha_{3-i}(s)}\int_{\beta_{3-i}(y_0)}^{\beta_{3-i}(y)} f_{(3-i)}(\sigma,t)\varphi_{3-i}\left(\psi^{-1}(\theta(\sigma,t))\right)dt\,d\sigma\right]ds\\&\leq \Phi(c_1(X,y)c_2(X,y))+\sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)} c_{(3-i)}(X,y)f_i(s,t)dt\,ds\\&\quad+\sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\left[\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dt\right)\right.\\&\quad\left.\times\int_{\alpha_{3-i}(x_0)}^{\alpha_{3-i}(s)}\int_{\beta_{3-i}(y_0)}^{\beta_{3-i}(y)} f_{(3-i)}(\sigma,t)\varphi_{3-i}\left(\psi^{-1}(\theta(\sigma,t))\right)dt\,d\sigma\right]ds,\end{aligned} \tag{2.9}$$

for all $(x,y) \in [x_0,X]\times[y_0,y_1)$, where using the definition of Φ in (2.1). Similarly to the above statement, we define a function $\omega(x,y)$ by the right-hand side of (2.9), then $\omega(x,y)$ is a positive and nondecreasing function in each variable, $\theta(x,y) \le \Phi^{-1}(\omega(x,y))$ and $\omega(x_0,y) = \Phi(c_1(X,y)c_2(X,y)) + \sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)} c_{(3-i)}(X,y)f_i(s,t)dt\,ds$. Differentiating $\omega(x,y)$ for x, by the relation among φ and φ_1,φ_2, we have

$$\begin{aligned}
\omega_x(x,y) &= \sum_{i=1}^{2}\alpha_i'(x)\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)dt\right) \\
&\quad\times \int_{\alpha_{(3-i)}(x_0)}^{\alpha_{(3-i)}(x)}\int_{\beta_{(3-i)}(y_0)}^{\beta_{(3-i)}(y)} f_{(3-i)}(\sigma,t)\varphi_{(3-i)}\left(\varphi^{-1}(\theta(\sigma,t))\right)dt\,d\sigma \\
&\le \sum_{i=1}^{2}\alpha_i'(x)\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)dt\right) \\
&\quad\times \int_{\alpha_{(3-i)}(x_0)}^{\alpha_{(3-i)}(x)}\int_{\beta_{(3-i)}(y_0)}^{\beta_{(3-i)}(y)} f_{(3-i)}(\sigma,t)\varphi\left(\varphi^{-1}\left(\Phi^{-1}(\omega(\sigma,t))\right)\right)dt\,d\sigma \\
&\le \varphi\left(\varphi^{-1}\left(\Phi^{-1}(\omega(x,y))\right)\right)\sum_{i=1}^{2}\alpha_i'(x)\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)dt\right) \\
&\quad\times \int_{\alpha_{(3-i)}(x_0)}^{\alpha_{(3-i)}(x)}\int_{\beta_{(3-i)}(y_0)}^{\beta_{(3-i)}(y)} f_{(3-i)}(\sigma,t)\ dt\,d\sigma, \quad \forall(x,y) \in [x_0,X]\times[y_0,Y_1],
\end{aligned} \tag{2.10}$$

where Y_1 is defined by (2.4). From (2.10), we have

$$\begin{aligned}
\frac{\omega_x(x,y)}{\varphi(\varphi^{-1}(\Phi^{-1}(\omega(x,y))))} &\le \sum_{i=1}^{2}\alpha_i'(x)\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)dt\right) \\
&\quad\times \int_{\alpha_{(3-i)}(x_0)}^{\alpha_{(3-i)}(x)}\int_{\beta_{(3-i)}(y_0)}^{\beta_{(3-i)}(y)} f_{(3-i)}(\sigma,t)dt\,d\sigma,
\end{aligned} \tag{2.11}$$

for all $(x,y) \in [x_0,X]\times[y_0,Y_1]$. By taking $s = x$ in (2.11) and then integrating it from x_0 to x, using the definition of Ψ in (2.1), we get

$$\begin{aligned}
\Psi(\omega(x,y)) &\le \Psi(\omega(x_0,y)) + \sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\left[\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dt\right)\right. \\
&\quad\left.\times \int_{\alpha_{(3-i)}(x_0)}^{\alpha_{(3-i)}(s)}\int_{\beta_{(3-i)}(y_0)}^{\beta_{(3-i)}(y)} f_{(3-i)}(\sigma,t)dt\,d\sigma\right]ds \\
&= \Psi\left(\Phi(c_1(X,y)c_2(X,y)) + \sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)} c_{(3-i)}(X,y)f_i(s,t)dt\,ds\right) \\
&\quad+ \sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\left[\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dt\right)\times\int_{\alpha_{(3-i)}(x_0)}^{\alpha_{(3-i)}(s)}\int_{\beta_{(3-i)}(y_0)}^{\beta_{(3-i)}(y)} f_{(3-i)}(\sigma,t)dt\,d\sigma\right]ds.
\end{aligned} \tag{2.12}$$

Using the fact $u(x,y) \le \varphi^{-1}(\theta(x,y))$ and $\theta(x,y) \le \Phi^{-1}(\omega(x,y))$, from (2.12) we obtain

$$\begin{aligned} u(x,y) &\le \varphi^{-1}(\theta(x,y)) \le \varphi^{-1}\left(\Phi^{-1}(\omega(x,y))\right) \\ &\le \varphi^{-1}\Bigg(\Phi^{-1}\Bigg(\Psi^{-1}\Bigg(\Psi\Bigg(\Phi(c_1(X,y)c_2(X,y)) \\ &\quad +\sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)} c_{(3-i)}(X,y)f_i(s,t)dt\,ds\Bigg) \\ &\quad +\sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\left[\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dt\right)\times\int_{\alpha_{(3-i)}(x_0)}^{\alpha_{(3-i)}(s)}\int_{\beta_{(3-i)}(y_0)}^{\beta_{(3-i)}(y)} f_{(3-i)}(\sigma,t)dt\,d\sigma\right]ds\Bigg)\Bigg)\Bigg). \end{aligned} \tag{2.13}$$

Let $x = X$, from (2.13)we observe that

$$\begin{aligned} u(X,y) &\le \varphi^{-1}\Bigg(\Phi^{-1}\Bigg(\Psi^{-1}\Bigg(\Psi\Bigg(\Phi(c_1(X,y)c_2(X,y)) \\ &\quad +\sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)} c_{(3-i)}(X,y)f_i(s,t)dt\,ds\Bigg) \\ &\quad +\sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\left[\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dt\right)\times\int_{\alpha_{(3-i)}(x_0)}^{\alpha_{(3-i)}(s)}\int_{\beta_{(3-i)}(y_0)}^{\beta_{(3-i)}(y)} f_{(3-i)}(\sigma,t)dt\,d\sigma\right]ds\Bigg)\Bigg)\Bigg). \end{aligned} \tag{2.14}$$

Since $X \in [x_0, X_1)$ is arbitrary, from (2.14), we get the required estimation (2.2).

3. Applications

In this section, we present an application of our result to obtain bound of the solution of a integral equation:

$$\begin{aligned} \psi(z(x,y)) = k&\left(a_1(x,y) - \int_{\alpha_1(x_0)}^{\alpha_1(x)}\int_{\beta_1(y_0)}^{\beta_1(y)} g_1(x-s,t)\varphi_1(z(s,t))dt\,ds\right) \\ &\times\left(a_2(x,y) + \int_{\alpha_2(x_0)}^{\alpha_2(x)}\int_{\beta_2(y_0)}^{\beta_2(y)} g_2(x-s,t)\varphi_2(z(s,t))dt\,ds\right), \quad \forall(x,y)\in\Delta, \end{aligned} \tag{3.1}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $\psi(0) = 0, |\psi(r)| = \psi(|r|) > 0$, and $\psi(t) \to \infty$ as $t \to \infty$, k is a given positive constant, $|a_1|, |a_2| : \Delta \to \mathbb{R}_+$ are bounded functions and nondecreasing in each variable, functions α_i and β_i satisfy hypothesis H_4, i=1,2, $g_i, z \in C^0(\Delta, \mathbb{R})$ and $\varphi_i \in C^0(\mathbb{R},\mathbb{R})$ is nondecreasing on $\mathbb{R}_+$ such that $|\varphi_i(u)| = \varphi_i(|u|), \varphi_i(u) > 0$ for $u > 0, i = 1, 2$.

The integral equation (3.1) is obviously more general than (1.7) considered in [10]. When keeping y fixed, let $\varphi(z(x,y)) = z(x,y), \varphi_i(z(x,y)) = z(x,y), \alpha_i(x) = x, i = 1,2, x_0 = 0$, then integral equation (3.1) reduces to integral equation (1.7) in [10].

Corollary 3.1. *Consider integral equation* (3.1) *and suppose that* $|g_i(x-s,t)| \le f_i(s,t), i = 1,2$, *where* $f_i \in C^0(\Delta, \mathbb{R}_+)$. *Then all solutions* $z(x,y)$ *of* (3.1) *have the estimate*

$$|z(x,y)| \le \varphi^{-1}\left(\Phi^{-1}\left(\Psi^{-1}(H(x,y))\right)\right), \tag{3.2}$$

for all $(x,y) \in [x_0, X_2) \times [y_0, Y_2)$, *where*

$$H(x,y) = \Psi(B(x,y)) + \sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\left[\left(\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dt\right) \times \int_{\alpha_{3-i}(x_0)}^{\alpha_{3-i}(s)}\int_{\beta_{3-i}(y_0)}^{\beta_{3-i}(y)} f_{(3-i)}(\sigma,t)dt\,d\sigma\right]ds,$$
$$B(x,y) = \Phi(|a_1(x,y)||a_2(x,y)|) + \sum_{i=1}^{2}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} |a_{(3-i)}(s,t)|f_i(s,t)dt\,ds. \tag{3.3}$$

Functions $\Phi, \Phi^{-1}, \Psi, \Psi^{-1}$ *are defined as in Theorem 2.1, and* $(X_2, Y_2) \in \Delta$ *is arbitrarily given on the boundary of the planar region*

$$\mathcal{R} := \left\{(x,y) \in \Delta : H(x,y) \in \mathrm{Dom}\left(\Psi^{-1}\right), \Psi^{-1}(H(x,y)) \in \mathrm{Dom}\left(\Phi^{-1}\right)\right\}. \tag{3.4}$$

Proof. From the integral equation (3.1), we have

$$\begin{aligned}
\varphi(|z(x,y)|) \le{}& \left(|a_1(x,y)| + \int_{\alpha_1(x_0)}^{\alpha_1(x)}\int_{\beta_1(y_0)}^{\beta_1(y)} |g_1(x-s,t)|\varphi_1(|z(s,t)|)dt\,ds\right) \\
&\times\left(|a_2(x,y)| + \int_{\alpha_2(x_0)}^{\alpha_2(x)}\int_{\beta_2(y_0)}^{\beta_2(y)} |g_2(x-s,t)|\varphi_2(|z(s,t)|)dt\,ds\right) \\
\le{}& \left(|a_1(x,y)| + \int_{\alpha_1(x_0)}^{\alpha_1(x)}\int_{\beta_1(y_0)}^{\beta_1(y)} f_1(s,t)\varphi_1(|z(s,t)|)dt\,ds\right) \\
&\times\left(|a_2(x,y)| + \int_{\alpha_2(x_0)}^{\alpha_2(x)}\int_{\beta_2(y_0)}^{\beta_2(y)} f_2(s,t)\varphi_2(|z(s,t)|)dt\,ds\right), \quad \forall (x,y) \in \Delta.
\end{aligned} \tag{3.5}$$

Clearly, inequality (3.5) is in the form of (1.8). Thus, the estimate (3.2) of the solution $z(x,y)$ in this corollary is obtained immediately by our Theorem 2.1. □

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